Representations and partition function zeros of the Potts model with and without boundaries

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WORK WITH JÉRÔME DUBAIL, HUBERT SALEUR, ALAN SOKAL

Summary

1. Representations of the Potts model
   - From Potts to Boundary Loop models
   - From Potts to RSOS models
   - Hierarchy of truncations between models
2. Decomposition of loop model partition functions
   - Transfer matrix structure
   - Eigenvalue amplitudes
3. Conformal properties
   - Reminder of 0BTL results
   - New results for 1BTL and 2BTL
4. Partition function zeros
   - Beraha-Kahane-Weiss theorem
   - Boundary chromatic polynomial

Summary

Fortuin-Kasteleyn's *random cluster* representation

- Q state model on graph $G = (V, E)$ with edge weights $v_e = \mathrm{e}^{J_e} - 1$

$$Z_G(Q, \mathbf{v}) = \sum_{A \subseteq E} Q^{C(A)} \prod_{e \in A} v_e$$

Distinguish boundary vertices

- Let G be planar; embed G in an annulus; distinguish vertices on left/right rims
- Let $C = \#$ any clusters, and $C_{\mathrm{l}}, C_{\mathrm{r}}, C_{\mathrm{b}} = \#$ clusters touching left/right/both rims
- Let $Q = \#$ bulk states, $Q_{\mathrm{l}}, Q_{\mathrm{r}} = \#$ boundary states, and $Q_{\mathrm{b}} = \#$ states common to left and right boundaries

$$Z_G(Q, \mathbf{v}) = \sum_{A \subseteq E} Q^C \left(\frac{Q_{\mathrm{l}}}{Q}\right)^{C_{\mathrm{l}}} \left(\frac{Q_{\mathrm{r}}}{Q}\right)^{C_{\mathrm{r}}} \left(\frac{Q_{\mathrm{b}}}{Q}\right)^{C_{\mathrm{b}}} \prod_{e \in A} v_e$$

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Loop representation

- Loops on medial lattice separate clusters and their duals
- Let $N = \#$ any loops, etc.
- Use Euler relation $C = (N + |V| - |E|)/2$

$$Z_G(Q, \mathbf{v}) = Q^{|V|/2} \sum_{A \subseteq E} Q^{N/2} \left(\frac{Q_\mathrm{l}}{Q}\right)^{N_\mathrm{l}} \left(\frac{Q_\mathrm{r}}{Q}\right)^{N_\mathrm{r}} \left(\frac{Q_\mathrm{b}}{Q}\right)^{N_\mathrm{b}} \prod_{e \in A} \frac{v_e}{\sqrt{Q}}$$

Boundary Loop model

- Weight n for bulk loops, and $n_\mathrm{l}, n_\mathrm{r}, n_\mathrm{b}$ for boundary loops

$$Q = n^2, \qquad Q_\mathrm{l} = nn_\mathrm{l}, \qquad Q_\mathrm{r} = nn_\mathrm{r}, \qquad Q_\mathrm{b} = nn_\mathrm{b}$$

- Could weight differently $(\ell, \ell_\mathrm{l}, \ell_\mathrm{r}, \ell_\mathrm{b})$ non-contractible loops

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Summary

RSOS MODEL (OR *graph homomorphism* MODEL)

$$Z_\Gamma^{\rm RSOS} = \sum_{\sigma \in {\rm Hom}(\Gamma, H)} W(\sigma)$$

- Sum over homomorphisms from graph $\Gamma = (\mathcal{V}, \mathcal{E})$ to (locally) finite target graph H

INTERACTION-ROUND-A-FACE (IRF) MODEL

- $\Gamma = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ is a connected plane quadrangulation
- Let $\mathcal{A}$ be the adjacency matrix of H
- Let $\mathcal{A}\psi = \lambda\psi$ with all $\psi_x \neq 0$

$$\begin{aligned} W(\sigma) &= \left(\prod_{i \in \mathcal{V}} W_i(\sigma_i)\right) \left(\prod_{F \in \mathcal{F}} W_F(\sigma_F)\right) \\ W_i(\sigma_i) &= \psi_{\sigma_i} \\ W_F(\sigma_{i_1}, \sigma_{i_2}, \sigma_{i_3}, \sigma_{i_4}) &= a_e \psi_{\sigma_{i_1}}^{-1} \delta(\sigma_{i_1}, \sigma_{i_3}) + b_e \psi_{\sigma_{i_2}}^{-1} \delta(\sigma_{i_2}, \sigma_{i_4}) \end{aligned}$$

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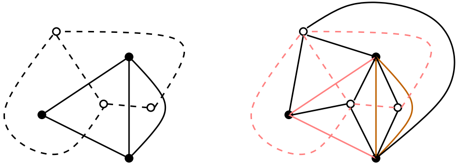
Potts-RSOS mapping [Kostov, JJ-Sokal]

- Let $G = (V, E)$ be a planar graph with medial $\mathcal{M}(G)$
- Let $\Gamma = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ be the quadrangulation $\mathcal{M}(G^*)$
- Let $Z_\Gamma^{\rm RSOS}$ with vertex and face weights as above

$$Z_\Gamma^{\rm RSOS}(\mathbf{a}, \mathbf{b}) = |\psi|^2 \lambda^{-|V|} \left(\prod_{e \in E} b_e \right) Z_G(\lambda^2, \lambda \mathbf{a}/\mathbf{b})$$



(a) (b)

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Potts to RSOS truncation

- In Potts loop model, Q can take any real value
- In RSOS model, not all values of $Q = \lambda^2$ are possible
  - If H is *connected* and *simple*, it is a Dynkin-Coxeter graph
  - Its Perron-Frobenius eigenvector has $0 < \lambda_{\mathrm{PF}} < 2$
  - In particular for H = line graph A_n

$$\lambda_{\mathrm{PF}} = 2\cos\left(\frac{\pi}{n}\right)$$

- Almost explains chromatic zeros at the Beraha numbers!

Complete truncation hierarchy

- 2BTL: Potts model with two distinguished boundaries
- 1BTL: As above, but $n_{\mathrm{r}} = n$ and $n_{\mathrm{b}} = n_{\mathrm{l}}$
- 0BTL: As above, but $n_{\mathrm{l}} = n$ (ordinary Potts model)
- RSOS: As above, but $n = \lambda$ takes particular discrete values

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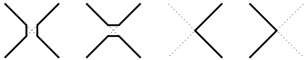
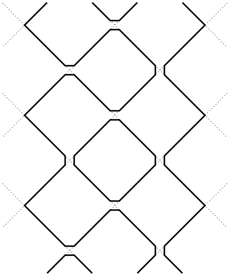
TRANSFER MATRIX (e.g. for 0BTL)

- Take regular lattice of fixed width on an annulus
- Built “upwards” by action of elementary generators



- Example on the square lattice



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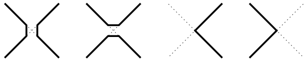

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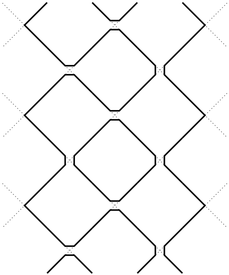
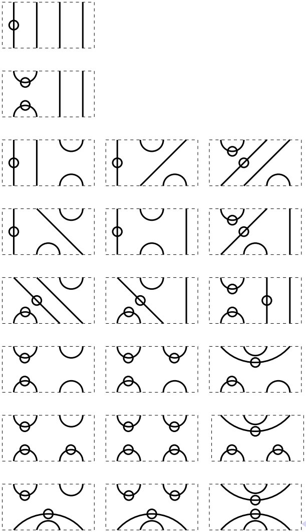


SPACE OF STATES ON 4 STRANDS (e.g. for 1BTL)



ALGEBRAIC STRUCTURE FOR 0BTL

- Temperley-Lieb (TL) algebra

$$
\begin{aligned}
e_i^2 &= n e_i \\
e_i e_{i\pm1} e_i &= e_i \\
[e_i, e_j] &= 0 \text{ for } |i-j| \geq 2
\end{aligned}
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FURTHER ALGEBRAIC STRUCTURE FOR 1BTL

- 1BTL algebra (or *blob algebra*)

$$
\begin{aligned}
b_1^2 &= b_1 \\
e_1 b_1 e_1 &= n_1 e_1 \\
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Algebraic structure for 0BTL

- Temperley-Lieb (TL) algebra

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Further algebraic structure for 1BTL

- 1BTL algebra (or *blob algebra*)

$$\begin{aligned} b_{\mathrm{l}}^2 &= b_{\mathrm{l}} \\ e_1 b_{\mathrm{l}} e_1 &= n_{\mathrm{l}} e_1 \\ [b_{\mathrm{l}}, e_i] &= 0 \text{ for } i = 2, 3, \ldots, N-1 \end{aligned}$$

TRANSFER MATRIX STRUCTURE (e.g. for 1BTL)

$$T = \left(\prod_{j=1}^{N/2-1} (I + e_{2j}) \right) \left(\prod_{j=1}^{N/2} (I + e_{2j-1}) \right) b_{\mathrm{l}}$$

- T is block-diagonal wrt lower half state
  - Hence all eigenvalues gotten by acting on upper half state
- T is lower block-triangular wrt # through lines
  - Hence fix # through lines: sectors T_ℓ
- T_ℓ lower block-triangular wrt blobbing of leftmost line
  - Hence fix blobbing status of leftmost line: sectors T_ℓ^{b} and T_ℓ^{u}

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Transfer matrix structure (e.g. for 1BTL)

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DECOMPOSITION OF PARTITION FUNCTION (e.g. for 1BTL)

- $Z_{N,M}$ for annulus of width $N = 2N_2$ and length M
- Not simply $\mathrm{Tr}\, T$—eigenvalues come with amplitudes

$$Z_{N,M} = \sum_{j=0}^{N_2-1} D^{\mathrm{u}}_{2j} K^{\mathrm{u}}_{2j} + \sum_{j=1}^{N_2} D^{\mathrm{b}}_{2j} K^{\mathrm{b}}_{2j}$$

$$K^{\alpha}_{\ell} \equiv \mathrm{Tr}\,(T^{\alpha}_{\ell})^M = \sum_{i=1}^{d^{\alpha}_{\ell}} \left(\lambda_i^{(\alpha,\ell)}\right)^M$$

- Combinatorial method for determining the D
  - Express K in terms of Z by counting *compatible states*
  - Then invert to get Z in terms of K

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RESULTS FOR EIGENVALUE AMPLITUDES

- In terms of $U_k(x) = k$th order Chebyshev of 2nd kind

$$\begin{aligned} D_j^{\rm uu} &= U_j(n/2) - (n_{\rm l} + n_{\rm r}) U_{j-1}(n/2) + n_{\rm l} n_{\rm r} U_{j-2}(n/2) \\ D_j^{\rm ub} &= n_{\rm r} U_{j-1}(n/2) - (1 + n_{\rm l} n_{\rm r}) U_{j-2}(n/2) + n_{\rm l} U_{j-3}(n/2) \\ D_j^{\rm bu} &= n_{\rm l} U_{j-1}(n/2) - (1 + n_{\rm l} n_{\rm r}) U_{j-2}(n/2) + n_{\rm r} U_{j-3}(n/2) \\ D_j^{\rm bb} &= n_{\rm l} n_{\rm r} U_{j-2}(n/2) - (n_{\rm l} + n_{\rm r}) U_{j-3}(n/2) + U_{j-4}(n/2) \end{aligned}$$

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$$D_j = U_j(n/2)$$

- RSOS case: trivial amplitudes (all $D = 1$)

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Summary

Parametrisation of unitary minimal models

- Central charge (for $p \geq 2$ integer)
$$c = 1 - \frac{6}{p(p+1)}$$
- Scaling dimensions (for $1 \leq r < p$ and $1 \leq s \leq p$)
$$h_{r,s} = \frac{[(p+1)r - ps]^2 - 1}{4p(p+1)}$$

Results for 0BTL

- Above c, when $n = 2\cos\gamma$ and $\gamma = \frac{\pi}{p+1}$
- Leading exponent of T_ℓ sector is $h_\ell = h_{1,1+\ell}$

Parametrisation of unitary minimal models

- Central charge (for $p \geq 2$ integer)

$$c = 1 - \frac{6}{p(p+1)}$$

- Scaling dimensions (for $1 \leq r < p$ and $1 \leq s \leq p$)

$$h_{r,s} = \frac{[(p+1)r - ps]^2 - 1}{4p(p+1)}$$

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Summary

1BTL CASE

- Set $n_{\rm l} = \frac{\sin(r_{\rm l}+1)\gamma}{\sin r_{\rm l}\gamma}$
- Exponent of $T_\ell^{\rm b}$ is $h_\ell^{\rm b} = h_{r,r+\ell}$
- Exponent of $T_\ell^{\rm u}$ is $h_\ell^{\rm u} = h_{r,r-\ell}$

2BTL CASE

- Setting $n_{\rm r} = \frac{\sin(r_{\rm r}+1)\gamma}{\sin r_{\rm r}\gamma}$, exponents for $\ell > 0$ are:

$$h_\ell^{\rm bb} = h_{r_{\rm l}+r_{\rm r}-1,r_{\rm l}+r_{\rm r}-1+\ell} \qquad h_\ell^{\rm bu} = h_{r_{\rm r}-r_{\rm l}-1,r_{\rm r}-r_{\rm l}-1+\ell}$$
$$h_\ell^{\rm ub} = h_{r_{\rm l}-r_{\rm r}-1,r_{\rm l}-r_{\rm r}-1+\ell} \qquad h_\ell^{\rm uu} = h_{r_{\rm l}+r_{\rm r}+1,r_{\rm l}+r_{\rm r}+1-\ell}$$

- For $\ell = 0$ we set $n_{\rm b} = \frac{\sin\left((r_{\rm l}+r_{\rm r}+1-r_{\rm b})\frac{\gamma}{2}\right)\sin\left((r_{\rm l}+r_{\rm r}+1+r_{\rm b})\frac{\gamma}{2}\right)}{\sin(r_{\rm l}\gamma)\sin(r_{\rm r}\gamma)}$
- Then the exponent is $h_0 = h_{r_{\rm b},r_{\rm b}}$

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Beraha-Kahane-Weiss theorem

- Partition function zeros accumulate (as $M \to \infty$) to
  a) Isolated points, if $D = 0$ for dominant eigenvalue
  b) Curves, if ≥ 2 dominant eigenvalues are equimodular

How to get real accumulation points from CFT

- Dominant sector $\leftrightarrow$ least exponent
  - Usually identity sector ($h = 0$): free bc's and $\ell = 0$
  - But not so if some exponent is *negative*
- Case a) of BKW: Inspect corresponding D_ℓ
- Case b) of BKW: Two different sectors have same h
  - Works even for finite L, due to quantum group symmetries!

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Summary

Chromatic polynomial and the Berker-Kadanoff phase

- Analytically continue to $n < 0$
- Use $(n, v) \to (-n, -v)$ symmetry to get back to $n > 0$
- Thus $Q = n^2$ Potts model in the *antiferromagnetic* region
- Here v is RG irrelevant, hence controls a whole region
- In particular part of the chromatic line $v = -1$
  - On square lattice, for $0 \le Q < 3$
  - On triangular lattice, for $0 \le Q < 3.8196717312\cdots$

Boundary chromatic polynomial (e.g. for 1BTL)

- Set $\gamma = \pi(t-1)/t$ so that t = Beraha number

$$h_\ell^{(\mathrm{b})} = \frac{1}{4t}\left(\ell^2 - 2r\ell(t-1) + (r^2-1)(t-1)^2\right)$$

$$D_\ell = \begin{cases} 1 & \text{for } L = 0 \\ n_{\mathrm{s}} U_{\ell-1}(n/2) - U_{\ell-2}(n/2) & \text{for } L > 0 \end{cases}$$

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BOUNDARY CHROMATIC POLYNOMIAL (*continued*)

- Level crossings involving dominant sector ($\ell \leq t-1$): $h_\ell = h_{\ell+2} \Leftrightarrow r = \frac{\ell+1}{t-1}$
- Dominant amplitude vanishes when ($\ell = 2, 4, 6, \ldots$) $D_L = 0 \Leftrightarrow r = \frac{\ell}{t-1}$

MAIN MESSAGE

- In the 0BTL case ($n_1 = n$) we have $r = 1$, hence chromatic zeros when t integer [well known]
- For fixed n, zeros for $r = s/(t-1)$ and integer $s \in (0, t]$
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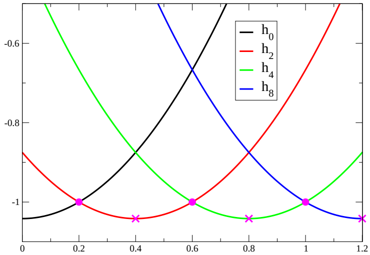
EXAMPLE FOR $t = 6$ (I.E. $Q = 3$)

- Level crossings (circles) and vanishing dominant amplitudes (crosses)



- Phase transitions at $r = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1, \frac{6}{5}$
- Corresponds to $Q_l = 0, 1, \frac{3}{2}, 2, 3, \infty$