Dominant traits in the zeros of two-variate two-terminal reliability polynomials

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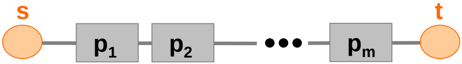
Outline

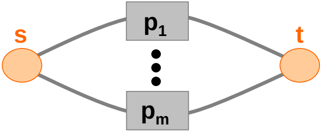
Two-terminal reliability

- Probability that two sites of a network are connected. Also known as pair connectedness or connectivity function in percolation theory
- Example 1 : Series configuration



$$R_{\text{series}} = p_1\ p_2\ \cdots\ p_m \to p^m$$

- Example 2 : Parallel configuration

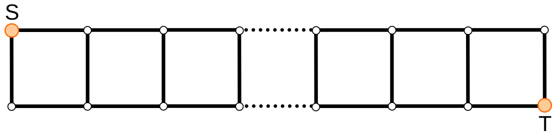


$$R_{\text{parallel}} = 1 - (1 - p_1)\,(1 - p_2)\ \cdots\ (1 - p_m) \to 1 - (1 - p)^m$$

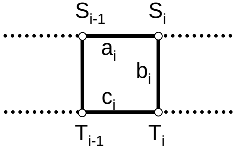
- For non series-parallel networks, (un)directed underlying graphs may be considered. Expressions are not invariants of the graph (contrary to the all-terminal reliability).

The simple ladder configuration (1)

- Architecture used in long-haul optical networks for IP connexions
  (nodes : IP routers or optical crossconnects; edges : fibre links).



- The elementary cell (edges and nodes are labelled by their *individual* reliabilities)



The simple ladder configuration (2)

- The general solution is

$$\mathcal{R}_{S_0 \to T_n} = (0 \;\; 1 \;\; 0)\, M_n \cdots M_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

with the i^{th} transfer matrix M_i given by

$$M_i = \begin{pmatrix} a_i\, S_i & b_i\, c_i\, S_i\, T_i & a_i\, b_i\, c_i\, S_i\, T_i \\ a_i\, b_i\, S_i\, T_i & c_i\, T_i & a_i\, b_i\, c_i\, S_i\, T_i \\ -a_i\, b_i\, S_i\, T_i & -b_i\, c_i\, S_i\, T_i & a_i\,(1 - 2\, b_i)\, c_i\, S_i\, T_i \end{pmatrix}$$

- If p and ρ are used, a single matrix needs be considered. The three eigenvalues are then $p\,\rho\,(1 - p\,\rho)$ and

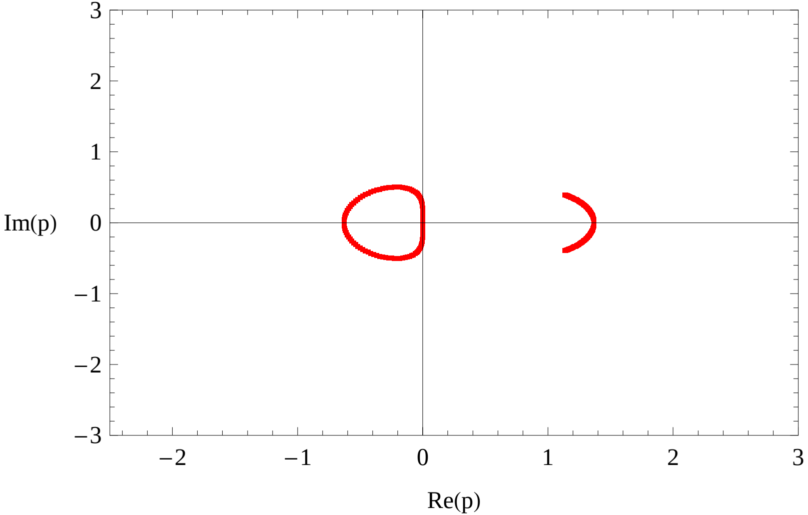
$$\zeta_\pm = \frac{p\,\rho}{2} \left(1 + 2\,p\,(1-p)\,\rho \pm \sqrt{1 + 4\,p^2\,\rho - 8\,p^3\,\rho^2 + 4\,p^4\,\rho^2}\right)$$

- The generating function $\mathcal{G}(z) = \sum_n^\infty \mathcal{R}_{S_0 \to T_n}\, z^n = \dfrac{\mathcal{N}(z)}{\mathcal{D}(z)}$ is a rational fraction.

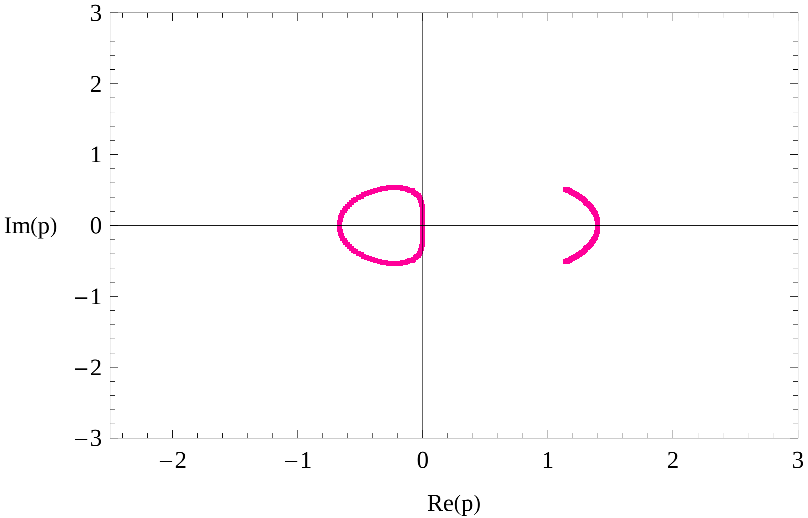
Complex zeros of polynomials

- Various polynomials occur in graph theory : chromatic, flow and all-terminal reliability polynomials. These polynomials are special instances of the Tutte polynomial. The location of their complex zeros has been given a lot of interest and effort by many speakers at the present workshop
- For recursive families of polynomials indexed by n, the zeros converge/aggregate on equimodular curves or isolated points (Beraha, Kahane, and Weiss)
- Here, for our two-variate two-terminal reliability polynomials, ρ has a fixed value and we plot the zeros in the complex plane for p. We take n large enough so that the limit is nearly reached. The polynomials are determined by recursion relations (using the coefficients of the characteric polynomial of the transfer matrix)
- Question : what happens when ρ decreases from 1 to 0?

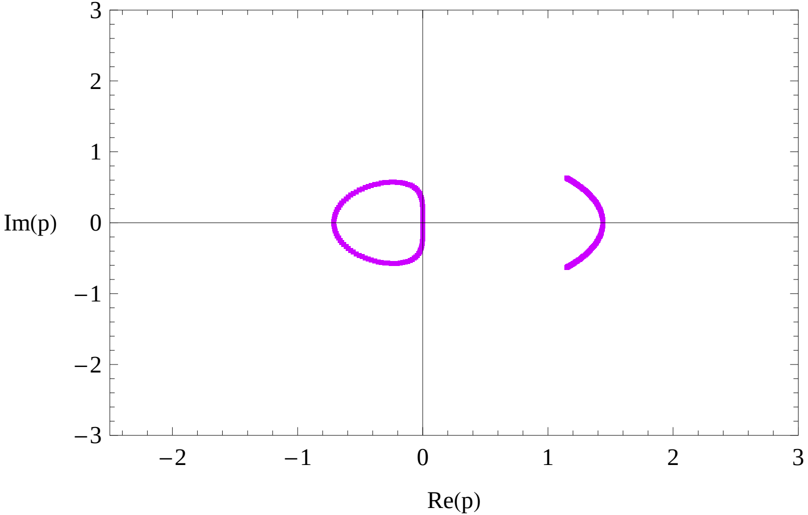
Complex zeros for the simple ladder ($n = 100$, $\rho = 1$)



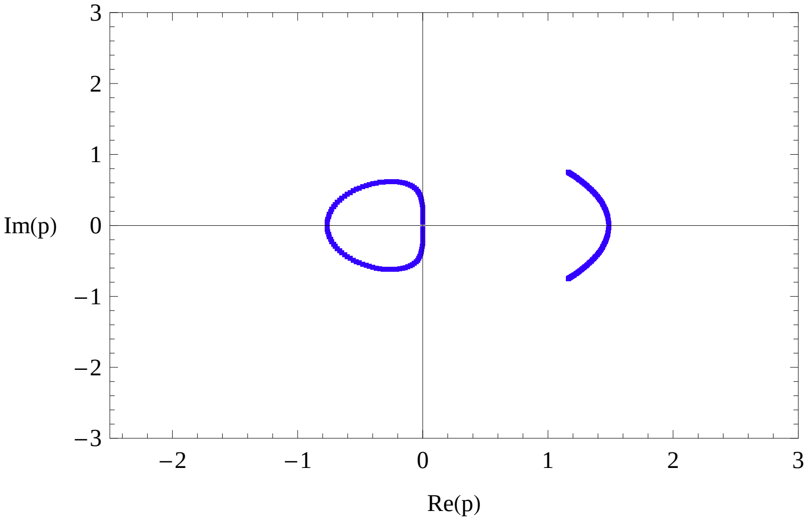
Complex zeros for the simple ladder ($n = 100$, $\rho = 0.9$)



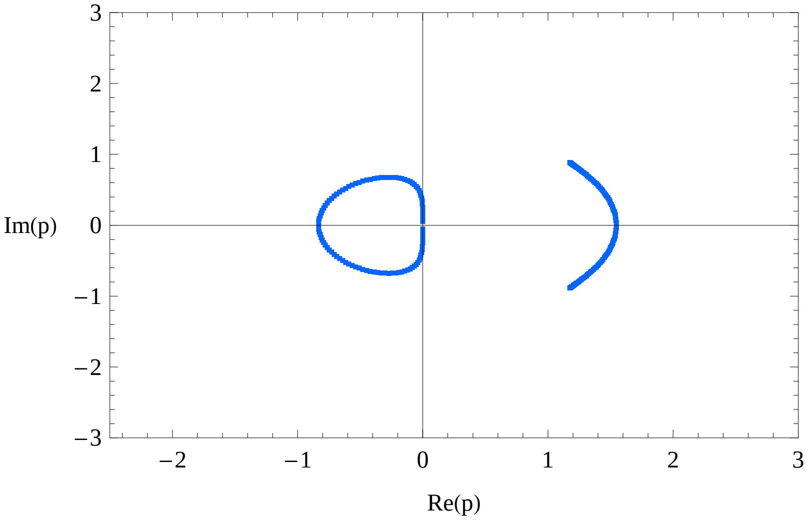
Complex zeros for the simple ladder ($n = 100$, $\rho = 0.8$)



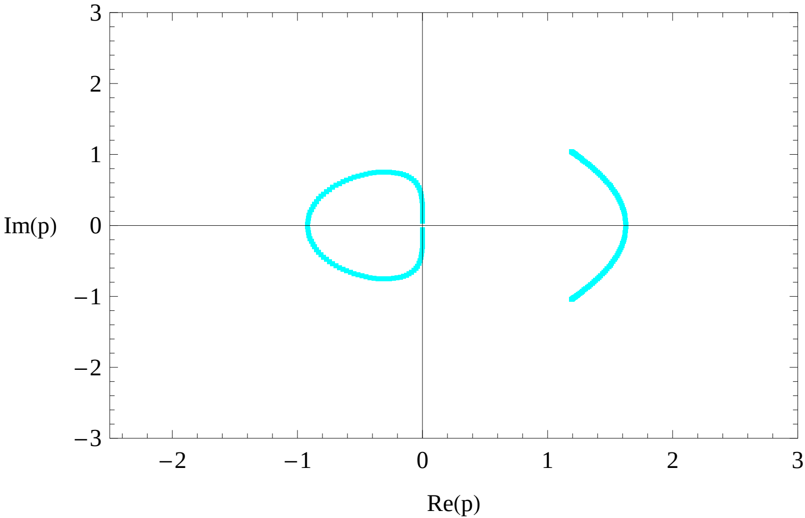
Complex zeros for the simple ladder ($n = 100$, $\rho = 0.7$)



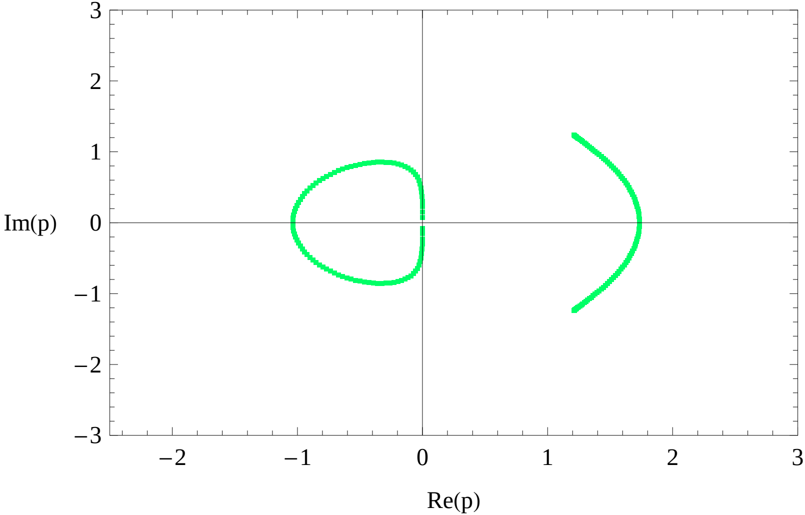
Complex zeros for the simple ladder ($n = 100$, $\rho = 0.6$)



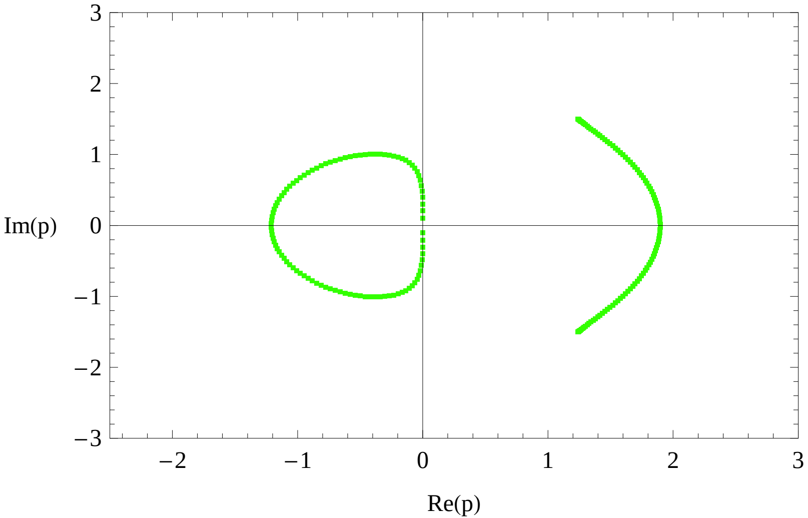
Complex zeros for the simple ladder ($n = 100$, $\rho = 0.5$)



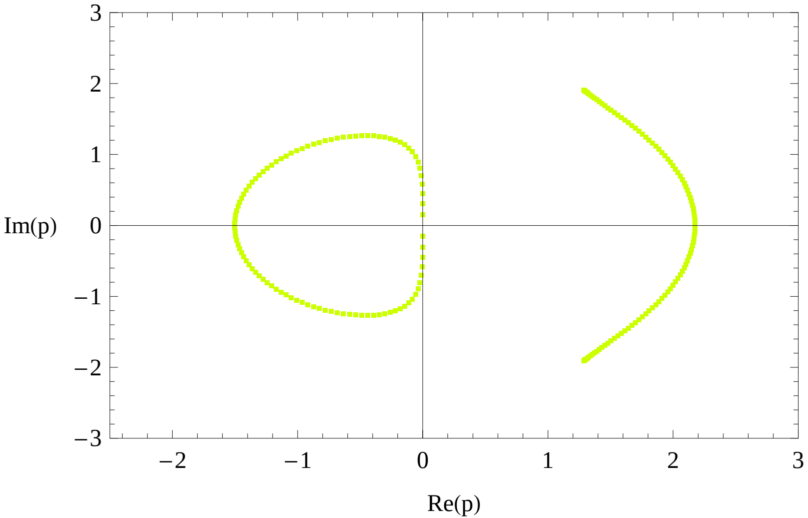
Complex zeros for the simple ladder ($n = 100$, $\rho = 0.4$)



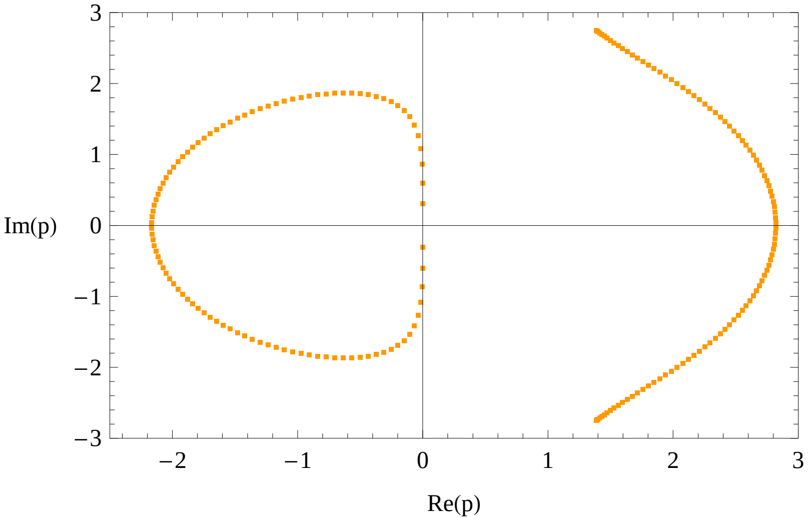
Complex zeros for the simple ladder ($n = 100$, $\rho = 0.3$)



Complex zeros for the simple ladder ($n = 100$, $\rho = 0.2$)



Complex zeros for the simple ladder ($n = 100$, $\rho = 0.1$)



Results for the simple ladder

- Not many changes...
- Structure expansion as ρ goes down to 0
- Double roots of $\mathcal{D}(z) = 0$ obey the constraint

$$1 + 4\,p^2\,\rho - 8\,p^3\,\rho^2 + 4\,p^4\,\rho^2 = 0$$

  Endpoints of the rightmost curve are asymptotically $\pm \frac{i}{(2\,\rho)^{1/2}}$

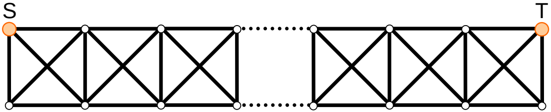
- Intersections with the real axis are given by

$$\frac{1}{2}\left(1 \pm \sqrt{1 + \frac{2}{\rho}}\right)$$

- Conclusion : smooth expansion as $(2\,\rho)^{-1/2}$ to a quasi-circular outermost shape

The K_4 ladder configuration

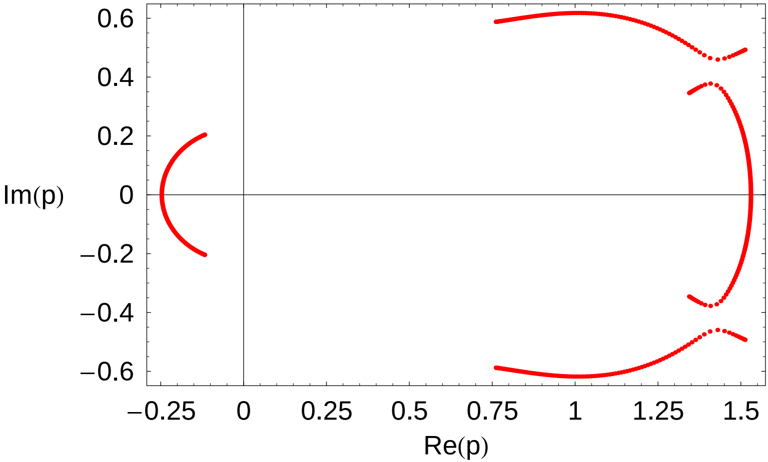
- Architecture (for each new cell, we add 5 edges and 2 nodes)



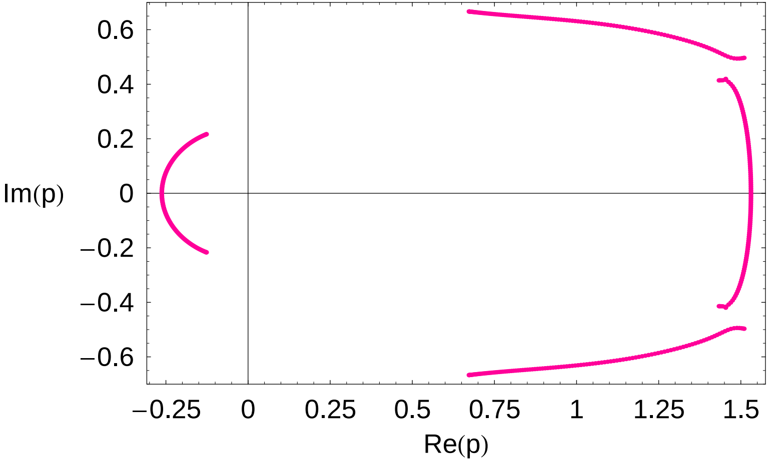
- The transfer matrix is now 5×5
  - but for p and ρ reliabilities, only three eigenvalues remain,
  - and only two eigenvalues for perfect nodes ($\rho = 1$) !

- If only one diagonal edge had been added, the transfer matrix would be 4×4, and there would be 4 eigenvalues

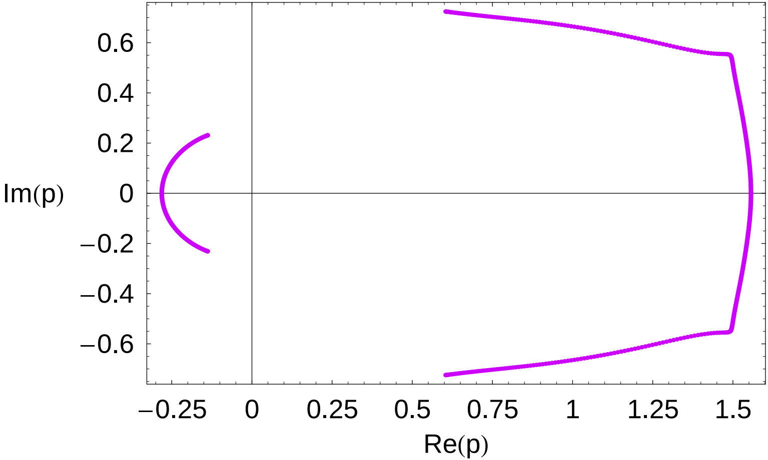
Complex zeros for K_4 ladder ($n = 120$, $\rho = 1$)



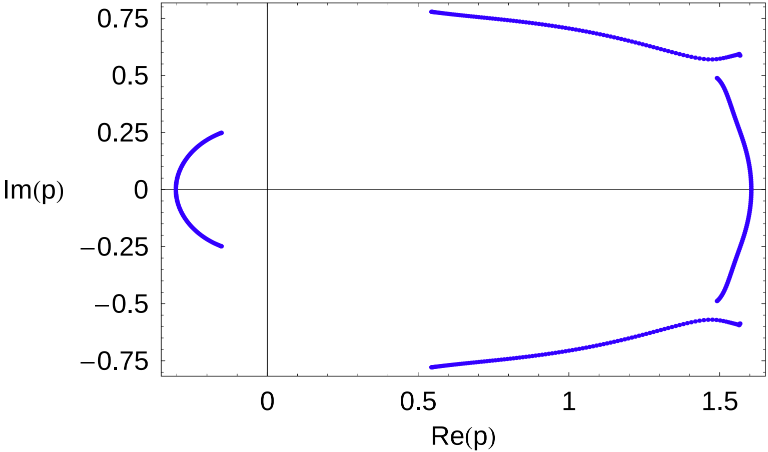
Complex zeros for K_4 ladder ($n = 120$, $\rho = 0.9$)



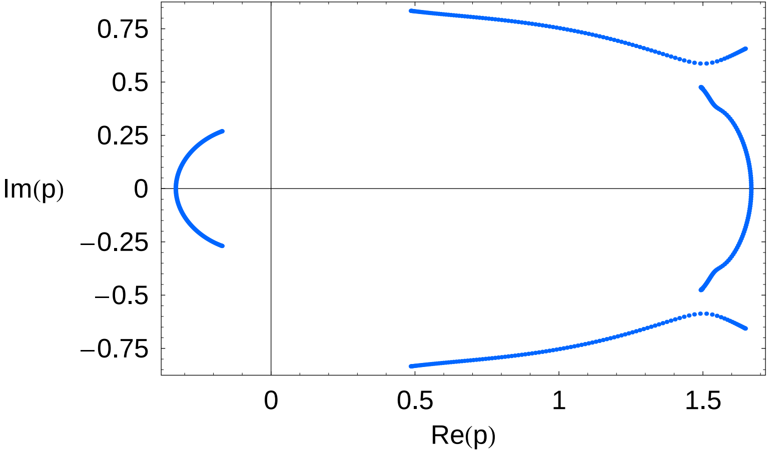
Complex zeros for K_4 ladder ($n = 120$, $\rho = 0.8$)



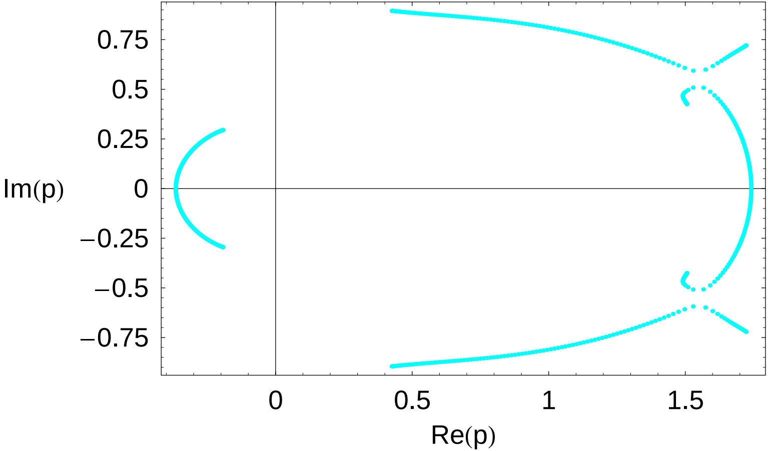
Complex zeros for K_4 ladder ($n = 120$, $\rho = 0.7$)



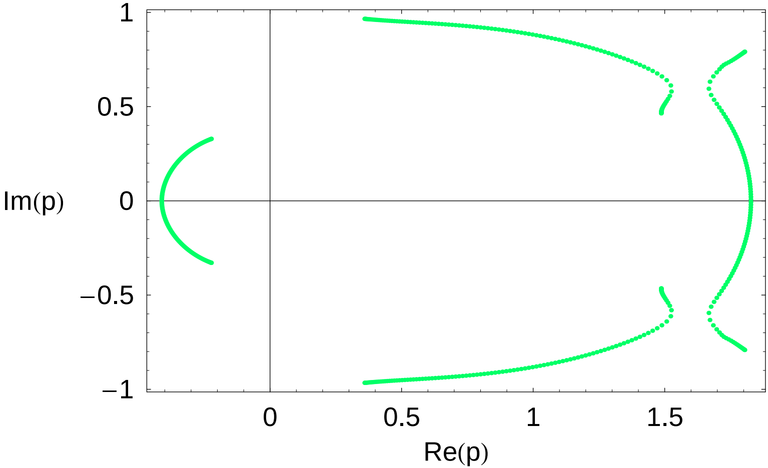
Complex zeros for K_4 ladder ($n = 120$, $\rho = 0.6$)



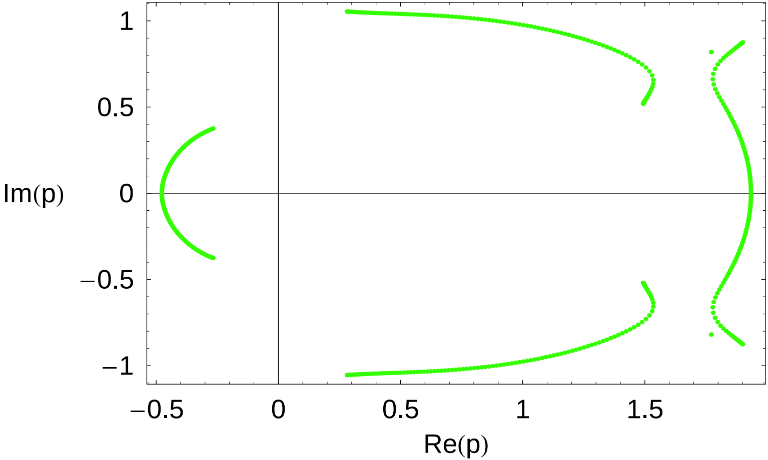
Complex zeros for K_4 ladder ($n = 120$, $\rho = 0.5$)



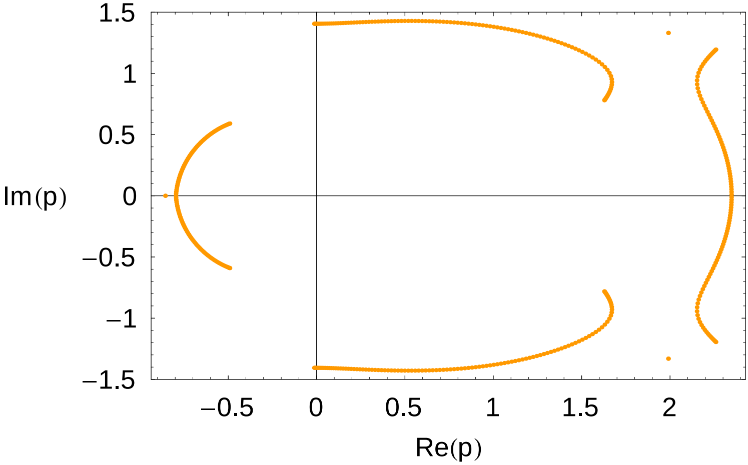
Complex zeros for K_4 ladder ($n = 120$, $\rho = 0.4$)



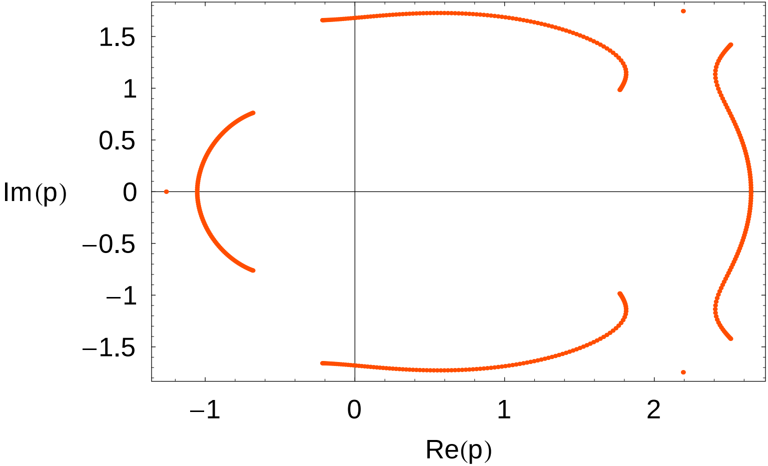
Complex zeros for K_4 ladder ($n = 120$, $\rho = 0.3$)



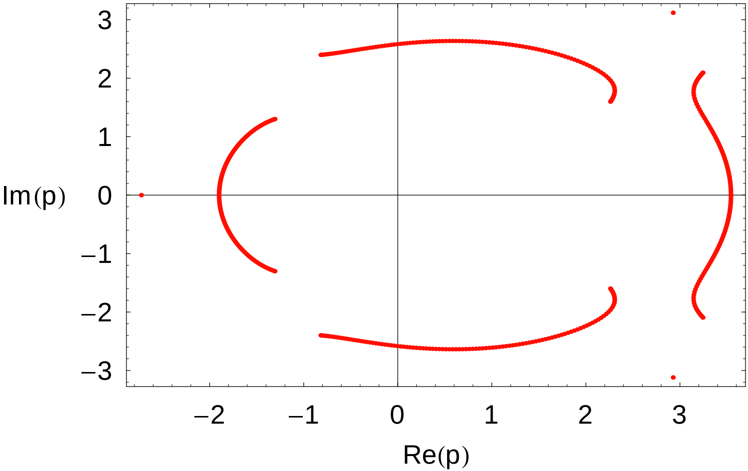
Complex zeros for K_4 ladder ($n = 120$, $\rho = 0.1$)



Complex zeros for K_4 ladder ($n = 120$, $\rho = 0.05$)



Complex zeros for K_4 ladder ($n = 120$, $\rho = 0.01$)



Results for the K_4-ladder

- No closed curve on the left of the complex plane
- Different structural changes
  - equimodular curves join for $\rho = 0.8$
  - isolated points appear when the residue of the generating function corresponding to the eigenvalue of highest modulus vanishes. This translates into a constraint between p and ρ

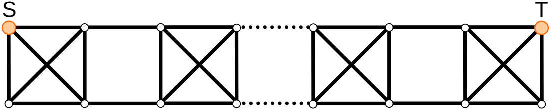
$$\begin{aligned} 0 &= 2 + 2\,\rho + 4\,(3\,\rho + 1)\,\rho\,p - (40\,\rho + 11)\,\rho\,p^2 \\ &\quad + (45\,\rho + 4)\,\rho\,p^3 - 20\,\rho^2\,p^4 + 3\,\rho^2\,p^5 \end{aligned}$$

  - the first two isolated points occur for $\rho \leq 0.406657811123$ (solution of an algebraic equation of degree 65 in ρ)
  - the third one appears when $\rho \leq 0.175221381869$ (solution of an equation of degree 10 in ρ)
- Critical points are determined by their obeying two constraints on p and ρ
- Different expansion rates for quasi-circular equimodular curves ($|p| \to (2\,\rho)^{-1/4}$) and isolated points ($-(2\,\rho)^{-1/3}\,e^{\frac{2\,i\,k\,\pi}{3}}$)
- All these expansions are derived from the generating function

Modification of the building blocks

- Structural changes differ for the simple- and K_4-ladders (isolated points, expansion rates)
- What happens if we mix the patterns (making full use of the general, multivariate result), so that the building blocks are not of unit length anymore ?
  - are there persistent features, which do not depend on the pattern ?
  - do new behaviours occur ?

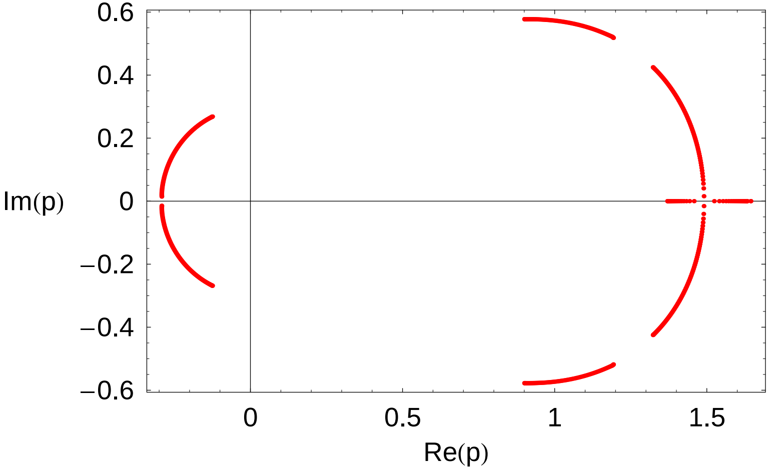
The K_4+simple ladder configuration



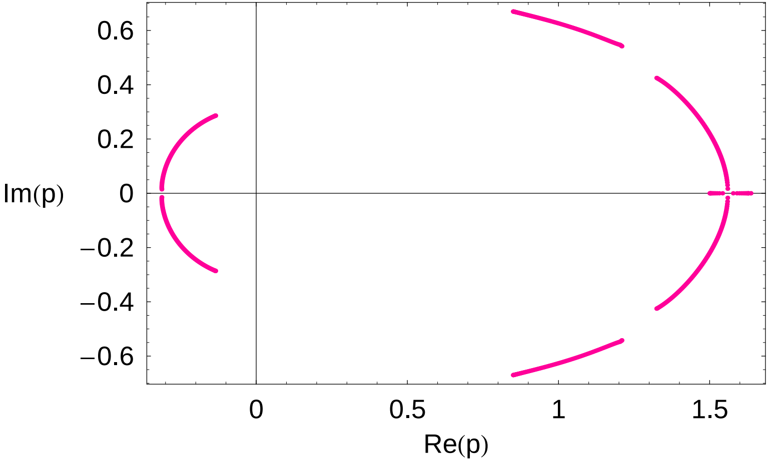
- K_4 and C_4 graphs are added in succession, so that the building block has length 2 (n is the number of repeated $(K_4 + C_4)$'s)
- Only two eigenvalues $\zeta_\pm$, even for imperfect modes !

$$
\begin{aligned}
\zeta_\pm &= \frac{p^2\,\rho^2}{2}\,(\mathcal{A} \pm \sqrt{\mathcal{B}}) \\
\mathcal{A} &= 2 + 4\,(1-p)\,p\,\rho + p^2\,\left(6 - 38\,p + 59\,p^2 - 36\,p^3 + 8\,p^4\right)\,\rho^2 \\
\mathcal{B} &= 4 + 16\,(1-p)\,p\,\rho \\
&\quad -4\,p^2\,\left(-6 + 16\,p + 3\,p^2 - 20\,p^3 + 8\,p^4\right)\,\rho^2 \\
&\quad +8\,(1-p)\,p^3\,\left(2 - 32\,p + 63\,p^2 - 40\,p^3 + 8\,p^4\right)\,\rho^3 \\
&\quad +p^4\,\left(4 - 352\,p + 1992\,p^2 - 4780\,p^3 + 6257\,p^4 - 4848\,p^5\right. \\
&\qquad \left. +2240\,p^6 - 576\,p^7 + 64\,p^8\right)\,\rho^4
\end{aligned}
$$

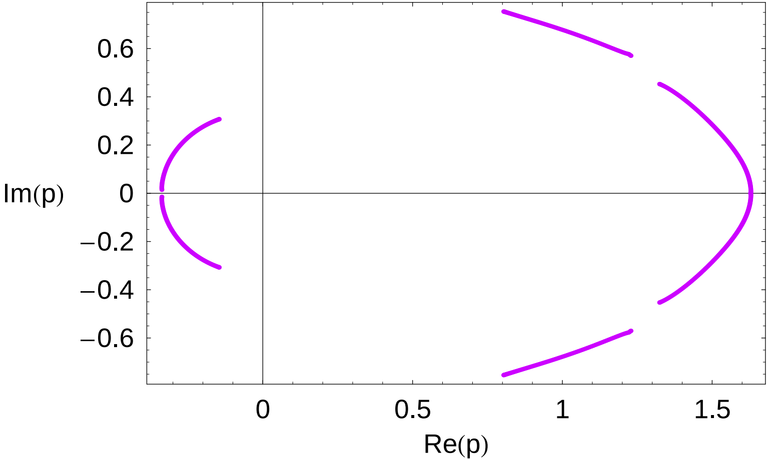
Complex zeros for K_4+simple ladder ($n = 100$, $\rho = 1$)



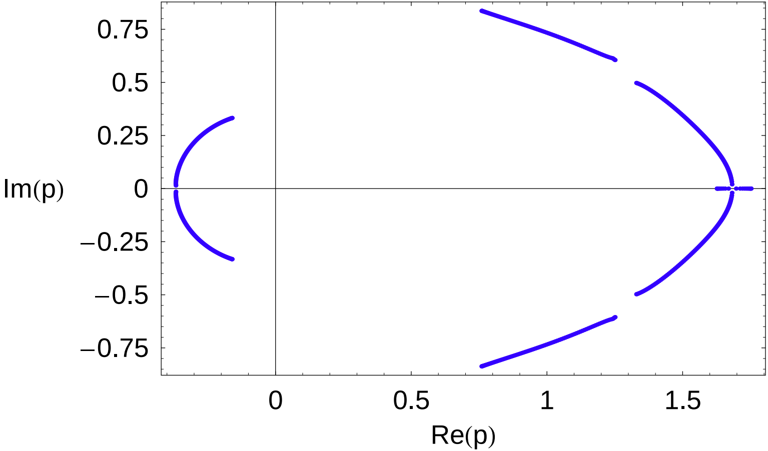
Complex zeros for K_4+simple ladder ($n = 100$, $\rho = 0.9$)



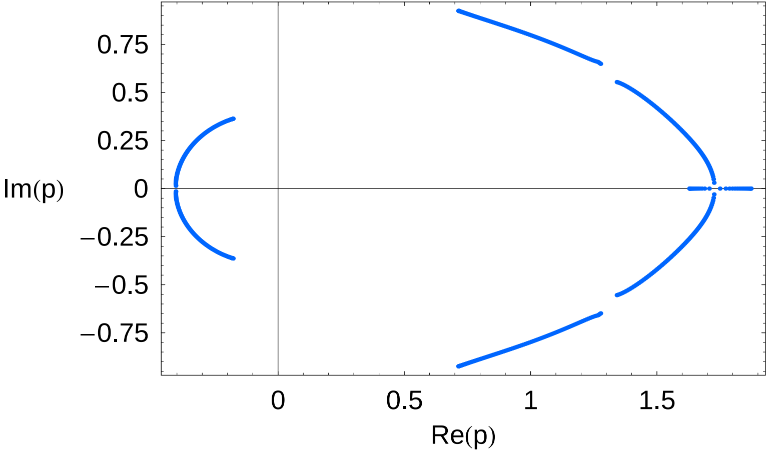
Complex zeros for K_4+simple ladder ($n = 100$, $\rho = 0.8$)



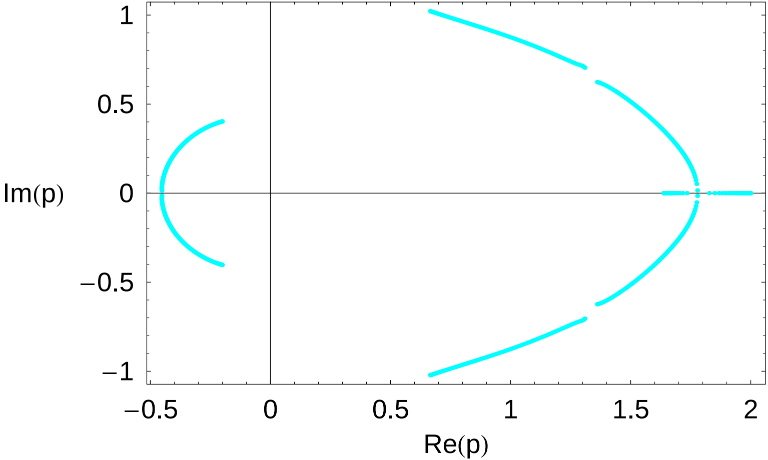
Complex zeros for K_4+simple ladder ($n = 100$, $\rho = 0.7$)



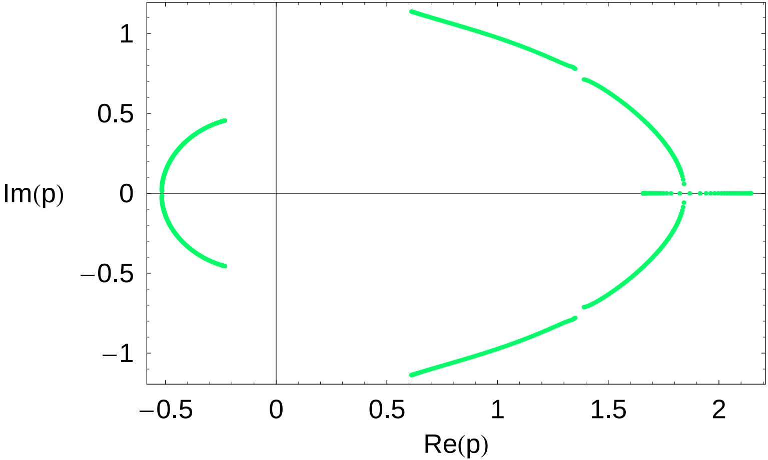
Complex zeros for K_4+simple ladder ($n = 100$, $\rho = 0.6$)



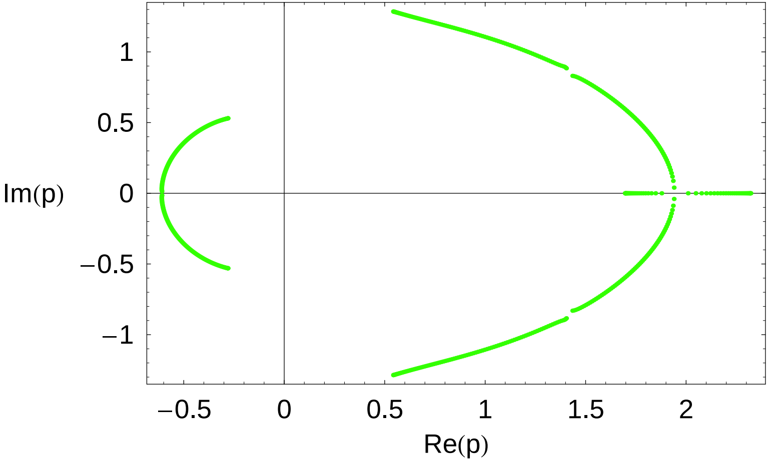
Complex zeros for K_4+simple ladder ($n = 100$, $\rho = 0.5$)



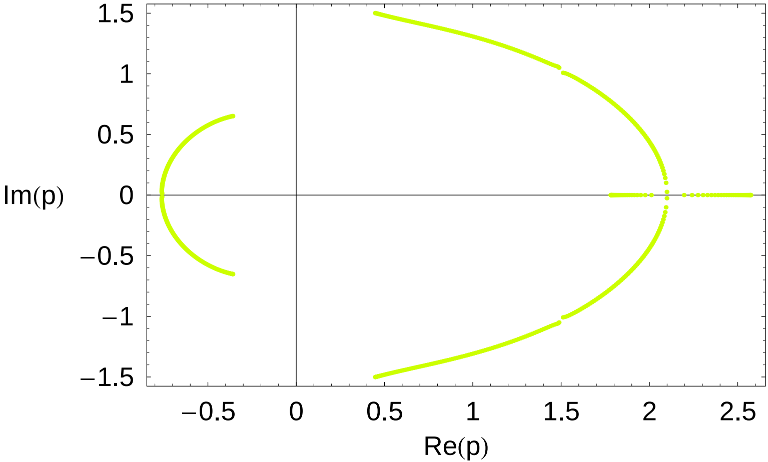
Complex zeros for K_4+simple ladder ($n = 100$, $\rho = 0.4$)



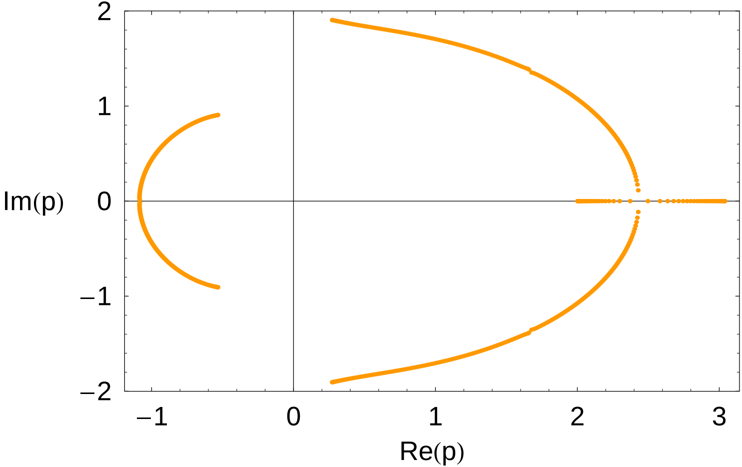
Complex zeros for K_4+simple ladder ($n = 100$, $\rho = 0.3$)



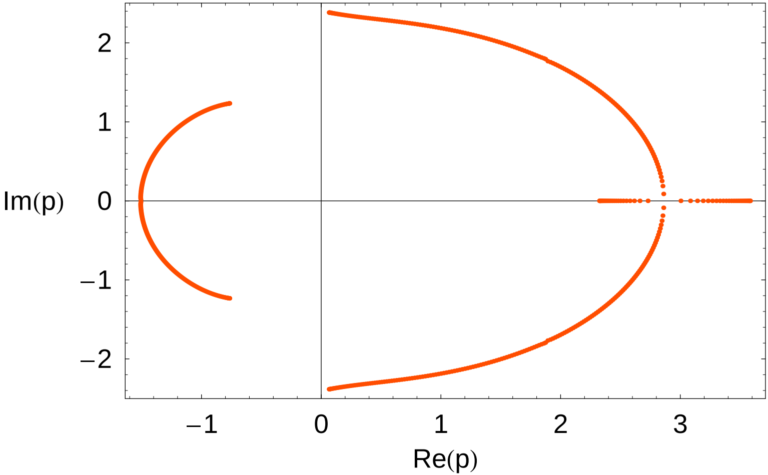
Complex zeros for K_4+simple ladder ($n = 100$, $\rho = 0.2$)



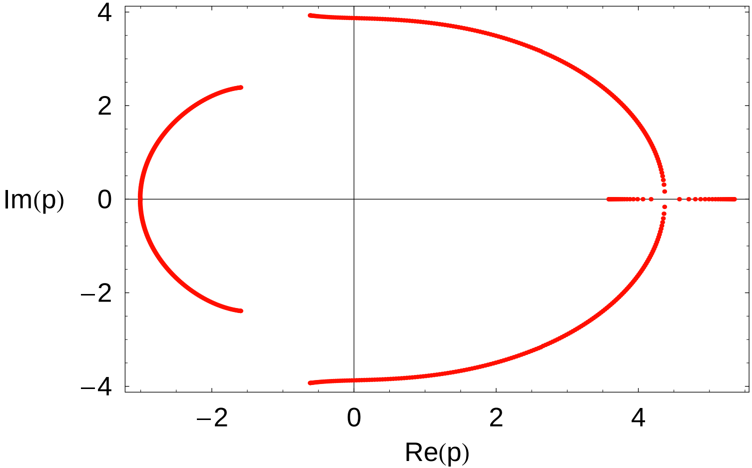
Complex zeros for K_4+simple ladder ($n = 100$, $\rho = 0.1$)



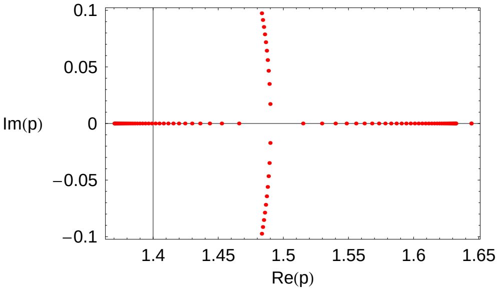
Complex zeros for K_4+simple ladder ($n = 100$, $\rho = 0.05$)



Complex zeros for K_4+simple ladder ($n = 100$, $\rho = 0.01$)



Isolated zero for K_4+simple ladder ($n = 150$, $\rho = 1$)



- the existence of isolated points is controlled by

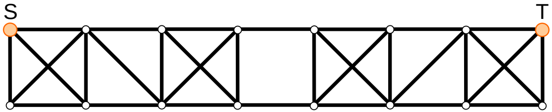
$$0 = -3 + 2\,p + p\,\rho\,(-7 + 18\,p - 15\,p^2 + 4\,p^3)$$

- for $\rho = 1$, the isolated zero lies asymptotically at 1.644299, while the segment is (1.370197,1.632580)

Results for the (K_4+simple)-ladder

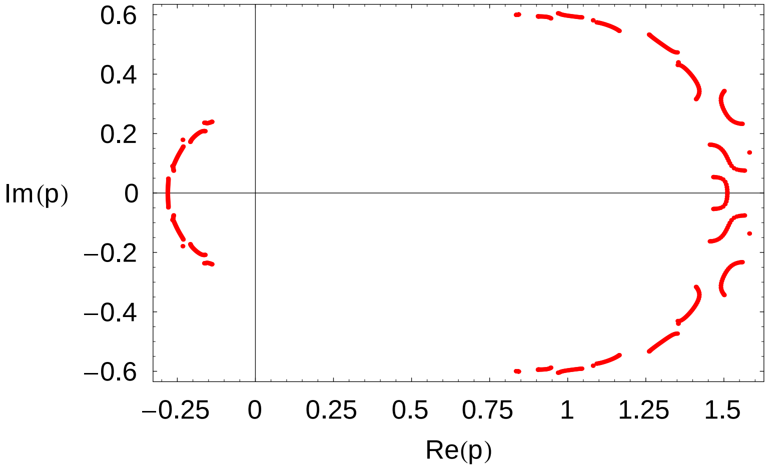
- Isolated point even for $\rho = 1$
- Apparition of a segment on the real axis, even for perfect nodes
  - it disappears (along with the isolated zero) for $\rho_c = \frac{3}{208}\left(71 - (681 + 416\sqrt{2})^{1/3} - (681 - 416\sqrt{2})^{1/3}\right)$, i.e., roughly 0.8026, but reappears and remains down to $\rho = 0$.
  - its extremities $p_\pm$ when $\rho \to 0$, are asymptotically $p_\pm \to \frac{1}{(2\,\rho)^{1/3}} \pm \frac{\sqrt{2}}{3\,(2\,\rho)^{1/6}} + \frac{3}{4}$
  - its intersection p_o with the rightmost curve is $p_0 \to \frac{1}{(2\,\rho)^{1/3}} + \frac{23}{36}$
- Quasi-circular equimodular curves : $|p| \to (2\,\rho)^{-1/3}$

The (long pattern) ladder configuration

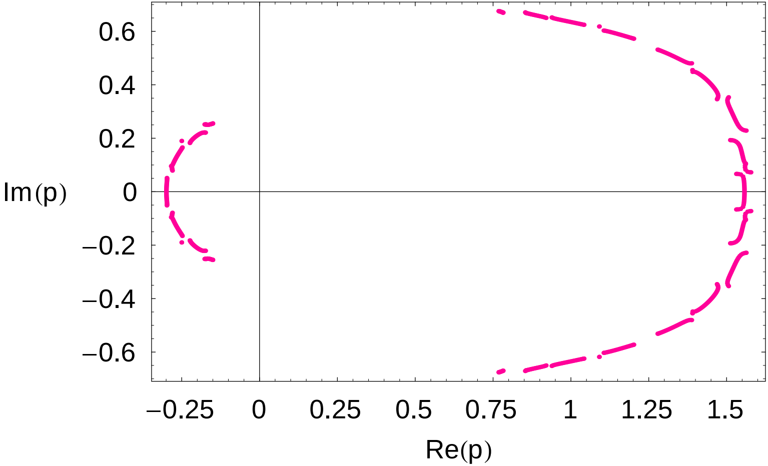


- Another elementary single-cell pattern has been added
- Still, only two eigenvalues : $\mathcal{R}_n = \alpha_+ \zeta_+^n + \alpha_- \zeta_-^n$!
- Unfortunately, the expressions would fill more than one page...
- What do you expect ?

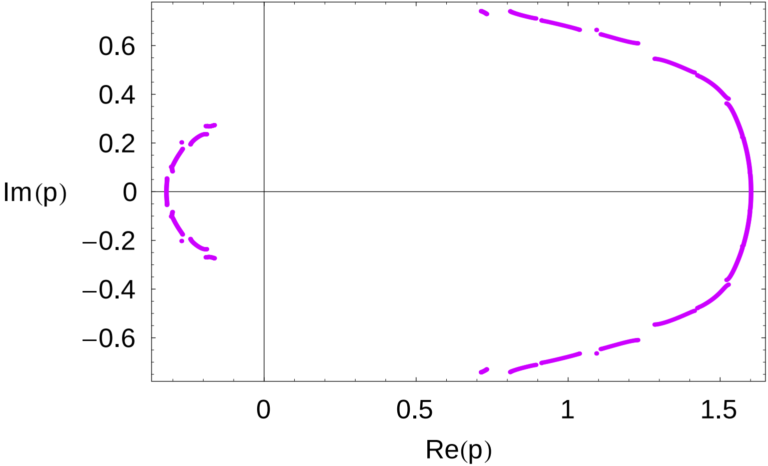
Complex zeros for mixed ladder ($n = 30$, $\rho = 1$)



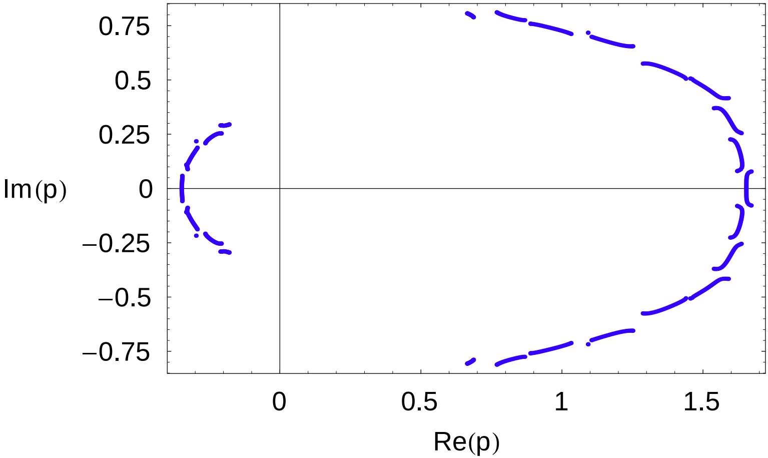
Complex zeros for mixed ladder ($n = 30$, $\rho = 0.9$)



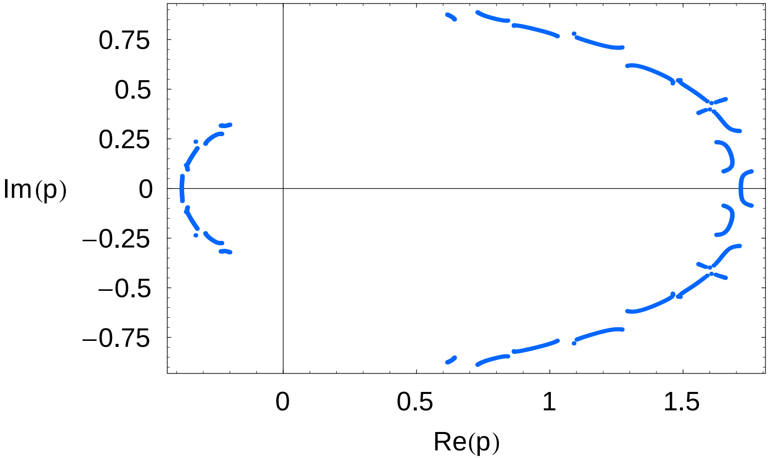
Complex zeros for mixed ladder ($n = 30$, $\rho = 0.8$)



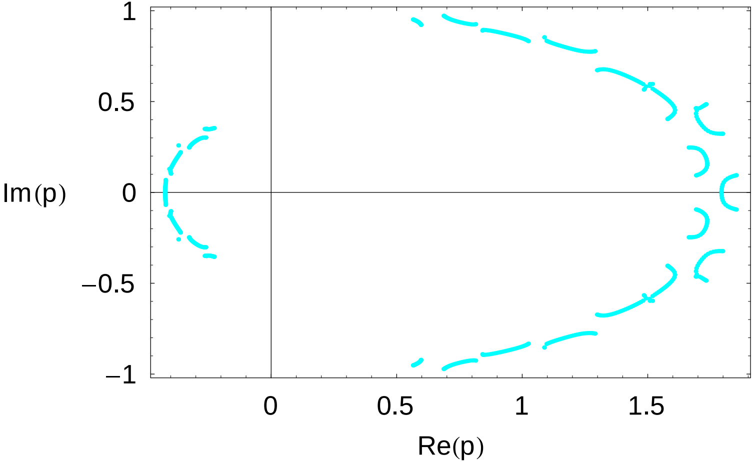
Complex zeros for mixed ladder ($n = 30$, $\rho = 0.7$)



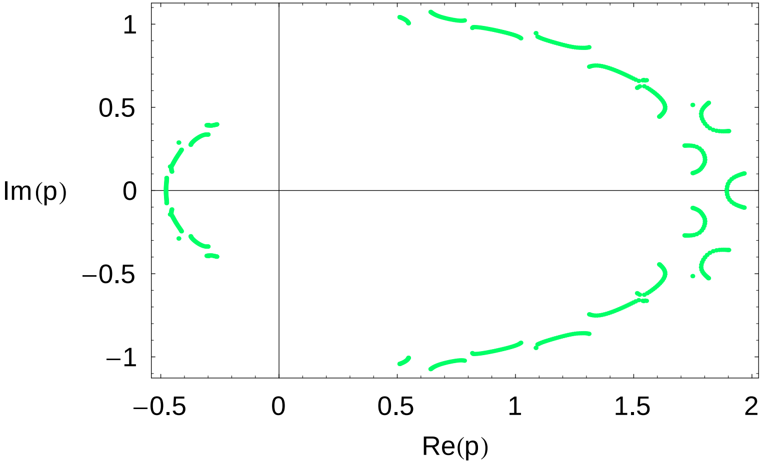
Complex zeros for mixed ladder ($n = 30$, $\rho = 0.6$)



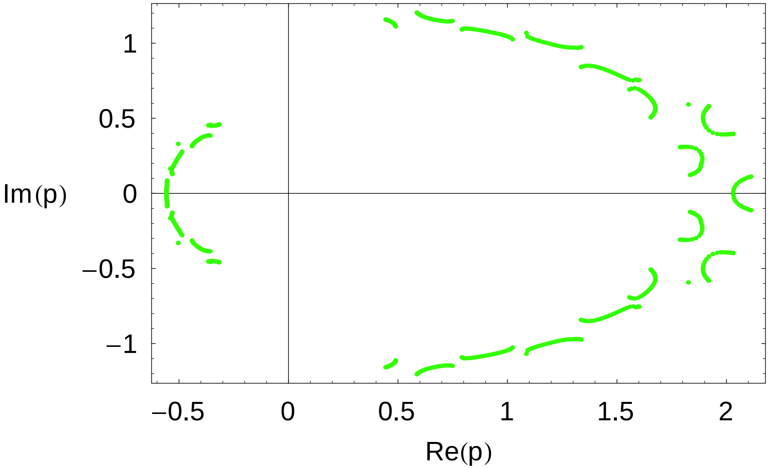
Complex zeros for mixed ladder ($n = 30$, $\rho = 0.5$)



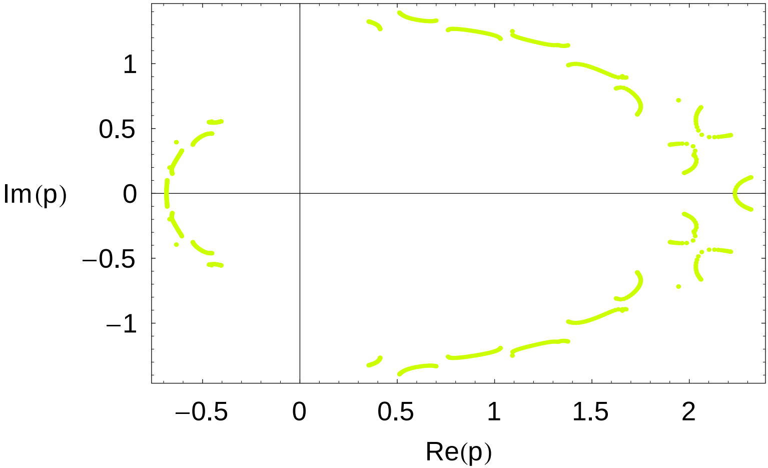
Complex zeros for mixed ladder ($n = 30$, $\rho = 0.4$)



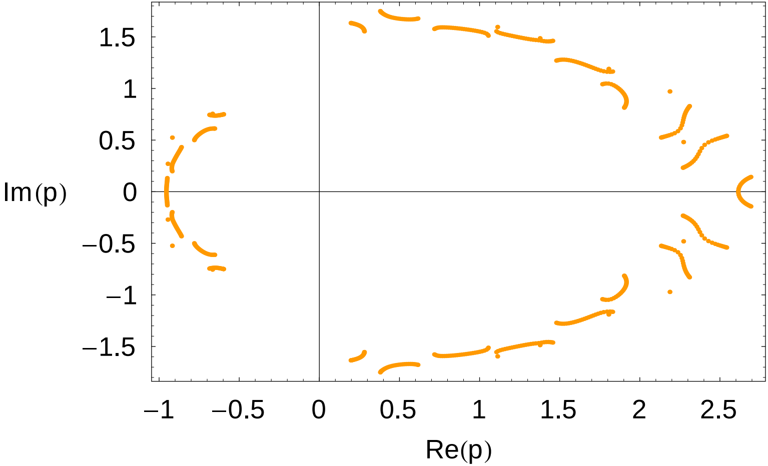
Complex zeros for mixed ladder ($n = 30$, $\rho = 0.3$)



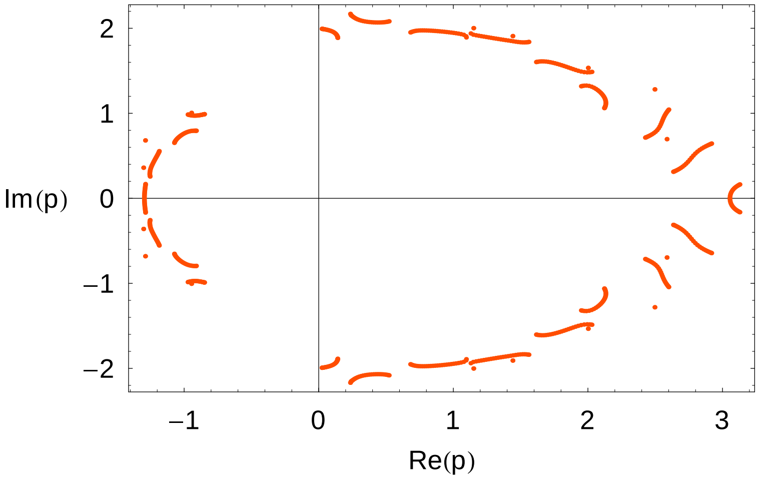
Complex zeros for mixed ladder ($n = 30$, $\rho = 0.2$)



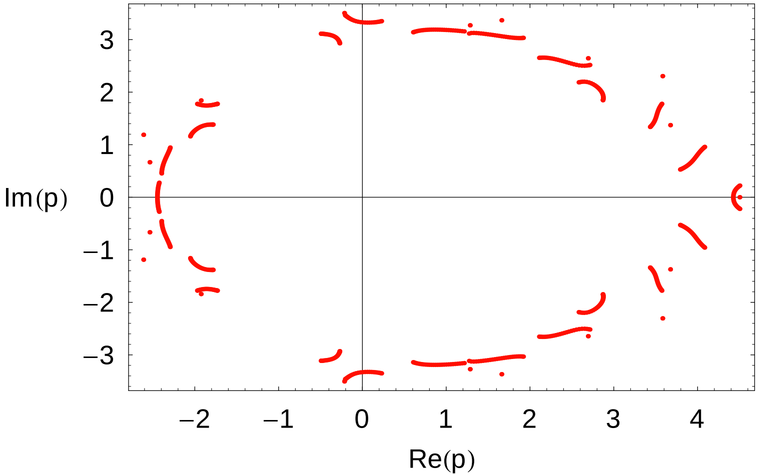
Complex zeros for mixed ladder ($n = 30$, $\rho = 0.1$)



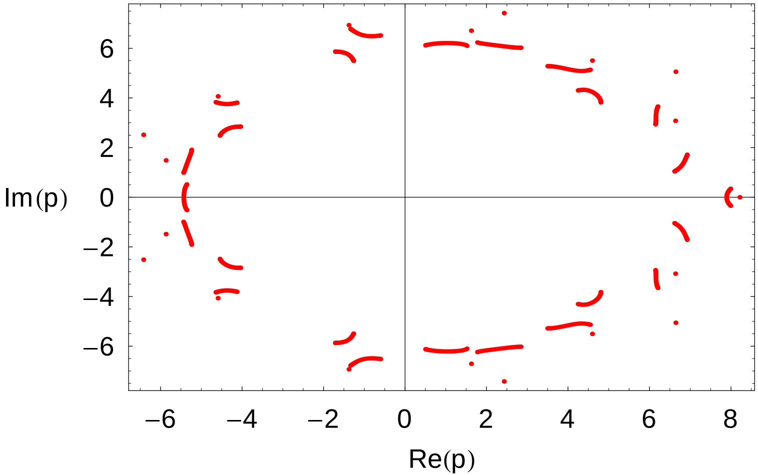
Complex zeros for mixed ladder ($n = 30$, $\rho = 0.05$)



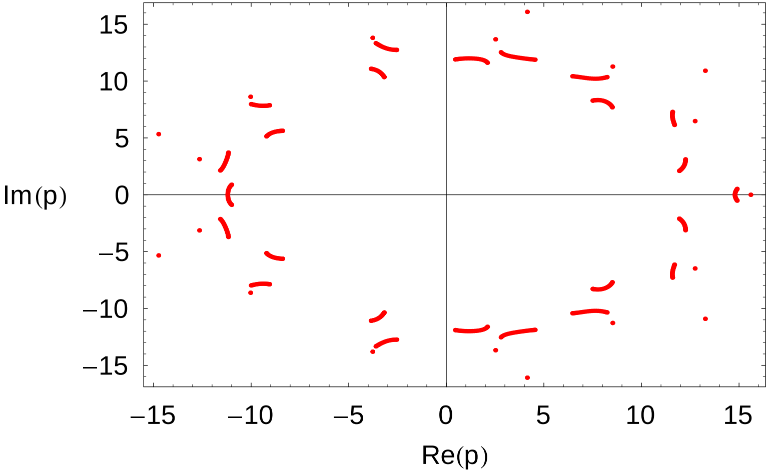
Complex zeros for mixed ladder ($n = 30$, $\rho = 10^{-2}$)



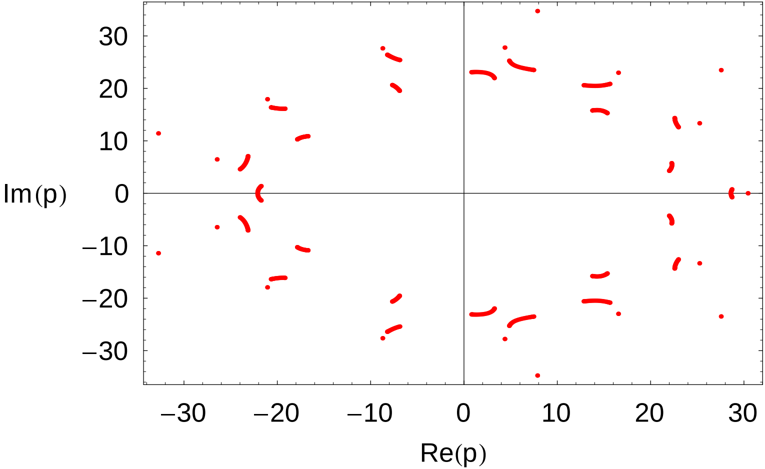
Complex zeros for mixed ladder ($n = 30$, $\rho = 10^{-3}$)



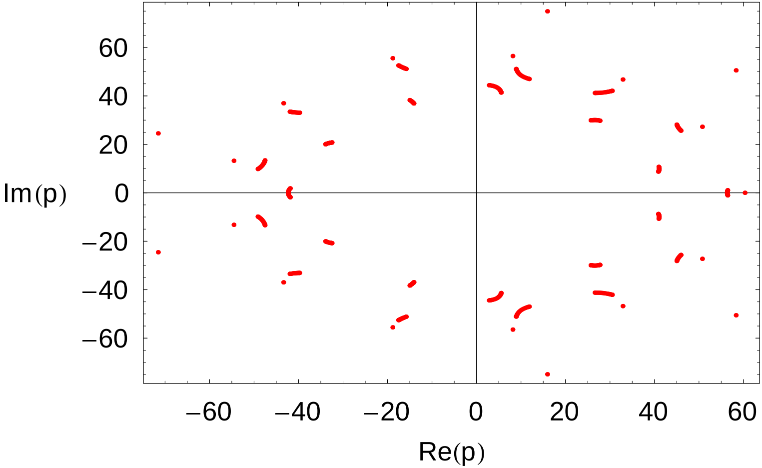
Complex zeros for mixed ladder ($n = 30$, $\rho = 10^{-4}$)



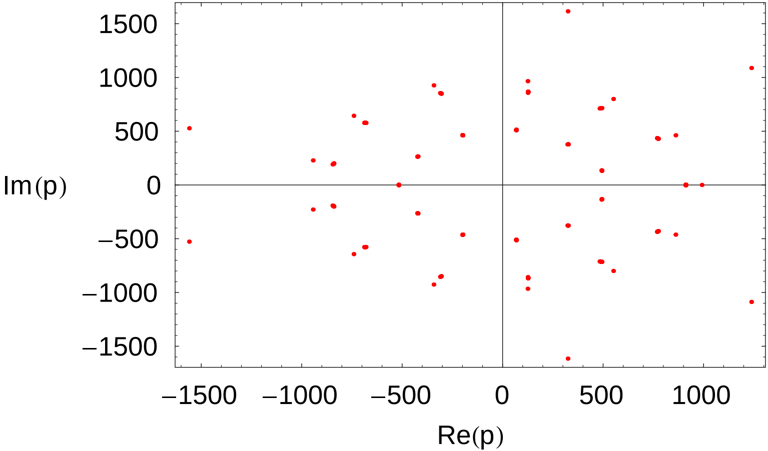
Complex zeros for mixed ladder ($n = 30$, $\rho = 10^{-5}$)



Complex zeros for mixed ladder ($n = 30$, $\rho = 10^{-6}$)



Complex zeros for mixed ladder ($n = 30$, $\rho = 10^{-10}$)



Results for the long mixed ladder

- No segment of the real axis
- "Explosion" of the other features
- Smoother structures for $\rho = 0.8$.
- Families of isolated points with different expansion rates and asymptotic symmetries
  - $p \to \left(\frac{-\frac{1}{4} + \frac{i}{4}\sqrt{\frac{11}{5}}}{\rho} \right)^{1/3} e^{2\,i\,k\,\pi/3}$ and complex conjugate.
  - $p \to \frac{1}{\left(\left(\frac{40}{3}\right)^{1/4} \rho\right)^{4/13}} e^{2\,i\,k\,\pi/13}$
- No asymptotic circular curves when $\rho \to 0$, but islands of equimodular curves with different expansion rates and symmetries
  - $|p| \to \frac{1}{\left(50^{1/4}\,\rho\right)^{4/13}}$ and $|p| \to \frac{1}{\left(\left(\frac{4}{3}\right)^{2/3} \rho\right)^{3/11}}$

Conclusion and outlook

- The two-terminal network reliability $\mathcal{R}_n$ can be expressed as a product of transfer matrices taking all individual edge and node reliabilities exactly into account : this is a consequence of the underlying algebraic graph structure
- For uniform p (edge) and ρ (node) reliabilities, we obtain a two-variate, two-terminal reliability polynomial $\mathcal{R}_n$
- The location of their complex zeros may exhibit spectacular structural transitions as a function of ρ. Critical values ρ_c may be deduced from the (rational) generating function
- When the building block has a nonsimple structure
  - The number of eigenvalues of a recursive family of graphs may be strongly linked to a particular substructure of the repeated pattern
  - Different expansion rates may coexist for isolated points as well as equimodular curves when $\rho \to 0$
  - Their asymptotic shape is not always circular
- More work is under way on "wider" graphs...

Thank you for your attention

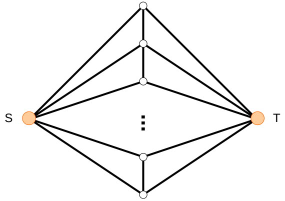
Bibliography

- I recommend the papers by Biggs, Chang, Jackson, Jacobsen, Royle, Salas, Shrock, Sokal, Wagner and collaborators (most of them are available online; see `arXiv`)
- Other "network" configurations have been studied for the two-terminal reliability polynomials, and their corresponding complex zeros (see `arXiv` and J. Phys. A: Math. Theor. 40 (2007) 14099–14116)

Appendix

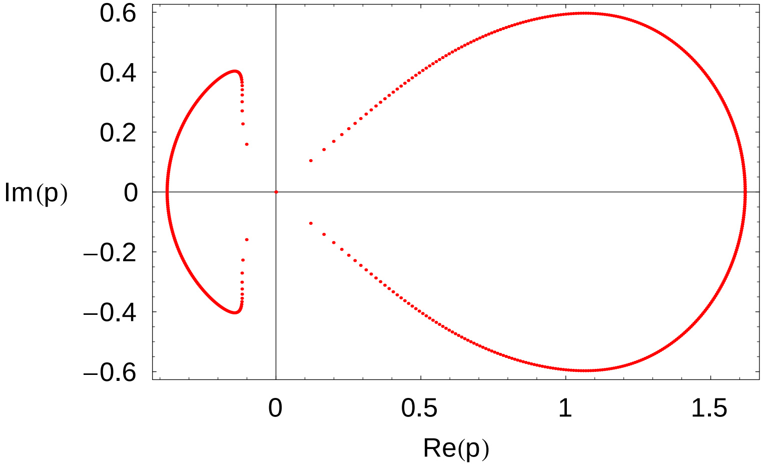
The double fan configuration

- Building block of the architecture

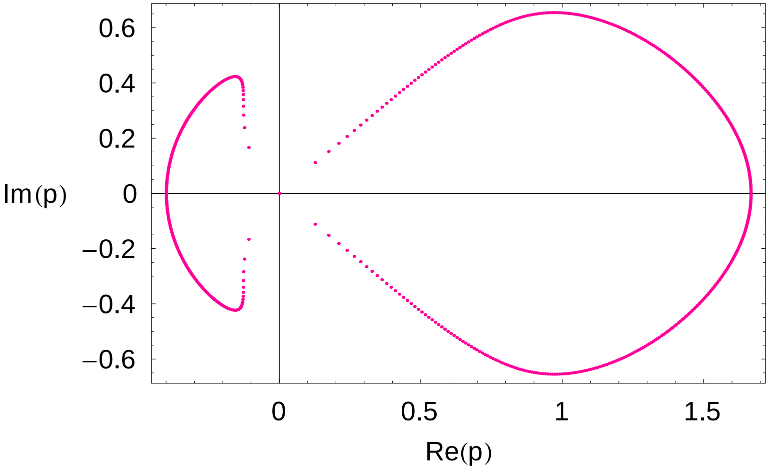


- this graph can be very "wide"
- a 3×3 transfer matrix may be defined
- only three eigenvalues — one of them is equal to 1 — remain when p and ρ are considered...
- ... still, various structures appears, as the dominant eigenvalues may change.

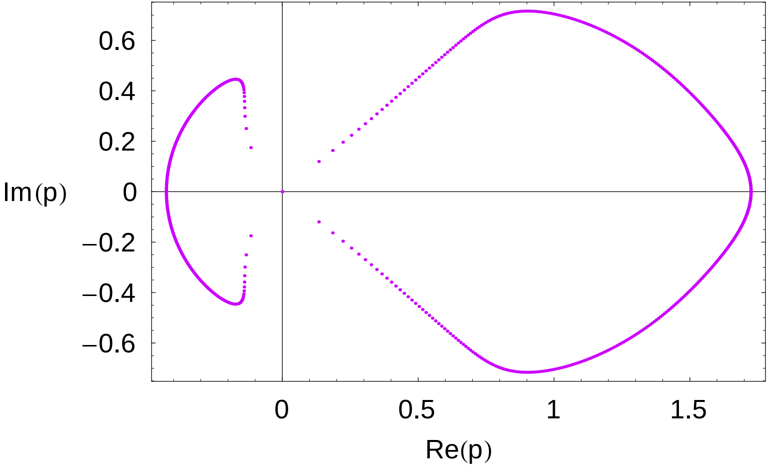
Complex zeros for double fan ($n = 200$, $\rho = 1$)



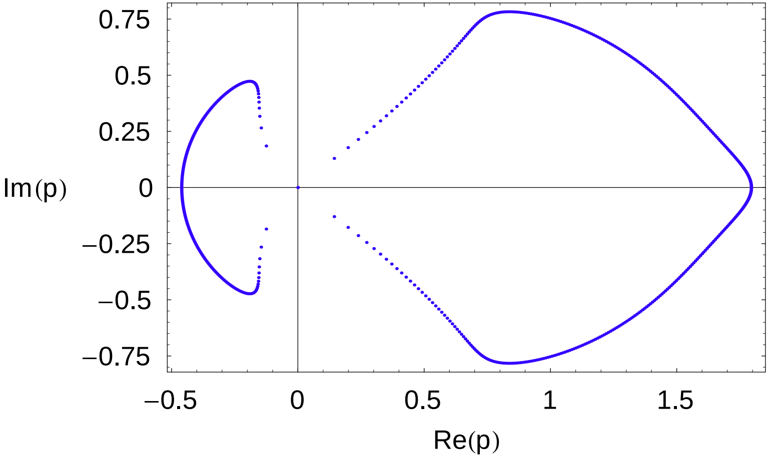
Complex zeros for double fan ($n = 200$, $\rho = 0.9$)



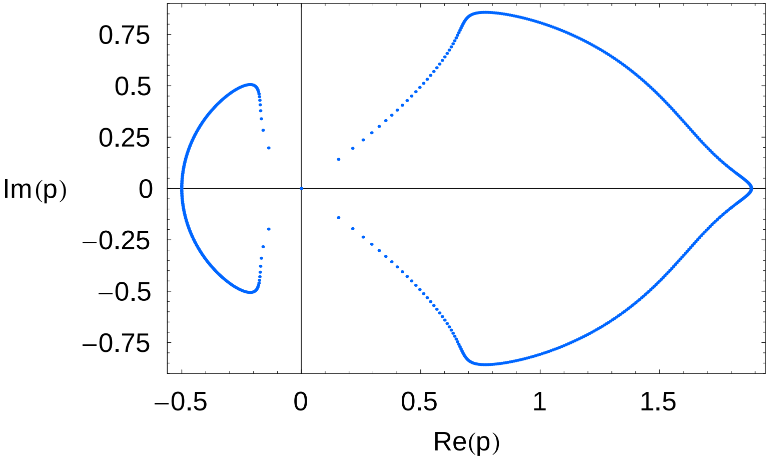
Complex zeros for double fan ($n = 200$, $\rho = 0.8$)



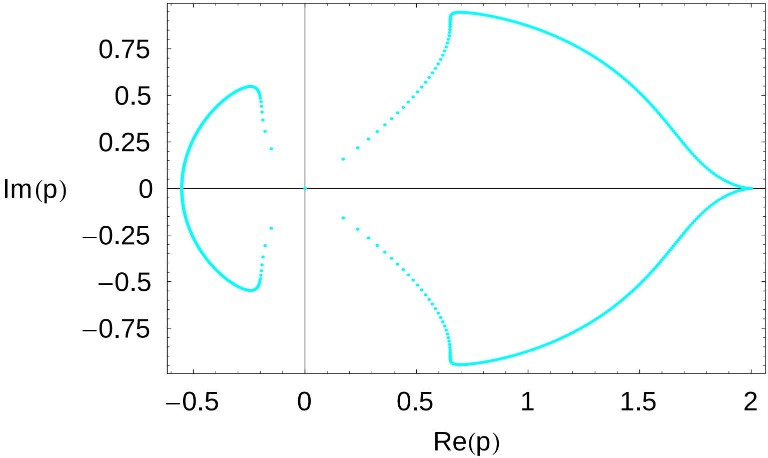
Complex zeros for double fan ($n = 200$, $\rho = 0.7$)



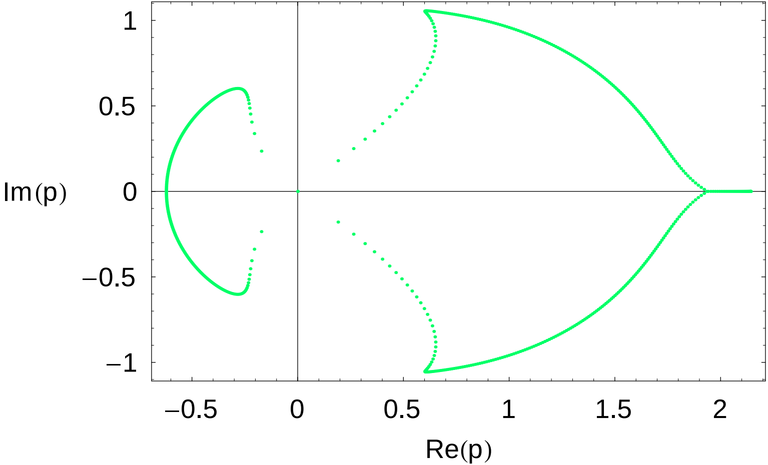
Complex zeros for double fan ($n = 200$, $\rho = 0.6$)



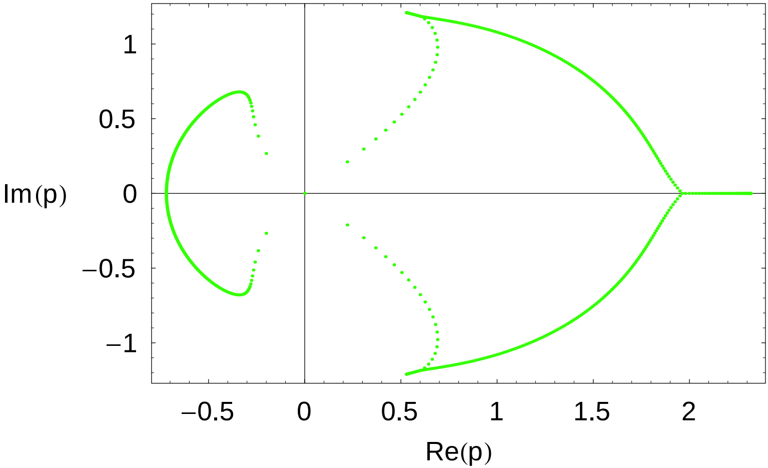
Complex zeros for double fan ($n = 200$, $\rho = 0.5$)



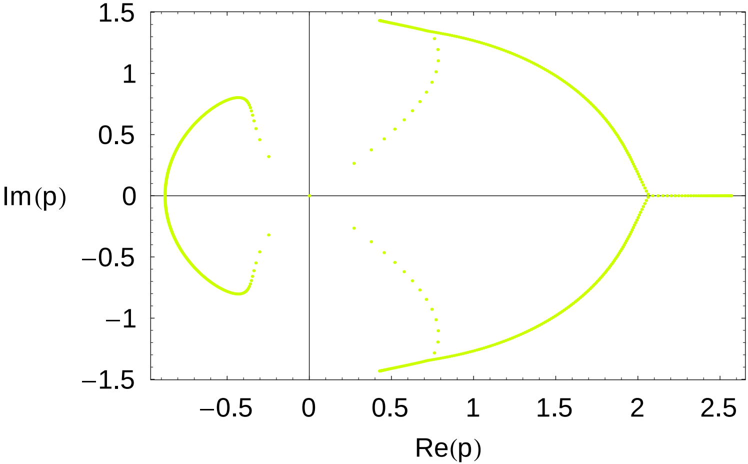
Complex zeros for double fan ($n = 200$, $\rho = 0.4$)



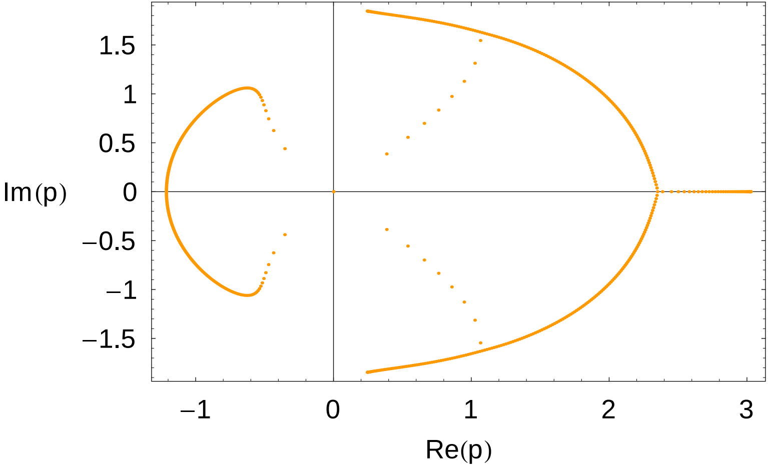
Complex zeros for double fan ($n = 200$, $\rho = 0.3$)



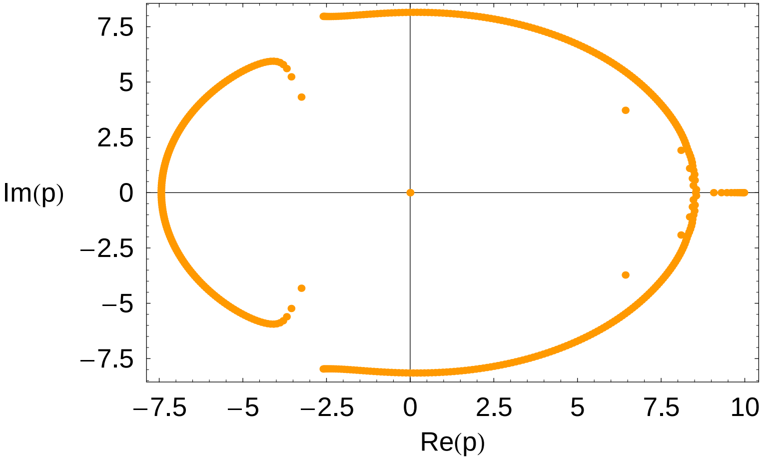
Complex zeros for double fan ($n = 200$, $\rho = 0.2$)



Complex zeros for double fan ($n = 200$, $\rho = 0.1$)



Complex zeros for double fan ($n = 150$, $\rho = 10^{-3}$)



Complex zeros for double fan ($n \to \infty$, $\rho = 10^{-3}$)