Integer Symmetric Matrices with Spectral Radius < 2.019

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Polynomials we consider:

$$\chi_A(x) = \det(xI - A) = \prod_{i=1}^{n}(x - \lambda_i)$$

A integer symmetric matrix

e.g. $A = A_G$ = adjacency matrix of graph G.

Spectral radius $R_A = \max|\lambda_i|$

*① $R_A \leq 2$ $\quad$ A <u>cyclotomic</u> matrix

as $z^n \chi_A(z + 1/z)$ is a cyclotomic polynomial

② $2 < R_A < 2.019$

① Find all cyclotomic matrices!

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Simplifying the Problem:

I. Decomposable matrices

<u>Defn</u> A is <u>decomposable</u> if by simultaneously reordering its rows & columns it becomes

$$\begin{pmatrix} \square & 0 \\ 0 & \square \end{pmatrix} \qquad (\geq 2 \text{ blocks})$$

Otherwise it is <u>indecomposable</u>.

1st simplification:

Can assume that A is <u>indecomposable</u>.

Simplifying the Problem:

II Matrix equivalence.

A cyclotomic [all $|\lambda_i| \leq 2$]

Then we can

- replace A by $-A$

or

- For some i multiply row i & column i by -1.

or

- Simultaneously reorder its rows & columns.

Then new matrix is again cyclotomic.

These operations generate an equivalence relation on set of cyclotomic matrices.

Enough to find equivalence class representatives.

Interlude: Cauchy Interlacing

Thm (Cauchy). A symmetric $_{n\times n}$, eigenvalues

$$\lambda_1 \le \lambda_2 \le \dots \le \lambda_n$$

$A' = A$ with row i & column i removed, eigenvalues

$$\lambda'_1 \le \lambda'_2 \le \dots \le \lambda'_{n-1}.$$

Then

$$\lambda_1 \le \lambda'_1 \le \lambda_2 \le \lambda'_2 \le \dots \le \lambda_{n-1} \le \lambda'_{n-1} \le \lambda_n.$$

Apply this to cyclotomic matrices.

Simplifying the Problem:

III Maximal cyclotomic matrices

Defn A cyclotomic matrix A' is non-maximal if A' can be obtained by deleting the ith row & column from some cyclotomic matrix A.

Otherwise it is maximal.

It turns out that we can restrict our attention to maximal cyclotomic matrices, because "Every non-maximal cyclotomic matrix is 'contained' in a maximal one".

From simplifications I, II, III :

It's enough to look for

equivalence class representatives of
indecomposable, maximal cyclotomic matrices.

Known Results: Graphs

GRAPH		MATRIX
Graph	$\longleftrightarrow$	$\{0,1\}$-symmetric matrix with 0s on diagonal "adjacency matrix".
connected	$\Longleftrightarrow$	indecomposable
vertices not labelled	$\Longleftrightarrow$	equivalence class
cyclotomic graph	$\Longleftrightarrow$	cyclotomic matrices

Example

$G' =$  $\longleftrightarrow$ $A' = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

non-maximal cyclotomic

$G =$  $\Longleftrightarrow$ $A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$

Maximal cyclotomic

Maximal connected cyclotomic Graphs

Theorem (J.H. Smith 1970). These are $\tilde{E}_6$, $\hat{E}_7$, $\tilde{E}_8$, $\tilde{A}_n$ $(n \geq 2)$, $\hat{D}_n$ $(n \geq 4)$.

We now extend Smith's results to general integer symmetric matrices.

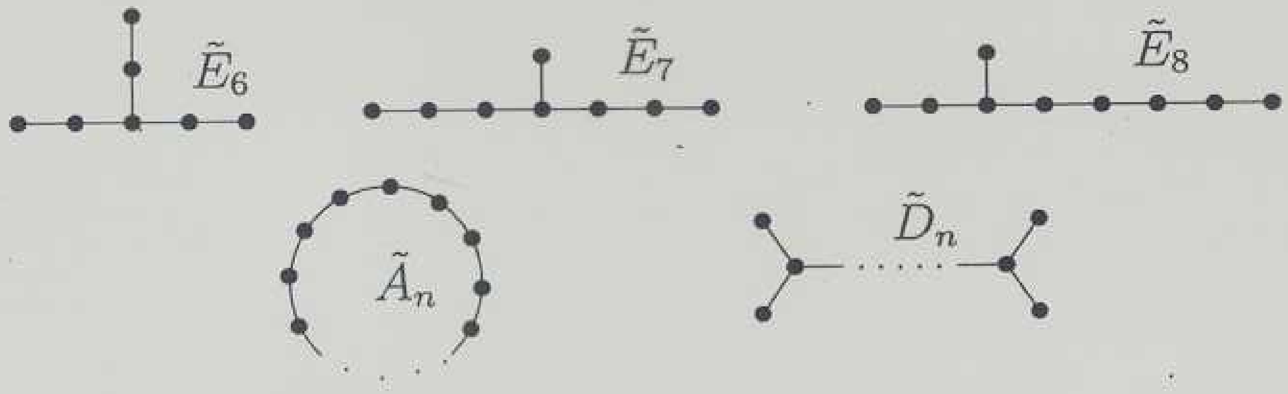


FIGURE 9. The maximal connected cyclotomic graphs $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8, \tilde{A}_n (n \geq 2)$ and $\tilde{D}_n (n \geq 4)$. The number of vertices is 1 more than the index. (From [MS]).

Step 1: Matrices containing an entry ≥ 2 in modulus.

Proposition. Every indecomposable maximal cyclotomic matrix having an entry ≥ 2 in modulus is equivalent either to the

- 1×1 matrix (2)

or the

- 2×2 matrix $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$.

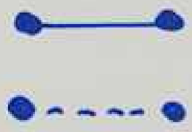
—||—

So from now on can assume that our cyclotomic matrix A has all entries in $\{-1, 0, 1\}$.

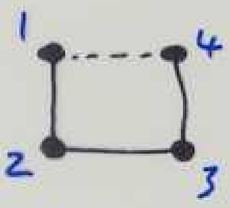
Step 2: $\{-1,0,1\}$-matrices with 0's on the diagonal:

Signed Graphs!

Signed graph = graph where edges have sign $+1$ or -1



Ex



$$\Longleftrightarrow \begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix}$$

signed graph — its adjacency matrix

Maximal Connected Cyclotomic Signed graphs.

<u>Theorem</u> There are:

- S_{14}
- S_{16}
- T_{2k} for all $k \geq 3$.

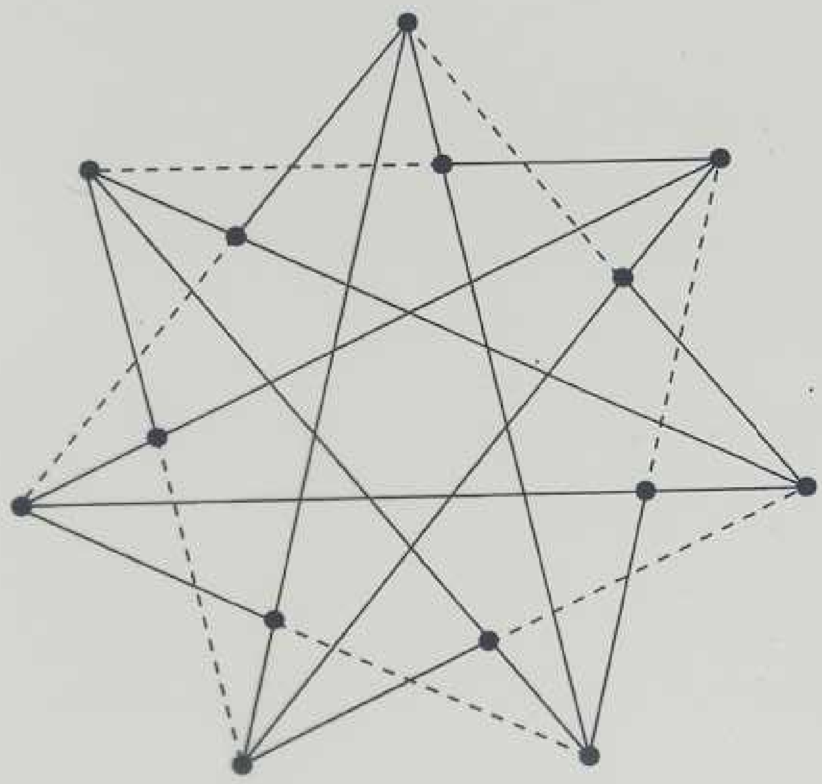


FIGURE 3. The 14-vertex sporadic maximal cyclotomic signed graph S_{14}. See also Section 12.2.

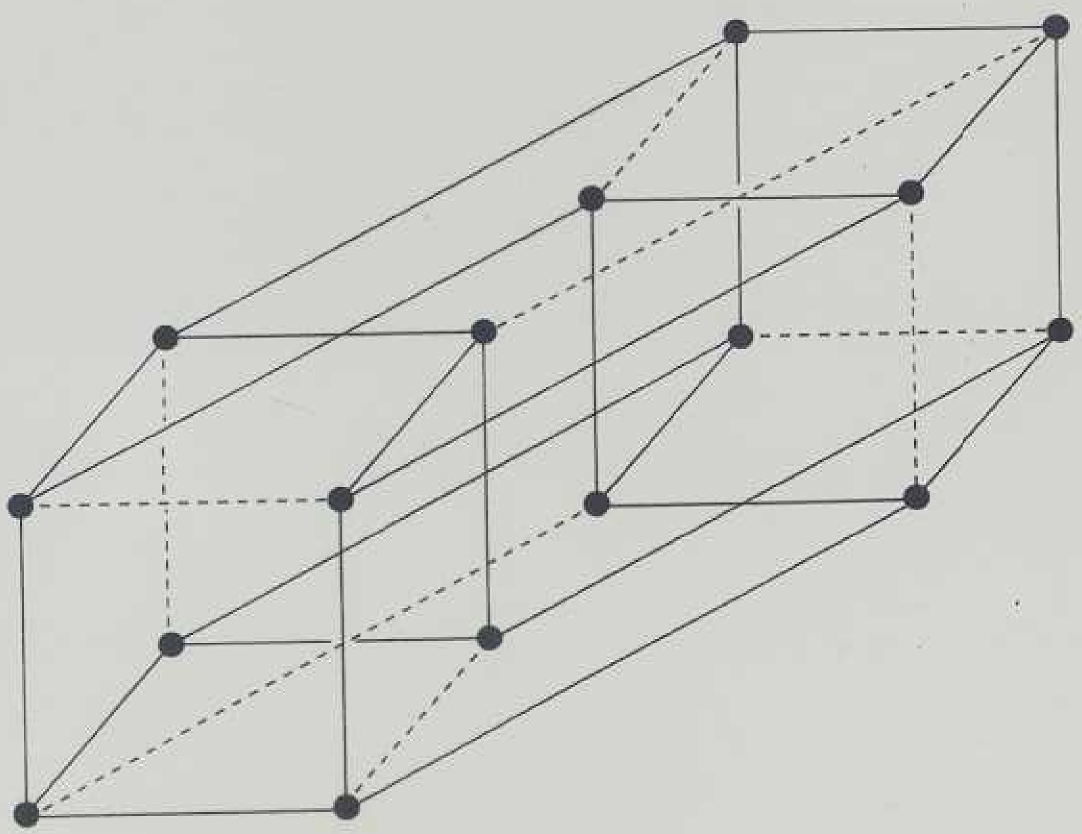


FIGURE 4. The hypercube sporadic maximal cyclotomic signed graph S_{16}.

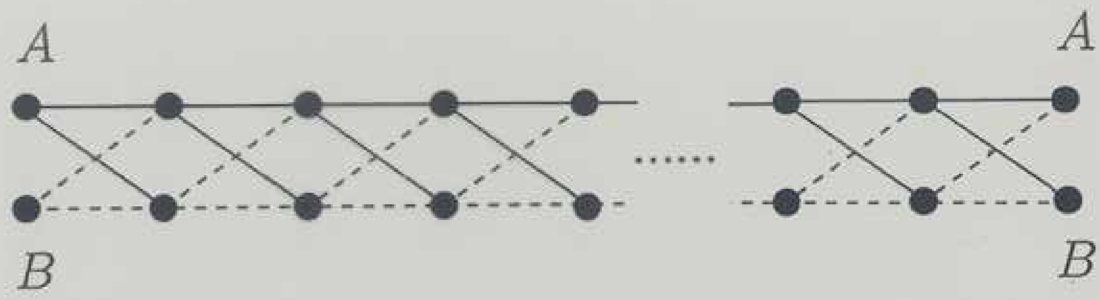


FIGURE 1. The family T_{2k} of $2k$-vertex maximal cyclotomic toral tesselations, for $k \geq 3$. (The two copies of vertices A and B should be identified.)

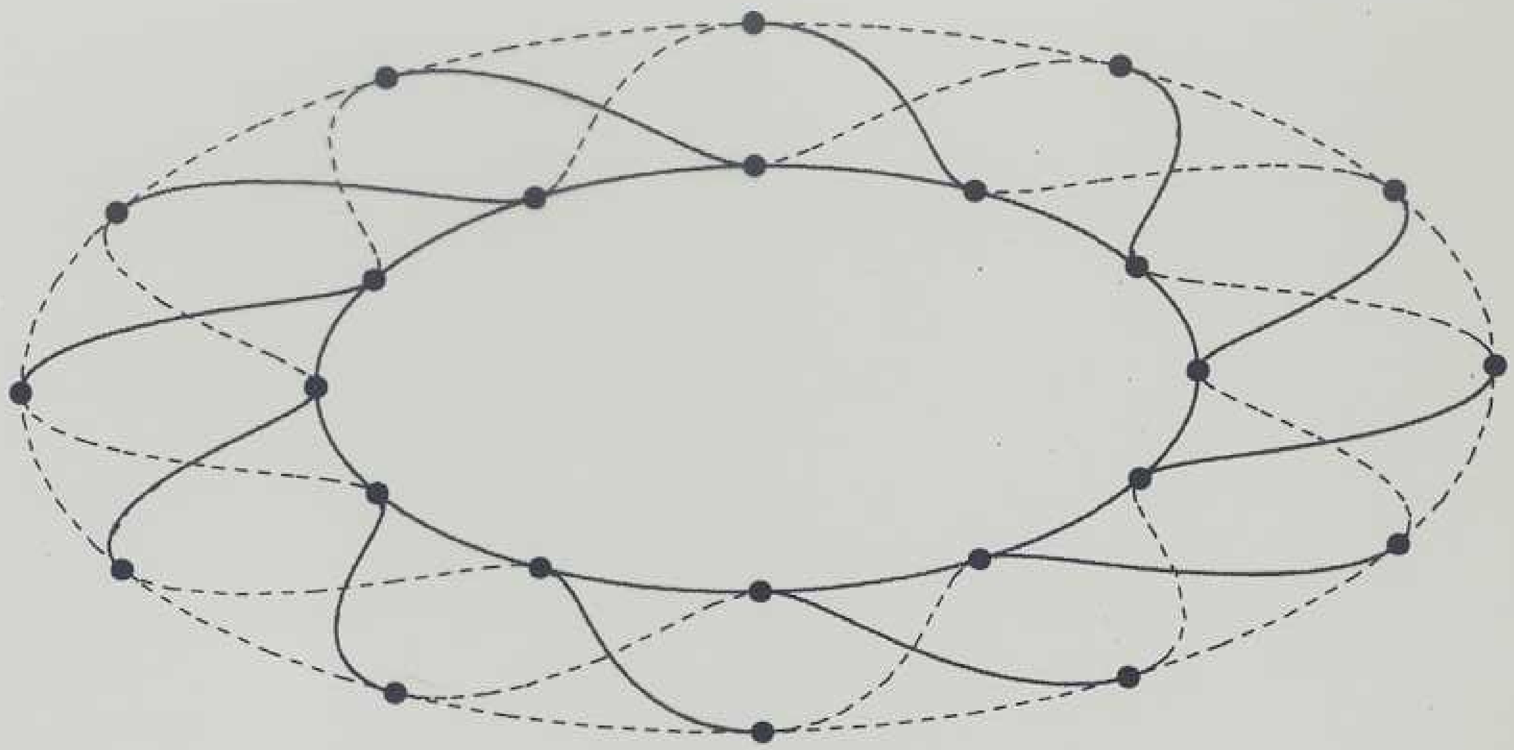


FIGURE 2. A typical toral tesselation T_{2k}: the signed graph T_{12}.

Maximal Connected Cyclotomic Charged signed graphs

0, ±1 on diagonal

<u>Theorem</u> These are:

- S_7
- S_8
- S_8'
- C_{2k}^{++} for $k \geq 2$
- C_{2k}^{+-} "   .

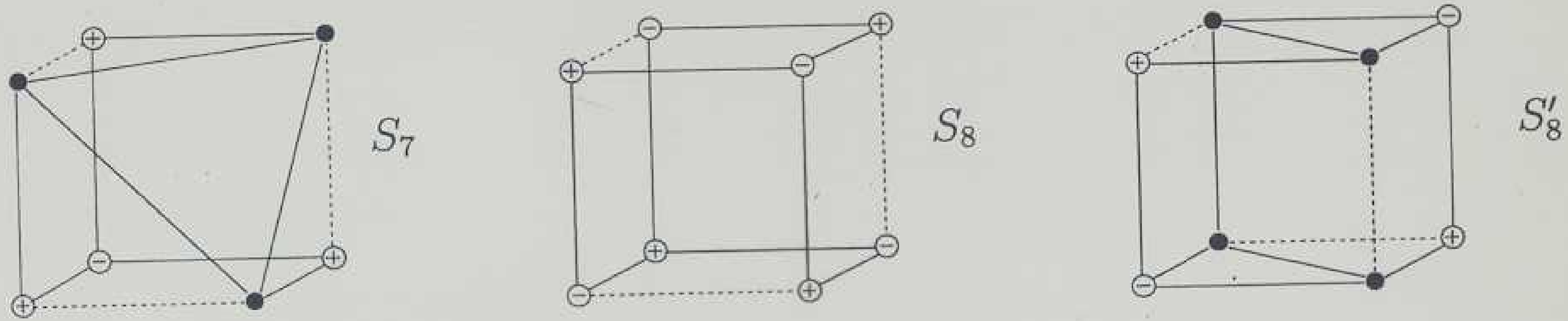


FIGURE 7. The three sporadic maximal cyclotomic charged signed graphs S_7, S_8, S'_8.

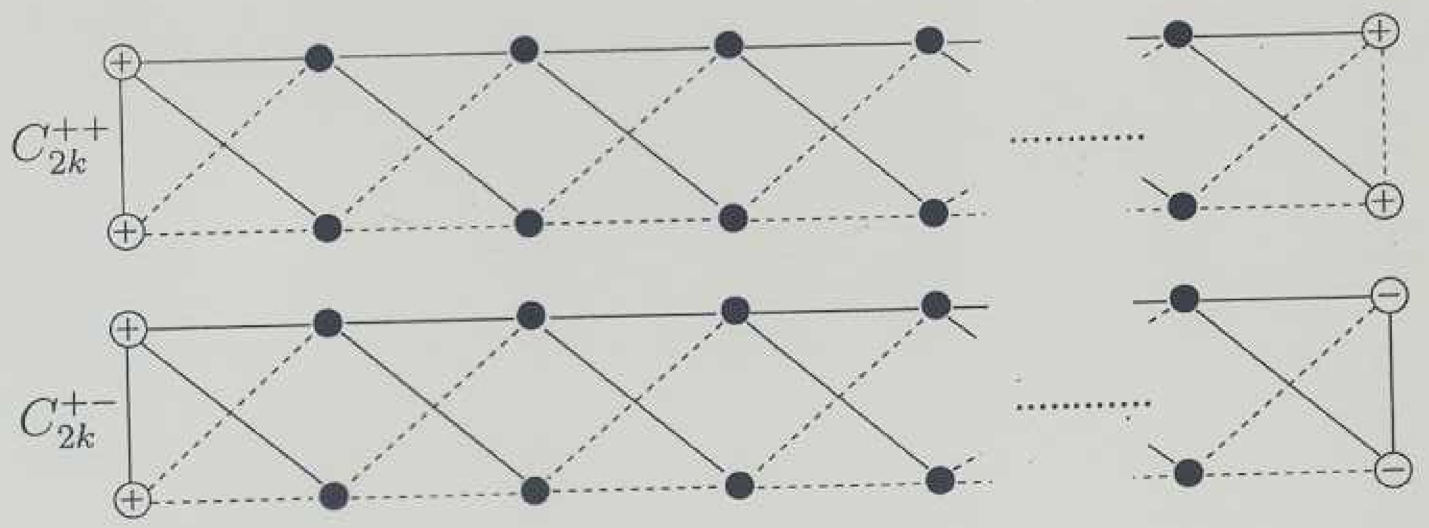


FIGURE 6. The families of $2k$-vertex maximal cyclotomic cylindrical tesselations C_{2k}^{++} and C_{2k}^{+-}, for $k \geq 2$.

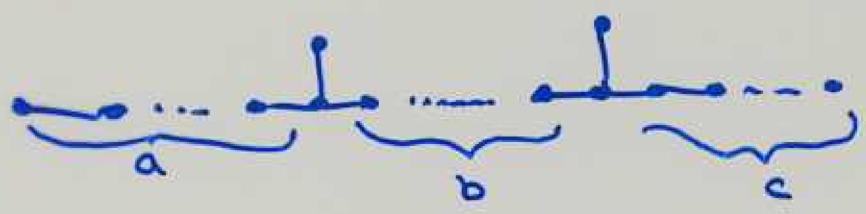
R_{A_q} for graphs

$R_{A_q} \leq 2$ cyclotomic: done earlier

$2 < R_{A_q} < \sqrt{2+\sqrt{5}} = 2.058\ldots$:

Graphs $T_{a,b,c}$ & $H_{a,b,c}$

$H_{a,b,c}$:



$R_{A_q} > \sqrt{2+\sqrt{5}}$: dense on $(\sqrt{2+\sqrt{5}}, \infty)$.

② $2 < R_A < 2.019$

$T_{a,b,c} =$

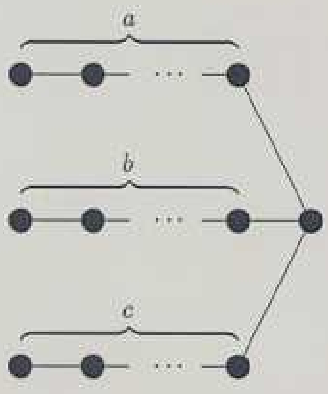
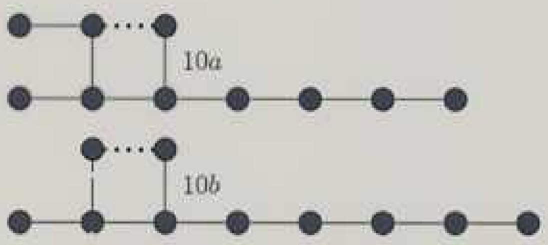
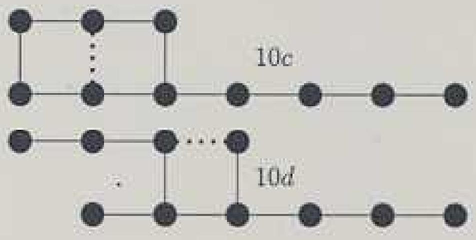


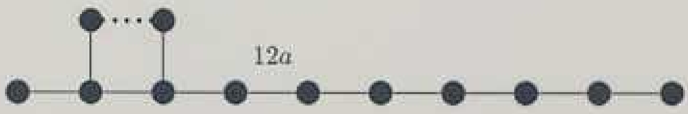
Figure 1: The tree $T_{a,b,c}$.





11a

11b



12a

R_A A

#	Maximum modulus of eigenvalues	Charged signed graph having corresponding spectral radius
1	2.00659...	$T_{1,2,6}$
2	2.00960...	$10b$
3	2.01075...	$T_{1,2,7}$
4	2.01348...	$T_{1,2,8}$, $10a$
5	2.01531...	$T_{1,3,4}$, $10c$, $11a$, $11b$, $T_{1,2,9}$
6	2.01657...	$10d$, $T_{1,2,10}$
7	2.01746...	$T_{1,2,11}$
8	2.01809...	$T_{1,2,12}$
9	2.01854...	$T_{1,2,13}$
10	2.01887...	$T_{1,2,14}$, $12a$

Table 1: The noncyclotomic connected charged signed graphs whose eigenvalues are at most 2.019 in modulus.

Thm All maximal indecomposable integer symmetric matrices A with $2 < R_A < 2.019$ are equivalent to one of

Salem Number τ :

"nearest thing to a root of unity":

all but two conjugates on $|z|=1$

Conjugates = roots of minimal polynomial.

Here = $\{\tau, \tau^{-1}, \tau_3, \ldots, \tau_d\}$

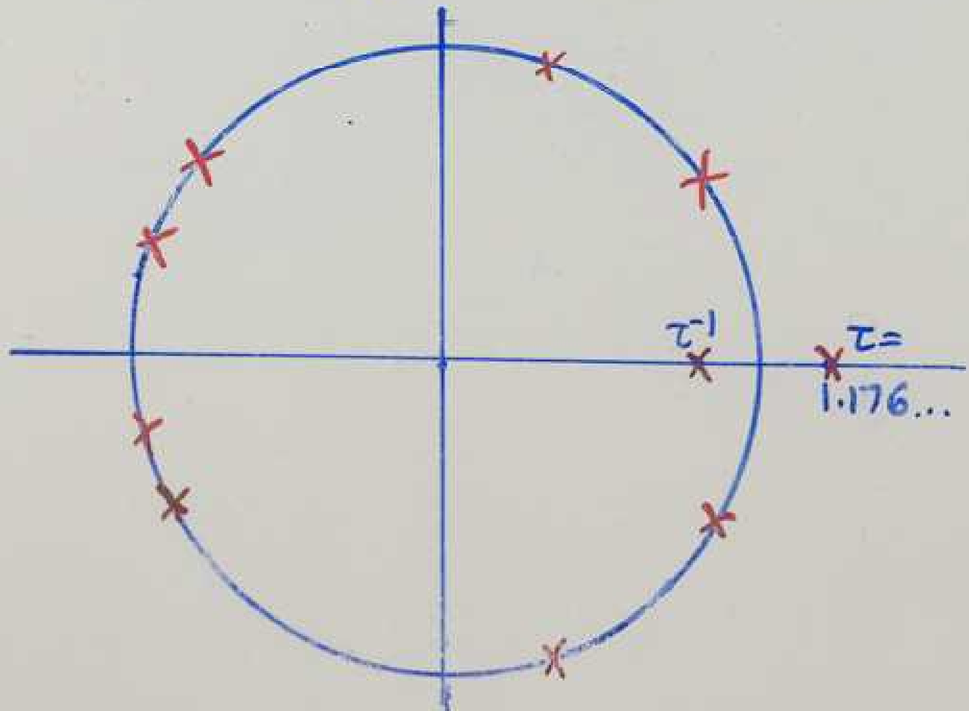
$\tau > 1, \quad |\tau_3| = \ldots = |\tau_d| = 1$

Example $\tau = 1.176\ldots$, root of

$$z^{10}+z^9-z^7-z^6-z^5-z^4-z^3+z+1$$

$$= (z-\tau)(z-\tau^{-1})\prod_{j=3}^{10}(z-\tau_j)$$

"Salem Polynomial"



Some graphs give Salem numbers:

G: 

$$z^5 \chi_G\left(\sqrt{z}+1/\sqrt{z}\right) = z^{10}+z^9-z^7-z^6-z^5-z^4-z^3+z+1$$