Orbital chromatic and flow polynomials

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1 Orbital polynomials

We are given a graph Γ and a group G of automorphisms of Γ. Take a graph polynomial which counts some features of Γ, e.g. the chromatic polynomial $\chi_\Gamma(x)$, whose value at the positive integer k is the number of proper k-colourings of Γ. We want to produce an *orbital polynomial*, which counts orbits of G on these features. For example, $O\chi_{\Gamma,G}$, whose value at k is the number of G-orbits of proper k-colourings of Γ.

The main questions in this largely untouched field are:

- Does such a polynomial exist?
- How much of the theory of the graph polynomial (e.g. location of roots) goes over to the orbital polynomials?
- What about generalisations, e.g. to matroids?
- There is a multivariate polynomial associated with orbit counting, the *cycle index* (Redfield–Pólya). Can one combine this with the structural aspects?

The main tool is the *Orbit-counting Lemma*, which states that if a (finite) group G acts on a (finite) set X, the number of G-orbits on X is equal to the average number of fixed points of its elements, i.e. the expected number of fixed points of a uniform random element of G:

$$\mathrm{orb}(G,X) = \frac{1}{|G|}\sum_{g\in G}\mathrm{fix}(g,X).$$

2 Orbital chromatic polynomial

2.1 Definition

By the orbit-counting lemma, we have to count the number of colourings of Γ fixed by an automorphism g. Now a colouring is fixed if and only if it is constant on the cycles of g. Thus a fixed colouring induces a colouring of the graph Γ/g obtained by shrinking each cycle of g to a point. (If a cycle contains an edge, then this shrinking creates a loop.) So the number of fixed colourings is $\chi_{\Gamma/g}(k)$, and we have

$$O\chi_{\Gamma,G}(x) = \frac{1}{|G|}\sum_{g\in G}\chi_{\Gamma/g}(x).$$

This shows that the orbital chromatic polynomial exists, and is a polynomial of degree equal to the number of vertices, with leading coefficient $1/|G|$.

2.2 Orbital chromatic roots

We recall some of the most significant facts about chromatic roots.

(a) If Γ has chromatic number m, then $0,\ldots,m-1$ are roots of χ.

(b) There are no real orbital chromatic roots in $(-\infty,0)$, $(0,1)$, or $(1,\frac{32}{27}]$, but they are dense in $[\frac{32}{27},\infty)$.

(c) Complex orbital chromatic roots are dense in the whole complex plane.

Part (a) extends to orbital chromatic roots, since there are no G-orbits on proper k-colourings if and only if there are no proper k-colourings. Part (c) also extends, since any chromatic root is an orbital chromatic root (for the trivial group). Things are different for (b):

(b′) Real orbital chromatic roots are dense in $\mathbb{R}$.

To see this, recall that the number of ways of choosing n objects from a set of k, if order is unimportant and repetition is allowed, is $\binom{n+k-1}{n}$. This already shows that negative roots occur: if Γ is the null graph on n vertices, and G the symmetric group S_n, then $O\chi_{\Gamma,G}(x) = \binom{x+n-1}{n}$, with roots $0,-1,\ldots,-(n-1)$.

Now let Γ consist of n disjoint triangles, and G the symmetric group permuting the triangles. Then $O\chi_{\Gamma,G}(x) = \binom{x(x-1)(x-2)+n-1}{n}$, which has roots where $x(x-1)(x-2) = -k$, for $0\le k\le n-1$. This equation has a unique negative root α_k; as

$k \to \infty$, these roots tend to $-\infty$, but the spacing between consecutive roots tends to zero. Now, just as with ordinary chromatic roots, we can shift orbital chromatic roots s places to the right by taking the join of Γ with K_s. This shows that the roots are dense.

Given this, two natural questions are:

(a) Can we restrict the positions of the roots by imposing some conditions on the graph or the group?

(b) How are orbital chromatic roots related to chromatic roots of the same graph?

I will describe the very small amount known about this.

Begin by recalling some more detail about the chromatic polynomial. Suppose that Γ has n vertices, c connected components, and b blocks. Then

(a) On $(-\infty, 0)$, $\chi_\Gamma(x)$ has the sign of $(-1)^n$.

(b) On $(0,1)$, $\chi_\Gamma(x)$ has the sign of $(-1)^{n+c}$.

(c) On $(1, \frac{32}{27}]$, χ_Γ has the sign of $(-1)^{n+c+b}$.

Also note that the number of vertices of Γ/g is equal to the number of cycles of g on vertices of Γ, and the parity of the difference between n and the number of cycles is the parity of the permutation g. Hence, if all elements of G are even permutations, the contributions $\chi_{\Gamma/g}$ will all have the same sign as χ_Γ on $(-\infty, 0)$.

This proves the first of the following observations, and the second is proved similarly.

(a) If every element of G acts as an even permutation on the vertex set of Γ, then $\chi_{\Gamma,G}$ has no zeros in $(-\infty, 0)$.

(b) If every element of G acts as an even permutation on the set of vertices and connected components of Γ, then $\chi_{\Gamma,G}$ has no zeros in $(0,1)$.

You might expect a similar statement involving blocks and the interval $(1, \frac{32}{27}]$; but this is not the case; blocks can behave differently. For example, consider the graph $K_{2,p}$ where p is odd, and let G be a cyclic group of order p, permuting the vertices in the second bipartite block in a single cycle and fixing those in the first. Clearly Γ is 2-connected, and G acts as even permutations of the vertices. But, if

g is a generator of G, then Γ_g is $K_{2,1}$, a path of length 2, hence not 2-connected. Indeed, if p is an odd prime, the orbital chromatic polynomial of Γ and G is

$$x(x-1)\left((x-2)^p+(x-1)^{p-1}+(p-1)(x-1)\right)/p,$$

which has a root in $(1,\frac{32}{27})$ if $p \geq 5$; indeed, the roots tend to 1 as $p \to \infty$.

Problem: Suppose that Γ is 2-connected, and that elements of G act as even permutations of the vertex set. Which points of the interval $(1,\frac{32}{27}]$ can occur as orbital chromatic roots under this hypothesis, and what are their limit points?

On the second question, a very limited amount of experimentation suggests that the real roots of $O\chi_{\Gamma,G}$ may be bounded above by the largest real root of χ_Γ. Is this true?

3 Orbital Tutte polynomial

The chromatic polynomial is, apart from a prefactor, a specialisation of the two-variable Tutte polynomial or Whitney rank function of the graph. We would like a polynomial which specialises in a similar way to the orbital chromatic polynomial. In this search, it is helpful to look at another important specialisation, the flow polynomial; this points out the difficulties that arise.

3.1 An example

Let A be a finite abelian group of order k, and Γ a graph. Choose an arbitrary but fixed orientation of the edges of Γ. An *A-flow* on Γ is a function from the oriented edges of Γ to A having the property that the flow into and out of any vertex are equal (when calculated in A). If we change the orientation by reversing an edge, we simply change the sign of the flow on that edge. A flow is *nowhere-zero* if it never takes the value 0 (the identity element of A). The number of nowhere-zero A-flows on Γ is the evaluation at $x=k$ of the *flow polynomial* $\phi_\Gamma(x)$, independent of the chosen orientation of Γ and (more surprisingly) of the structure of the abelian group A.

If G is a group of automorphisms of Γ, then G acts on the set of nowhere-zero A-flows. (If an automorphism reverses the orientation of an edge, we ask that it negates the value of the flow on that edge.) So we can ask: Is there an orbital flow polynomial for Γ and G?

The following example shows that the number of orbits can depend on the structure of A, not just its order. In the graph, edges are oriented downwards.

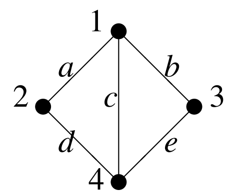

Figure 1: An example

We take G to be the full automorphism group of Γ, consisting of the identity, $(2,3)$, $(1,4)$ and $(1,4)(2,3)$. Flow values are taken from an abelian group A of order k.

The conditions for a flow are $d = a$, $e = b$, and $a+b+c = 0$. So for a nowhere-zero flow, there are $k-1$ choices for a, and $k-2$ for b (any element except zero and $-a$; then c is determined. So there are $(k-1)(k-2)$ nowhere-zero flows. Now

- A flow is fixed by $(2,3)$ if and only if $a = b$; then $c = -2a$, and we require $2a \neq 0$. If the number of solutions of $2x = 0$ in A is α, there are $k-\alpha$ choices for a, then everything is determined.

- A flow is fixed by $(1,4)$ if and only if $a = -a$ and $b = -b$. So a and b are distinct non-zero elements of solution set of $2x = 0$, and there are $(\alpha-1)(\alpha-2)$ choices.

- A flow is fixed by $(1,4)(2,3)$ if and only if $a = -b$. But then $c = -(a+b) = 0$, so there are no nowhere-zero flows fixed by this element.

So the number of orbits on nowhere-zero flows is

$$\frac{1}{4}\left((k-1)(k-2)+(k-\alpha)+(\alpha-1)(\alpha-2)\right).$$

We see that this depends on the structure of A, not just on its order.

The flow polynomial is regarded as being the dual of the chromatic polynomial. So why does one have an orbital version and not the other? It turns out that what is really dual to flows are tensions, defined as follows. With a chosen orientation of Γ as before, an A-*tension* on Γ is a function f on oriented edges with the property that, for any cycle C, the sum of the values of f on positively-directed edges of C is equal to the sum on negatively-directed edges. (We choose an orientation of the cycle; an edge is positively or negatively directed if its orientation agrees or disagrees with that of the cycle.)

Now the "gradient" of a nowhere-zero A-tension on Γ is a proper colouring of Γ, with colour set A, and conversely. So the number of proper k-colourings of Γ is k^c times the number of nowhere-zero A-tensions, where c is the number of connected components of Γ, and A is an abelian group of order k.

3.2 The orbital Tutte polynomial

The following theorem is proved in [1]. I won't give the proof here. If A is a finite abelian group and i a non-negative integer, we denote by $\alpha_i(A)$ the number of solutions of $ix = 0$ in A. Note that $\alpha_0 = |A|$ and $\alpha_1 = 1$.

Theorem 1 *Let G be a group of automorphisms of a graph Γ. There is a polynomial $OT_{\Gamma,G}$ in two infinite sequences of variables $(x_i),(x_i^*)$ (indexed by the natural numbers) such that*

(a) the number of orbits of G on nowhere-zero A-flows is obtained from $OT_{\Gamma,G}$ by the substitution

$$x_i = \alpha_i(A), \qquad x_i^* = -1.$$

(b) the number of orbits of G on nowhere-zero A-tensions is obtained from $OT_{\Gamma,G}$ by the substitution

$$x_i = -1, \qquad x_i^* = \alpha_i(A).$$

(c) if Γ is connected, then the number of orbits of G on proper k-colourings is obtained from $OT_{\Gamma,G}$ by the substitution

$$x_i = -1, \quad x_0^* = k, \quad x_i^* = 1 \text{ for } i > 0,$$

followed by multiplication by k.

3.3 The orbital flow polynomial

The above theorem suggests defining the *orbital flow polynomial* of Γ and G to be the polynomial $O\phi_{\Gamma,G}(x)$ obtained from $OT_{\Gamma,G}$ by the substitution

$$x_0 = x, \quad x_i = 1 \text{ for } i > 0, \quad x_i^* = -1.$$

It is known that, if a variable x_i or x_i^* for $i > 0$ actually occurs in $OT_{\Gamma,G}$, then either $i = 0$, or G contains an element of order i. Hence, if $|G|$ and $|A|$ are coprime, then for any such i the number of solutions of $ia = 0$ in A is 1; so the number of orbits on nowhere-zero flows is $O\phi_{\Gamma,G}(k)$ in this case.

Now that we have a univariate polynomial, we can ask about its zeros. Using arguments like those for the orbital chromatic polynomial, it is possible to show that orbital flow roots are dense in the negative real axis. The situation for positive roots is more mysterious. As with ordinary flow roots, we don't have an example of a bridgeless graph having an orbital flow root greater than 4.

There are flow roots in $(0,1)$, but they are not known to be dense. Here is a construction showing that reciprocals of positive integers are limit points of orbital flow roots. These are the only such limit points known (to me!). These arise from the graph with k components, each component being n parallel edges, with the action of the symmetric group S_k; as $n \to \infty$ for fixed k, there are orbital flow roots tending to $1/(k+1)$.

3.4 Other specialisations

The orbital flow polynomial counts orbits on nowhere-zero A-flows in the case when $|A|$ and $|G|$ are coprime. If one defines an orbital tension polynomial $O\tau_{\Gamma,G}$ similarly, by the substitution

$$x_i = -1, \quad x_0^* = x, \quad x_i^* = 1 \text{ for } i > 0,$$

then it counts orbits on nowhere-zero A-tensions if $|A|$ and $|G|$ are coprime; and, if Γ is connected, then the orbital chromatic polynomial is obtained by multiplying the orbital tension polynomial by x.

There are other ways of getting a univariate polynomial. Here is one suggested by Bill Jackson.

Let p be a prime. We define the p-modular orbital flow (or tension) polynomial to be the polynomial whose value at p^m is the number of G-orbits on nowhere-zero

A-flows (or A-tensions), where A is an elementary abelian group of order p^m (the direct product of m cyclic groups of order p). Now for this group A we have

$$\alpha_i(A) = \begin{cases} |A| & \text{if } p \text{ divides } i, \\ 1 & \text{otherwise;} \end{cases}$$

so the p-modular polynomials are obtained from the orbital Tutte polynomial by the substitutions

$$x_i = x \text{ if } p \mid i, \quad x_i = 1 \text{ otherwise}, \quad x_i^* = -1,$$

and

$$x_i = -1, \quad x_i^* = x \text{ if } p \mid i, \quad x_i^* = 1 \text{ otherwise},$$

respectively.

In our example, the 2-modular flow polynomial is obtained by putting $k = \alpha = x$, and is $\frac{1}{2}(x-1)(x-2)$. Note that the leading coefficient is larger than $1/|G|$.

3.5 An algebraic view

In fact we might get more insight from an algebraic view. As noted earlier, the construction of the orbital Tutte polynomial uses a dual pair of matrices over $\mathbb{Z}$; the variables x_i and x_i^* record invariant factors. For graphs, we use the vertex-edge and cycle-edge incidence matrices over the integers. Reducing the integers mod p we obtain the p-modular flow and tension polynomials defined above; embedding $\mathbb{Z}$ in $\mathbb{Q}$ gives the (ordinary) orbital flow and tension polynomials. In the same framework, one could define p-adic orbital polynomials.

References

[1] P. J. Cameron, B. Jackson and J. D. Rudd, Orbit-counting polynomials for graphs and codes, *Discrete Math.* **308** (2008), 920–930; doi: 10.1016/j.disc.2007.07.108

[2] P. J. Cameron and K. K. Kayibi, Orbital chromatic and flow roots, *Combinatorics, Probability and Computing* **16** (2007), 401–407; doi: 10.1017/S0963548306008200