Set-theoretic Solutions of the Yang-Baxter Equation

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Certain matrix solutions of the braid or Yang-Baxter equations lead to braided categories, knot invariants, quantum groups and other important constructions. However, these equations are also very interesting at the level of set maps $r : X \times X \to X \times X$, where X is a set and r is a bijection, a line of study proposed by Drinfeld (1990)

Set theoretic solutions extend to special linear solutions but also lead to

- a great deal of combinatorics - group action on X, cyclic conditions,
- matched pairs of groups, matched pairs of semigroups
- semigroups of I type with a structure of distributive lattice
- special graphs
- algebras with very nice algebraic and homological properties such as being:
- Artin-Schelter regular rings;
- Koszul;
- Noetherian domains;
- with PBW k-bases;
- with good computational properties -the theory of noncommutative Groebner bases is applicable.

YBE, QYBE and set-theoretic YBE

Let V be a vector space over a field k, R be a linear automorphism of $V \otimes V$.

- R is *a solution of YBE* if $R^{12}R^{23}R^{12} = R^{23}R^{12}R^{23}$,
- R is a solution of QYBE if $R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$, where both equalities hold in $V \otimes V \otimes V$, and R^{ij} means R acting on the i-th and j-th component.
- Let $X \neq \emptyset$ be a set. A bijective map $r : X \times X \longrightarrow X \times X$ is *a set-theoretic solution of YBE*, if the braid relation

$$r^{12}r^{23}r^{12} = r^{23}r^{12}r^{23}$$

  holds in $X \times X \times X$, $r^{12} = r \times id_X$, $r^{23} = id_X \times r$. In this case (X, r) is called *a braided set*.
- Each set-theoretic solution of YBE induces naturally a solution to the YBE and QYBE.

Quadratic sets (X,r)

A quadratic set (X, r) is a nonempty set X with a bijective map $r : X \times X \longrightarrow X \times X$. The formula

$$r(x, y) = ({}^{x}y, x^{y}).$$

defines a "left action" $\mathcal{L} : X \times X \longrightarrow X$, and a "right action" $\mathcal{R} : X \times X \longrightarrow X$, on X as:

$$\mathcal{L}_x(y) = {}^{x}y, \quad \mathcal{R}_y(x) = x^{y} \quad \text{for all} \quad x, y \in X.$$

- r is *nondegenerate*, if $\mathcal{R}_x$ and $\mathcal{L}_x$ are bijective for each $x \in X$, i.e. $\mathcal{L}_x, \mathcal{R}_x \in Sym(X)$;
- r is *square-free* if $r(x, x) = (x, x)$ for all $x \in X$;
- (X, r) is *a braided set* if $r^{12}r^{23}r^{12} = r^{23}r^{12}r^{23}$;
- A braided set (X, r) with r involutive is called *a symmetric set*.

Associated algebraic objects to (X, r)

These are generated by X and with quadratic defining relations $\Re = \Re(r)$:

$$xy = zt \in \Re \quad \Longleftrightarrow \quad r(x, y) = (z, t).$$

- *The monoid* $S = S(X, r) = \langle X; \Re(r) \rangle$;
- *The group* $G = G(X, r) = {}_{gr}\langle X; \Re(r) \rangle$;
- *The k-algebra* $A = A(k, X, r) = k\langle X \rangle / (\Re(r))$, where k is a field
- The group of left action $\mathcal{G} = \mathcal{G}(X, r)$ defined as the subgroup $\mathcal{L}(G(X, r))$ of $Sym(X)$.

Conditions l1,r1,lr3 and the group actions

(X, r) is a braided set *iff* the following three conditions hold for all $x, y, z \in X$.

- **l1** : $\quad {}^{x}({}^{y}z) = {}^{{}^{x}y}({}^{x^{y}}z)$,
- **r1** : $\quad (x^{y})^{z} = (x^{{}^{y}z})^{y^{z}}$,
- **lr3** : $\quad ({}^{x}y)^{({}^{x^{y}}(z))} = {}^{((x)^{{}^{y}z})}(y^{z})$,

When (X, r) is nondegenerate, **l1** implies that $G = G(X, r)$ acts on X on the left, respectively, **r1** implies a right action of G on X.
Let $\mathcal{L} : G(X, r) \longrightarrow Sym(X)$ be the group homomorphism defined via the left action.
$\mathcal{G} = \mathcal{G}(X, r)$ will denote the subgroup $\mathcal{L}(G(X, r))$ of $Sym(X)$.

- $\mathcal{G}(X, r) = \{id_X\}$ *iff* (X, r) is the trivial solution.

Example of a nondegenerate square-free solution of mp level 2

Let $X = \{x_1, x_2, x_3, x_4, b, c\}$, define r as:

$$\begin{array}{lll} r(b,x_1) = (x_2,b) & r(b,x_2) = (x_1,b) & r(b,x_3) = (x_4,b) \\ r(b,x_4) = (x_3,b) & r(c,x_1) = (x_3,c) & r(c,x_3) = (x_1,c) \\ r(c,x_2) = (x_4,c) & r(c,x_4) = (x_2,c), & r(b,c) = (c,b) \\ r(c,b) = (b,c) & r(x_j,x_i) = (x_i,x_j), & 1 \leq i,j \leq 4. \end{array}$$

$\Re$ consists of 15 defining relations

$$\begin{array}{llll} bx_1 = x_2b & bx_2 = x_1b & bx_3 = x_4b & bx_4 = x_3b \\ cx_1 = x_3c & cx_3 = x_1c & cx_2 = x_4c & cx_4 = x_2c \\ x_jx_i = x_ix_j, & 1 \leq i < j \leq 4, & cb = bc. & \end{array}$$

A square-free nondeg. symmetric set (X, r) is uniquely defined via the left action

In our example, r is uniquelly defined via

$$\mathcal{L}_b = (x_1x_2)(x_3x_4), \quad \mathcal{L}_c = (x_1x_3)(x_2x_4), \quad \mathcal{L}_{x_i} = id_X, 1 \leq i \leq 4$$

($\mathcal{R}_z = \mathcal{L}_z^{-1}$ always hold for nondegenerate square-free symmetric sets).
Then $\mathcal{G}(X, r) = \langle \mathcal{L}_b \rangle \times \langle \mathcal{L}_c \rangle$, so it is isomorphic to the *Klein's group* $\mathbb{Z}_2 \times \mathbb{Z}_2$. Direct computations show that the set of automorphisms consists of the following eight elements:

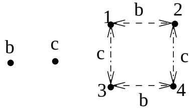
$$\begin{array}{lll} id_X, & \tau_1 = (bc)(x_2x_3), & \tau_2 = (bc)(x_1x_2x_4x_3), \\ \tau_3 = (bc)(x_1x_3x_4x_2), & \tau_4 = (bc)(x_1x_4), & \mathcal{L}_b, \quad \mathcal{L}_c, \quad \mathcal{L}_b \circ \mathcal{L}_c. \end{array}$$

Furthermore, one has

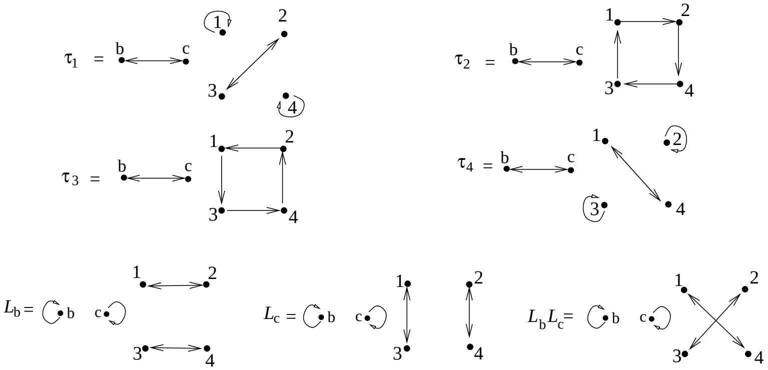
$$\mathrm{Aut}(X, r) = {}_{gr}\langle \tau_1, \tau_2 \mid \quad \tau_1^2 = 1, \tau_2^4 = 1, \tau_1\tau_2\tau_1^{-1} = \tau_2^3 \rangle \approx D_4.$$

$\mathrm{Aut}(X, r)$ is a proper subgroup of $\mathrm{Nor}_{\mathrm{Sym}(X)}(\mathcal{G})$.

(a) (X,r) $X=\{x_1, x_2, x_3, x_4, b, c\}$

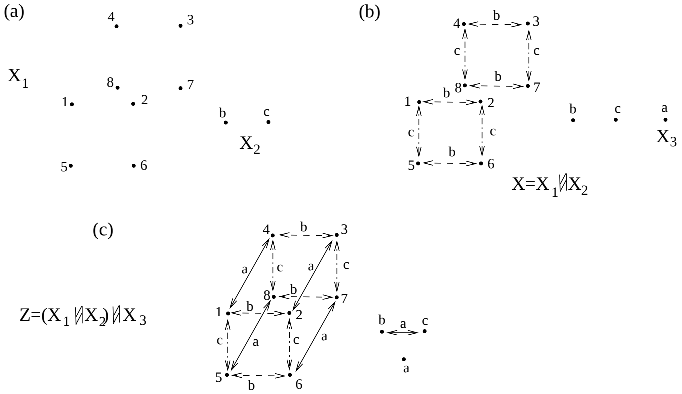


(b) Aut(X,r)

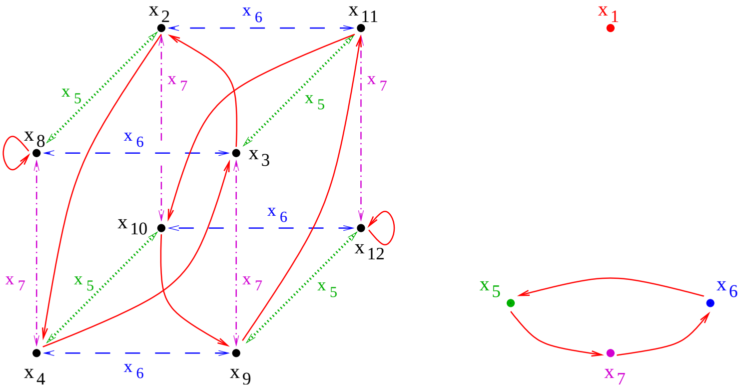


$\natural$ is not associative, $mpl(Z) = 3$

$(Z, r) = (X_1 \natural X_2) \natural X_3) = (X_2 \natural X_1) \natural X_3)$.
(Z, r) is an YB extension of $Y = X_1 \natural X_3$ with (X_2, r_2), but **this is not a strong twisted union** of Y and X_2.



The Crystal



x 2
x 6
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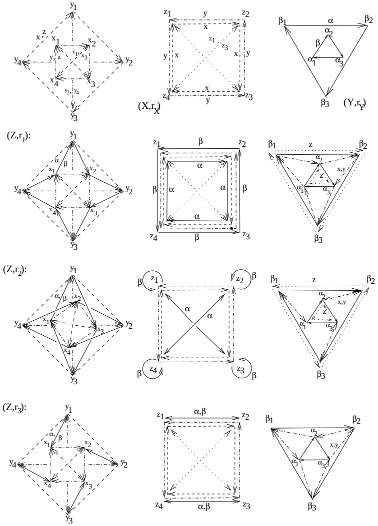
Example of extensions of solutions



Retraction

Let (X, r) be a nondegenerate symmetric sets. An equivalence relation $\sim$ is defined on X as

$$x \sim y \quad iff \quad \mathcal{L}x = \mathcal{L}y.$$

$[X] = X/_{\sim}$ denotes the set of equivalence classes $[x]$.

- The left and the right actions of X onto itself induce naturally left and right actions on the retraction $[X]$, via

$$^{[\alpha]}[x] := [^{\alpha}x] \quad [\alpha]^{[x]} := [\alpha^{x}], \text{ for all } \alpha, x \in X.$$

- The new actions define (as usual) a canonical map $r_{[X]} : [X] \times [X] \longrightarrow [X] \times [X]$
- $([X], r_{[X]})$ is a nondegenerate symmetric set, called the *retraction of* (X, r), and denoted $Ret(X, r)$. Furthermore,
- (X, r) cyclic $\Longrightarrow$ $([X], r_{[X]})$ cyclic.
- (X, r) is **lri** $\Longrightarrow$ $([X], r_{[X]})$ is **lri**.
- (X, r) square-free $\Longrightarrow$ $([X], r_{[X]})$ square-free.

Multipermutation solutions. Conjectures

- The solution $Ret(X,r) = ([X],[r])$ is called the *retraction of* (X,r).
- For all integers $m \geq 1$, $Ret^m(X,r)$ is defined recursively as $Ret^m(X,r) = Ret(Ret^{m-1}(X,r))$.
- (X,r) is *a multipermutation solution of level* m, $mpl(X,r) = m$ if m is the minimal number (if any), such that $Ret^m(X,r)$ is the trivial solution on a set of one element.
- By definition (X,r) is *a multipermutation solution of level* 0 *iff* X is a one element set.

Conjecture I. (TGI, 2004)

1. *Every finite nondegenerate square-free symmetric set* (X,r) *is retractable.*
2. *Every finite nondegenerate square-free symmetric set* (X,r) *is multipermutation solution, with* $mpl(X,r) <\mid X \mid$.

Clearly, these conjectures are equivalent.

???The solvable length $sl(G(X,r)) = mpl(X,r)$???

Question 1.

Suppose (X,r) is a multipermutation symmetric set. Is there any relation between $mpl(X,r)$ and the algebraic properties of $S(X,r), G(X,r), \mathcal{A}(X,r)$?

Fact. [ESS], [GI]. *Suppose (X,r) is a nondegenerate symmetric set. Then the group $G(X,r)$ is solvable.*

Theorem

Let (X,r) be a finite nondegenerate square-free solution of order $n > 1$. Suppose (X,r) is a multipermutation solution of level m. Then the the solvable length of $G(X,r)$ is at most m.

Conjecture II.

Suppose (X,r) is a finite nondegenerate square-free symmetric set with multipermutation level m. Then the solvable length of $G(X,r)$ is exactly m.

For a nondegenerate square-free symmetric set (X, r) the following conditions are equivallent.

- $mpl(X, r) = 1$.
- (X, r) is the trivial solution.
- ${}^{x}y = y$, for all $x, y \in X$.
- $S(X, r)$ is the free abelian monoid generated by X.
- $G(X, r)$ is the free abelian group generated by X.
- $\mathcal{G}(X, r) = \{id_X\}$.

Proposition.

- $mpl(X, r) = 2 \;\Leftrightarrow\; \mathcal{G}(X, r)$ *is abelian* $\;\Rightarrow\; slG(X, r) = 2$.
- $mpl(X, r) = 3 \Rightarrow sl\mathcal{G}(X, r) = 2$.

Extensions and twisted unions

Definition (GI-M)

Let (X, r_X) and (Y, r_Y) be disjoint quadratic sets. A quadratic set (Z, r) is *a (general) extension of* $(X, r_X), (Y, r_Y)$, if $Z = X \bigcup Y$ as sets, and r extends the maps r_X and r_Y. (Z, r) is *a YB-extension of* $(X, r_X), (Y, r_Y)$ if r obeys YBE.

Example (ESS)

Let $(X, r_X), (Y, r_Y)$ be nondegenerate symmetric sets, $\sigma \in Aut(X, r), \rho \in Aut(Y, r)$. Define the nondegenerate involutive extension $(Z, r) = X \natural_0 Y$ via r_X, r_Y and the formulae

$$r(\alpha, x) = (\sigma(x), \rho^{-1}(\alpha)), \quad r(x, \alpha) = (\rho(\alpha), \sigma^{-1}(x)).$$

(Z, r) is a symmetric set called a *twisted union*. One has

$$\mathcal{L}_{x|Z} = \mathcal{L}_{x|X}.\rho, \quad \mathcal{L}_{\alpha|Z} = \mathcal{L}_{\alpha|Y}.\sigma, \quad \forall x \in X, \alpha \in Y$$

The trivial extension (Z, r) of $(X, r_X), (Y, r_Y)$ is defined by $\sigma = id_X, \rho = id_Y$.

Strong twisted unions

Definition (GI-M)

A nondegenerate involutive extension (Z, r) is a *strong twisted union* of (X, r_X) and (Y, r_Y) if

- The assignment $\alpha \longrightarrow {}^{\alpha}\bullet$ extends to a left action of $G(Y, r_Y)$ on X, and the assignment $x \longrightarrow \bullet^{x}$ extends to a right action of $G(X, r_X)$ on Y;
- The actions satisfy

$$\textbf{stu}: \qquad {}^{\alpha^y}x = {}^{\alpha}x; \qquad \alpha^{{}^{\beta}x} = \alpha^x, \quad \text{for all} \quad x, y \in X, \alpha, \beta \in Y$$

We shall use notation $(Z, r) = (X, r_X)\natural(Y, r_Y)$

Lemma

Every extension (Z, r) of two disjoint trivial solutions (X, r_1) and (Y, r_2) is a strong twisted union. Furthermore, if Z is a nontrivial extension of (X, r_1) and (Y, r_2), then $mpl(Z, r) = 2$.

Theorem (GI-M)

1. *A strong twisted union of solutions* $(Z,r) = (X,r_X)\natural(Y,r_Y)$ *obeys YBE iff*
   - *The assignment* $\alpha \longrightarrow {}^{\alpha}\bullet$ *extends to a group homomorphism*
     $$\varphi : G(Y,r_Y) \longrightarrow Aut(X,r); \quad \textit{and}$$
   - *The assignment* $x \longrightarrow \bullet^{x}$ *extends to a group homomorphism*
     $$\psi : G(X,r_X) \longrightarrow Aut(Y,r).$$

   The pair of group homomorphisms is uniquely determined by r.
2. *Furthermore, there is a one-to-one correspondence between the sets* $Ext^{\natural}(X,Y)$ *and* $Hom(G(Y,r_Y),Aut(X,r_X)) \times Hom(G(X,r_X),Aut(Y,r_Y))$ *using (1).*

Main Theorem

[GI-M] Let (X, r) be a finite nondegenerate square-free symmetric set of order ≥ 2, $\mathcal{G}(X, r), \Gamma(X, r)$ in the usual notations. Let $\Gamma_1, \Gamma_2, \cdots, \Gamma_s$ be all connected components, and $X_i = \mathcal{V}(\Gamma_i), 1 \leq i \leq s$, be their sets of vertices. TFAEQ.

1. (X, r) is multipermutation solution with $mpl(X, r) = 2$.
2. $\mathcal{G}(X, r)$ is an abelian group of order ≥ 2.
3. $\mathcal{G}(X, r)$ is a nontrivial subgroup of the automorphism group $\mathrm{Aut}(X, r)$.
4. The set of nontrivial components is nonempty. Every nontrivial connected component Γ_i is a graph of first type, and $\forall i, j, 1 \leq i, j \leq s, \forall a, b \in X_j$ one has $\mathcal{L}_{a|X_i} = \mathcal{L}_{b|X_i}$.
5. (X, r) can be split into disjoint r-invariant subsets $X_i, 1 \leq i \leq s$, where each $(X_i, r_{|X_i})$ is a trivial solution and $X = X_1 \natural X_2 \natural \cdots \natural X_s$, in the sense that we can put parentheses and "apply" $\natural$ in any order. In particular,

$$X = X_i \natural \left(\bigcup_{1 \leq j \leq s, j \neq i} X_j \right)$$

The "increasing" sequence (X_m, r_m) with $mplX_m = m$

Theorem

There exists an infinite sequence of involutive nondegenerate square-free solutions

$$(X_0, r_0), (X_1, r_1), \cdots, (X_m, r_m), \cdots$$

such that
(i) for each m, $m = 0, 1, 2, \cdots, X_m \subset X_{m+1}$ is an r_{m+1}-invariant subset of X_{m+1}. Furthermore,
(ii) For each $m \geq 0$ (X_m, r_m) is a finite multipermutation solution with $mpl(X_m, r_m) = m$, and of order N_m, s.t.
$N_0 = 1, N_1 = 2, N_2 = 3$, and for $m \geq 2$, $N_{m+1} = 2N_m + 1$.
(iii) For each $m > 1$, the group $G = G(X_m, r_m)$ is a wreath product $G_m = (G_{m-1})_{wr}\mathbb{Z}_2$, and has solvable length $sl(G_m) = m$

Corollary. For every integer $m \geq 1$ there exist a multipermutation square-free symmetric set (X_m, r_m), with $mplX_m = m$, and $slG(X_m, r_m) = m$.

Lemma

Let $(X_1, r_1), (X_2, r_2)$ be multipermutation square-free solutions. Then every twisted union $(Z, r) = (X_1, r_1)\natural_0(X_2, r_1)$ is a multipermutation solution of level $mplZ = max\{mplX_1, mplX_2\}$

Lemma

Suppose $mpl(X, r) = m$, and there exists an automorphism $\tau \in Aut(X, r) \setminus \mathcal{G}(X, r)$, and (Y, r_0) is the trivial solution on the one element set $Y = \{a\}$, where a is not in X. Let $(Z, r_Z) = X\natural\{a\}$ be the strong twisted union defined via $\mathcal{L}_a = \tau$, $\mathcal{L}_{x|Y} = id_Y$, for all $x \in X$. Then $mpl(Z) = m + 1$.

Recursive construction of the sequence

Let $X = \{x_n \mid 1 \leq n\}$ be a countable set.

- (X_0, r_0) is the one element trivial solution with $X_0 = \{x_1\}$.
- (X_1, r_1) is the trivial solution on the set $X_1 = \{x_1, x_2\}$.
- We set $(X_2, r_2) = X_1 \natural \{x_3\}$, where $\mathcal{L}_{x_3} = (x_1 x_2)$. Clearly, $mplX_2 = 2$.
- Construction of (X_3, r_3). Let (X_2', r_2') be an isomorphic copy of (X_2, r_2), where $X_2' = \{x_4, x_5, x_6\}$, and the map $\tau : (X_2, r_2) \longrightarrow (X_2', r_2')$ with $\tau(x_i) = x_{i+3}, 1 \leq i \leq 3$ is an isomorphism of solutions. Let $(Y_2, r_{Y_2}) = X_2 \natural_0 X_2'$ be the trivial extension. We set $(X_3, r_3) = Y_2 \natural \{x_7\}$, where the map r_3 is defined via the left action $\mathcal{L}_{x_7} = (x_1 x_4)(x_2 x_5)(x_3 x_6)$. One has $\mathcal{L}x_7 \in Aut(Y_2, r_{Y_2} \setminus \mathcal{G}(Y_2, r_{Y_2})$, so $mplX_3$ =3.

The recursive construction

Assume we have constructed the sequence $(X_0, r_0), (X_1, r_1), \cdots, (X_m, r_m)$, satisfying conditions (i) and (ii). We shall construct effectively (X_{m+1}, r_{m+1}) so that (i) and (ii) are satisfied. Let $N = N_m$. $X'_m = \{x_{N+1}, \cdots x_{2N}, \}$ and let $(X'_m, r_{X'_m})$ be the solution isomorphic to (X_m, r_m) via the isomorphism $\tau : X_m \longrightarrow X'_m$ with $\tau(x_i) = x_{i+N}, 1 \leq i \leq N$. We denote by (Y_m, r_{Y_m}) the trivial extension $X_m \natural_0 X'_m$, and set $X_{m+1} = (Y_m, r_{Y_m}) \natural \{x_{2N+1}\}$ and r_{m+1} defined via the action $\mathcal{L}_{x_{m+1}} = (x_1 x_{N+1})(x_2 x_{N+2}) \cdots (x_N x_{2N})$. One can show that $mpl(X_{m+1}, r_{m+1}) = m + 1$.

Conjecture III.

Let (X, r_X) and (Y, r_Y), be square-free multipermutation solutions. Suppose the strong twisted union $(Z, r) = X\natural Y$ is a nondegenerate symmetric set. Then $mpl Z \leq max\{mpl X, mpl Y\} + 1$.

The following Theorem is true for arbitrary braided sets.

Theorem (GI-M)

Let (X, r_X), (Y, r_Y) be disjoint solutions of the YBE, with YB- groups $G_X = G(X, r_X)$, and $G_Y = G(Y, r_Y)$. Let (Z, r) be a regular YB-extension of (X, r_X), (Y, r_Y), with a YB-group $G_Z = G(Z, r)$. Then

- *G_X, G_Y is a matched pair of groups with actions induced from the braided group (G_Z, r).*
- *G_Z is isomorphic to the double crossed product $G_X \bowtie G_Y$.*

Proposition. (ESS)

If $Z \in Ext^+(X, Y)$, then $G_Z \simeq G_Y \triangleright < G_X$, where the semidirect product is formed using the action of G_Y on X via $\alpha \to \mathcal{L}_\alpha$.

$Ext^+(X, Y)$ is the set of all YB-extensions (Z, r) of the symmetric sets $(X, r_1), (Y, r_2)$ with $r(x, \alpha) = (\alpha, x^\alpha)$.

Open questions

Question 2.

Suppose (Z, r) is a YB strong twisted union of the finite square-free symmetric sets $(X, r_X), (Y, r_Y)$. How exactly are related the groups G_Z, G_X, G_Y?

Question 3.

Suppose (Z, r) is a finite nondegenerate multipermutation square-free symmetric set, $mplZ = m$. Is it true that (Z, r) can be always presented as a strong twisted union of invariant subsets of multipermutation level $< m$?