A Markov chain for certain triple systems

Peter J. Cameron
Queen Mary, University of London
and
Gonville & Caius College, Cambridge

INI Workshop on Markov Chain Monte Carlo methods
March 2008

Latin squares

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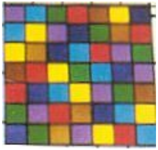
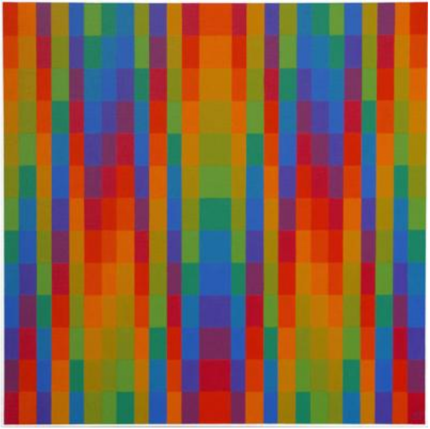


R. A. Fisher memorial window, Gonville & Caius College

Commercial break



Raymond Brownell
Great Expectations Gallery, London

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We don't know. There are upper and lower bounds of the form $(n/c)^{n^2}$ (with different values of c).

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The exact numbers are known for $n \leq 11$ (McKay and Wanless). The number of Latin squares of order 11 is

$$776\,966\,836\,171\,770\,144\,107\,444\,346\,734\,230\,682\,311\,065\,600\,000.$$

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2. If we rearrange the columns so that the first row is the identity permutation, then the second row is a derangement (a fixed-point-free permutation). Are all derangements (approximately) equally common?
3. Are the parities of the rows approximately independent (so that the number of rows which are odd permutations is approximately binomial $\mathrm{Bin}(n, \frac{1}{2})$)?

Transversals, orthogonal mates, intercalates

1	2	3	4
2	1	4	3
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- Transversal: set of cells, one for each row, column, symbol.

Transversals, orthogonal mates, intercalates

1*a*	2*b*	3*c*	4*d*
2*c*	1*d*	4*a*	3*b*
3*d*	4*c*	1*b*	2*a*
4*b*	3*a*	2*d*	1*c*

- Orthogonal mate: Latin square whose symbols are transversals to the original.

Transversals, orthogonal mates, intercalates

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- Intercalate: subsquare of order 2 (two rows, two columns, two symbols).

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Such a method was found by Jacobson and Matthews.

Improper Latin squares

The method is a random walk around an enlarged space containing both Latin squares and "improper" Latin squares. We need to know that not too infrequently we are at a proper Latin square, and that in the limit all proper Latin squares are equally likely to occur.

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Suppose we have a Latin square in which the entry in row i and column j is k, and we want to change this entry to k'. The entry k' aready occurs in row i (say in column j'), and in column j (say in row i'), and these entries must be changed to k. We need to remove k from position (i', j') and replace it with k'.

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This is OK if the entry there is k, otherwise we create a "fault".

Improper Latin squares

		j		j'	
i		k		k'	
i'		k'		k''	

$\longrightarrow$

		j		j'	
i		k'		k	
i'		k		$k' + k'' - k$	

The random walk

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From an improper Latin square, say with entry $k' + k'' - k$ somewhere, we must correct the fault by replacing k by either k' or k'' there. There are two rows through this cell containing k, and two columns also. So there are 8 moves.
The random walk is reversible and connected; so in the limiting distribution, any two Latin squares have the same probability (as do any two improper Latin squares).

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The most important open question about this method is how rapidly we converge to the uniform distribution (what is the mixing time).

Some exploration

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Häggkvist and Janssen showed that the probability that all rows have the same parity is exponentially small, but couldn't get the right constant. An interesting problem remains here!

First generalisation: Other kinds of triple system

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Triple systems

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Take a composition $(k_1, \ldots, k_m)$ of 3 (an ordered sum of positive integers summing to 3, that is, $(1, 1, 1)$, $(2, 1)$ or (3)). Take sets $X_1, \ldots, X_m$, one for each part of the composition. Our design will be a subset

$$\mathcal{B} \subseteq \binom{X_1}{k_1} \times \cdots \times \binom{X_m}{k_m}$$

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with the following property:

We require that, for any tuple of integers of the same length as the given composition and having sum 2, say $(t_1, \ldots)$, and any choice of subsets T_i of X_i with $|T_i| = t_i$, there is a unique $(K_1, \ldots, K_m) \in \mathcal{B}$ with $T_i \subseteq K_i$ for $i = 1, \ldots, m$.

The case $(1, 1, 1)$

We have three sets X_1, X_2, X_3, which we regard as indexing the rows, columns and symbols. Choose one element in any two of these sets; there should be a unique triple in $\mathcal{B}$ containing these elements.

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This is just a Latin square looked at a different way!

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We have a single set X_1, and $\mathcal{B}$ is a collection of 3-element subsets of X_1 with the property that any 2-element subset of X_1 is contained in a unique member of $\mathcal{B}$.

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In other words, a Steiner triple system!

The case $(2, 1)$

We have two sets X_1, X_2 which we think of as the set of vertices of a complete graph and the set of colours in an edge-colouring. The design $\mathcal{B}$ consists of pairs $(\{x_1, x_2\}, y)$ – a 2-subset of X_1 and an element of Y. We require:

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So we have a 1-factorisation of a complete graph K_n, i.e. a proper edge-colouring with the minimum number $n - 1$ of colours (so each colour is a 1-factor).

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But in the other two cases, we have no proof of connectedness.

Second generalisation: Larger values of λ

In the definition of triple systems, we can replace the condition that any pair is in a unique triple by the condition that any pair is in a constant number λ of triples. The resulting generalisation of Steiner triple systems consists of 2-designs, or "balanced incomplete-block designs" with block size 3; the generalisation of Latin squares consists of orthogonal arrays of strength 2 and degree 3. For 1-factorisations, there seems to be no previous account of these structures (which might be called λ-factorisations of the λ-fold complete multigraph).

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Further generalisations could be imagined.

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- Choose a random filled Sudoku grid (of arbitrary size).
- Choose a random BIBD with larger block size.