Glauber Dynamics for Ising Model on the Complete Graph

David A. Levin, *U. Oregon*
Malwina Luczak, *London School of Economics*
Yuval Peres, *Berkeley* and *Microsoft Research*

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Ising model

Let $G_n = (V_n, E_n)$ be a graph with $n = |V_n| < \infty$ vertices.

The nearest-neighbor *Ising model* on G_n is the probability distribution on $\{-1, 1\}^{V_n}$ given by

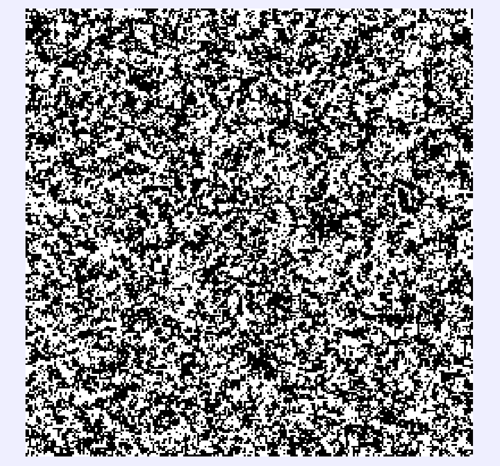
$$\mu(\sigma) = Z(\beta)^{-1} \exp\left(\beta \sum_{(u,v)\in E_n} \sigma(u)\sigma(v)\right),$$

where $\sigma \in \{-1, 1\}^{V_n}$.

The interaction strength β is a parameter which has physical interpretation as $1/T$, where T = temperature.

Three regimes

High temperature ($\beta < \beta_c$):



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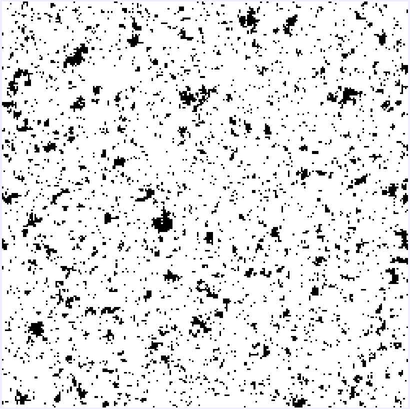

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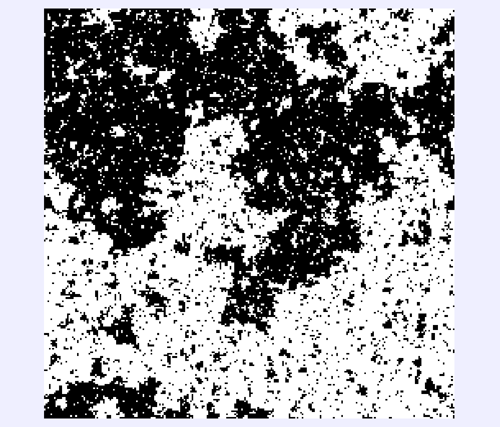

Glauber dynamics

The (single-site) *Glauber dynamics* for μ is a Markov chain (X_t) having μ as its stationary distribution.

Transitions are made from state σ as follows:

1. a vertex v is chosen uniformly at random from V_n.
2. The new state σ' agrees with σ everywhere except possibly at v, where $\sigma'(v) = 1$ with probability

$$\frac{e^{\beta S(\sigma,v)}}{e^{\beta S(\sigma,v)} + e^{-\beta S(\sigma,v)}}$$

where

$$S(\sigma, v) := \sum_{w:w\sim v} \sigma(w).$$

Note the probability above equals the μ-conditional probability of a positive spin at v, given that that all spins agree with σ at vertices different from v.

Note that the Glauber dynamics is reversible with respect to the Gibbs measure μ.

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Mixing time

For a Markov chain (X_t) on state-space Ω and with stationary distribution π, write

$$d(t) = \max_{x \in \Omega} \|\mathbb{P}\{X_t \in \cdot \mid X_0 = x\} - \pi\|_{\mathrm{TV}}$$

Define

$$t_{\mathrm{mix}}(\epsilon) := \min\{t \geq 0 \ : \ d(t) \leq \epsilon\}.$$

Consider now a *sequence* of Markov chains, (X_t^n), and write $d_n(t)$ and $t_{\mathrm{mix}}^n(\epsilon)$.
A sequence of Markov chains has a *cutoff* if

$$\frac{t_{\mathrm{mix}}^n(\epsilon)}{t_{\mathrm{mix}}^n(1-\epsilon)} \to 1 \quad \text{as} \quad n \to \infty.$$

Cutoff more precisely

The Glauber dynamics is said to exhibit a *cut-off* at $\{t_n\}$ with *window* $\{w_n\}$ if $w_n = o(t_n)$ and

$$\lim_{\gamma\to\infty} \liminf_{n\to\infty} d_n(t_n - \gamma w_n) = 1,$$

$$\lim_{\gamma\to\infty} \limsup_{n\to\infty} d_n(t_n + \gamma w_n) = 0.$$

A conjecture

For the Glauber dynamics on graph sequences with bounded degree, $t^n_{\text{mix}} = \Omega(n \log n)$.
(T. Hayes and A. Sinclair)

Conjecture (due to Y. Peres): If the Glauber dynamics for a sequence of transitive graphs satisfies $t^n_{\text{mix}} = O(n \log n)$, then there is a cut-off.

Mean field case

Take $G_n = K_n$, the complete graph on the n vertices: $V_n = \{1, \ldots, n\}$, and E_n contains all $\binom{n}{2}$ possible edges.

The total interaction strength should be $O(1)$, so replace β by β/n. The probability of updating to a +1 is then

$$\frac{e^{\beta(S-\sigma(v))/n}}{e^{\beta(S-\sigma(v))/n} + e^{-\beta(S-\sigma(v))/n}}$$

where S is the *total magnetization*

$$S = \sum_{i=1}^{n} \sigma(i).$$

The statistic S is almost sufficient for determining the updating probability.

Now, in this case $S_t = S(X_t)$ is a Markov chain in its own right; (S_t) will be key to analysis of the dynamics.

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Mean field has $t_{\text{mix}} = O(n \log n)$

A consequence (see, e.g., Aizenman and Holley (1987)) of the Dobrushin-Shlosman uniqueness criterion: For the Glauber dynamics on K_n, if $\beta < 1$, then

$$t_{\text{mix}} = O(n \log n).$$

(See also Bubley and Dyer (1998).)

High temperature mean-field

Theorem (L.-L.-P.)

Let (X_t^n) be the Glauber dynamics for the Ising model on K_n. If $\beta < 1$, then $t_{\mathrm{mix}}(\epsilon) = (1 + o(1))\frac{n \log n}{2(1-\beta)}$ and there is a cut-off.

In fact, we show that there is *window* of size $O(n)$ centered about

$$t_n = \frac{1}{2(1-\beta)} n \log n.$$

That is,

$$\limsup_n d_n(t_n + \gamma n) \to 0 \quad \text{as} \quad \gamma \to \infty.$$

and

$$\liminf_n d_n(t_n + \gamma n) \to 1 \quad \text{as} \quad \gamma \to -\infty.$$

Critical temperature mean-field

Theorem (L.-L.-P.)

Let (X_t^n) be the Glauber dynamics for the Ising model on K_n. If $\beta = 1$, then there are constants c_1 and c_2 so that

$$c_1 n^{3/2} \leq t_{\text{mix}} \leq c_2 n^{3/2}.$$

Low temperature mean-field

If $\beta > 1$, then

$$t_{\text{mix}}^{n} > c_1 e^{c_2 n}.$$

This can be established using Cheeger constant – there is a bottleneck going between states with positive magnetization and states with negative magnetization.

Our results show that once this barrier to mixing is removed, the mixing time is reduced to $n \log n$.

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Lower bound via Cheeger constant

Let

$$Q(A, A^c) = \sum_{x \in A, y \in A^c} \pi(x) P(x, y).$$

If

$$\Phi_A = \frac{Q(A, A^c)}{\pi(A)},$$

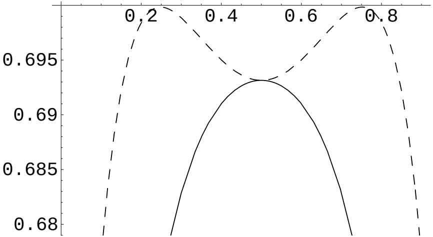
then for $\pi(A) \leq 1/2$,

$$t_{\text{mix}} \geq \frac{1}{4\Phi_A}.$$

Let A_k be those configurations with k spins equal to +1.Then

$$\mu(A_{\lfloor \alpha n \rfloor}) = \frac{1}{Z(\beta)} e^{-n[\phi(\alpha)+o(1)]}.$$

The function ϕ changes shape at $\beta = 1$:



The bottleneck

If Ω^+ are configurations with strictly positive magnetization,

$$\frac{Q(\Omega^+, (\Omega^+)^c)}{\pi(A_{\lfloor n/2 \rfloor})} \leq \frac{\exp n[\phi(1/2) + o(1)]}{\exp n[\phi(\alpha_0) + o(1)]}.$$

If $\beta > 1$, there is α_0 so that $\phi(\alpha_0) > \phi(1/2)$ and then

$$\phi_S \leq c_1 e^{-c_2 n}.$$

Truncated dynamics for low temperature mean-field

If the bottleneck at zero magnetization is removed by truncating the dynamics at zero magnetization, then the chain converges fast:

Theorem (L.-L.-P.)

Let $\beta > 1$. Let (X_t) be the Glauber dynamics on K_n, restricted to the set of configurations with non-negative magnetization. Then $t_{\text{mix}}^n = O(n \log n)$.

Proof idea for low temperature

Use coupling: Show that for arbitrary starting states, can run together two copies of the chain so that the chains meet with high probability in $O(n \log n)$ steps.

- First show that the magnetizations will agree after $O(n \log n)$ steps, when chains are run independently. (Hard part – involves hitting time calculations.)
- After magnetizations agree, couple the chains as below.

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- After magnetizations agree, couple the chains as below.

Write (X_t) and $(\tilde{X}_t)$ for the two chains. We assume that $S(X_t) = S(\tilde{X}_t)$.

Let J be the vertex selected for updating in X_t, and let $s \in \{-1, 1\}$ be the spin used to update $X_t(J)$.

The $\tilde{X}$-chain will also be updated with the spin s at a site $\tilde{J}$ which has $\tilde{X}_t(\tilde{J}) = X_t(J)$, although it will not always be that $J = \tilde{J}$.

If $X_t(J) = \tilde{X}_t(J)$, then update both chains at J.

If $X_t(J) \neq \tilde{X}_t(J)$, then pick $\tilde{J}$ uniformly at random from

$$\{i : \tilde{X}_t(i) \neq X_t(i) \text{ and } \tilde{X}_t(i) = X_t(J)\}.$$

A coupling (any temperature)

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A coupling, continued

If D_t is the number of sites where X_t and $\tilde{X}_t$ disagree, then when $D_t \geq 0$,

$$\mathbb{E}[D_{t+1} \mid \mathcal{F}_t] \leq \left[1 - \frac{c_1}{n}\right] D_t.$$

It takes $O(n \log n)$ steps to drive this expectation down to ϵ.

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Low temperature in more detail

How exactly do we restrict the dynamics? Whenever in a state with non-negative magnetization and a move is proposed to a state η with negative magnetization, we move to $-\eta$ instead.

Show there is a coupling $(X_t^+, \tilde{X}_t^+)$ of restricted dynamics started from $\sigma, \tilde{\sigma}$ such that

$$\limsup_{n\to\infty} \mathbb{P}_{\sigma,\tilde{\sigma}}(\tau_{\text{mag}} > cn\log n) \to 0 \quad \text{as } c \to \infty.$$

Here τ_{mag} is the first time t such that $S_t^+ = \tilde{S}_t^+$.

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Low temperature continued

We couple both these, S_t^B and S_t^T with an equilibrium copy, S_t, and we try to get them to "cross over".

For this, we need to consider hitting times:

$$\begin{aligned}\tau_1 &= \min\{t \geq 0 : S_t^T \leq s^* + c_1 n^{-1/2}\}\\ \tau_2 &= \min\{t \geq 0 : S_t^B \geq s^* + c_2 n^{-1/2}\},\end{aligned}$$

where s^* is the unique positive solution of $\tanh(\beta s) = s$.

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Birth-and-death chain hitting times

Let (Z_t) be a birth-and-death chain on $\{0, \dots, N\}$ with transition probabilities

$$\begin{aligned} p_k &= \mathbb{P}(Z_{t+1} - Z_t = +1 \mid Z_t = k), \quad k = 0, \dots, N-1, \\ q_k &= \mathbb{P}(Z_{t+1} - Z_t = -1 \mid Z_t = k), \quad k = 1, \dots, N, \\ r_k &= \mathbb{P}(Z_{t+1} - Z_t = 0 \mid Z_t = k), \quad k = 0, \dots, N. \end{aligned}$$

Let $\pi = (\pi(k))$ be the stationary measure.

Let $Z_t^{(\ell)}$ be a restriction of Z_t to $\{0, \dots, \ell\}$. Note that $\pi^{(\ell)}(k) \sim \pi(k)$ for $k = 0, \dots, \ell$.

Then for $\ell = 0, 1, \dots, N-1$,

$$\frac{1}{\pi^{(\ell)}(\ell)} = 1 + q_\ell \mathbb{E}_{\ell-1}(\tau_\ell).$$

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Hitting times for magnetization

Let $\ell^* = \lfloor ns^* \rfloor$. Then

$$\begin{aligned}
\mathbb{E}_{\ell-1}[\tau_\ell] &\leq \sqrt{n}(1 + O(n^{-1/2})), \quad 1 \leq \ell \leq C\sqrt{n} \\
\mathbb{E}_{\ell-1}[\tau_\ell] &\leq \frac{C_1 n}{\ell}, \quad C\sqrt{n} \leq \ell \leq \ell^*/2 \\
\mathbb{E}_{\ell-1}[\tau_\ell] &\leq C_2 n \ell^* - \ell, \quad \ell^*/2 \leq \ell \leq \ell^* - C\sqrt{n} \\
\mathbb{E}_{\ell-1}[\tau_\ell] &\leq \sqrt{n}(1 + O(n^{-1/2})), \quad \ell^* - C\sqrt{n} \leq \ell \leq \ell^* + C\sqrt{n}
\end{aligned}$$

Magnetization chain: key equation

If $S_t = \sum_{i=1}^n X_t(i)$, then for $S_t \geq 0$,

$$\mathbb{E}[S_{t+1} - S_t \mid \mathcal{F}_t] \approx -\left[\frac{S_t}{n} - \tanh(\beta S_t / n)\right].$$

$\beta < 1$

When $\beta < 1$, using the inequality $\tanh(x) \leq x$ for $x \geq 0$ shows that for $S_t \geq 0$,

$$\mathbb{E}[S_{t+1} \mid \mathcal{F}_t] \leq S_t \left(1 - \frac{1-\beta}{n}\right)$$

Need $[2(1-\beta)]^{-1} n \log n$ steps to drive $\mathbb{E}[S_t]$ to $\sqrt{n}$.

Additional $O(n)$ steps needed for magnetization to hit zero. (Compare with simple random walk.)

Can couple two versions of the chain so that the magnetizations agree by the time the magnetization of the top chain hits zero.

Once magnetizations agree, use a two-dimensional process to bound time until full configurations agree. Takes an additional $O(n)$ steps.

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When $\beta < 1$, using the inequality $\tanh(x) \leq x$ for $x \geq 0$ shows that for $S_t \geq 0$,

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Variance estimates: Markov chain lemma

Let (Z_t) be a Markov chain taking values in $\mathbb{R}$ and with transition matrix P. We will write $\mathbb{P}_z$ and $\mathbb{E}_z$ for its probability measure and expectation, respectively, when $Z_0 = z$. Suppose that there is some $0 < \rho < 1$ such that for all pairs of starting states $(z, \tilde{z})$,

$$|\mathbb{E}_z[Z_t] - \mathbb{E}_{\tilde{z}}[Z_t]| \leq \rho^t |z - \tilde{z}|.$$

Then $v_t := \sup_{z_0} \mathrm{var}_{z_0}(Z_t)$ satisfies

$$v_t \leq v_1 \min\{t, (1 - \rho^2)^{-1}\}.$$

This lemma enables us to show that $\mathrm{var}(S_t) = O(n)$ for $\beta < 1$ and $\mathrm{var}(S_t) = O(t)$ for $\beta = 1$.

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Conjecture

Consider the d-dimensional torus $(\mathbb{Z}/n\mathbb{Z})^d$. Let β_c be the critical temperature.

Conjectures:

1. For $\beta < \beta_c$, there is a cut-off.
2. For $\beta = \beta_c$, the mixing time is polynomial in n.
   Stronger: $t_{\text{mix}} = O(|V_n|^{3/2})$ for $d > d_c$.
3. For $\beta > \beta_c$, if the dynamics are truncated, the mixing time is polynomial in n.
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