Ising Model on K_n: Mixing time for Glauber dynamics at critical β

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Mean-field Ising model

Take $G_n = K_n$, the complete graph on the n vertices: $V_n = \{1, \dots, n\}$, and E_n contains all $\binom{n}{2}$ possible edges.

Ising measure on $\{-1, 1\}^{K_n}$ is

$$\pi(\sigma) = \frac{\exp\left(\frac{\beta}{n} \sum_{1 \le i \ne j \le n} \sigma(i)\sigma(j)\right)}{Z(\beta)}.$$

Glauber dynamics: Pick $v \in \{1, 2, \dots, n\}$ uniformly at random, update $\sigma(v)$.

The probability of updating to a $+1$ is then

$$\frac{e^{\beta(S-\sigma(v)/n)}}{e^{\beta(S-\sigma(v)/n)} + e^{-\beta(S-\sigma(v)/n)}}$$

where S is the *average magnetization*

$$S = \frac{1}{n} \sum_{i=1}^{n} \sigma(i).$$

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Critical temperature mean-field

Theorem (D.A.L., Luczak, Peres '07)

Let (X_t^n) be the Glauber dynamics for the Ising model on K_n. If $\beta = 1$, then there are constants c_1 and c_2 so that

$$c_1 n^{3/2} \leq t_{\text{mix}} \leq c_2 n^{3/2}.$$

Contrast with (Cf. M. Luczak's talk)

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Proof outline

1. First show that can couple together two copies of the magnetization chain (S_t) so that they agree in $O(n^{3/2})$ steps.
   - Show that one copy of the chain (S_t) hits zero in order $O(n^{3/2})$ steps.
   - Use the *reflection coupling* to show that (S_t) can be coupled to a second $(\tilde{S}_t)$ so that $S_t = \tilde{S}_t$ by the time (S_t) hits zero.
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$\beta = 1$. Why $n^{3/2}$?

The time τ_0 for one copy of (S_t) to hit zero is $O(n^{3/2})$.

To prove, two phases: First drive expectation $s_t = E(S_t)$ down, then allow for the fluctuations of (S_t) to finish the job.

Phase 1: (S_t) has drift

$$\mathbb{E}[S_{t+1} - S_t \mid \mathcal{F}_t] \approx -\left[\frac{S_t}{n} - \tanh(\beta S_t/n)\right].$$

Here $\beta = 1$. Expanding $\tanh(x) = x - x^3/3 + \cdots$ in the key equation yields

$$\mathbb{E}[S_{t+1} - S_t \mid \mathcal{F}_t] \approx -\frac{1}{3}\left(\frac{S_t}{n}\right)^3.$$

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Thus,

$$s_{t+1} - s_t \approx -\frac{s_t^3}{3n}.$$

Let $b_k = 2^{-k}$. If $s_t \le b_k$, then the number of additional steps u needed for $s_{t+u} \le b_{k+1}$ is not more than

$$\frac{3n(b_{k+1} - b_k)}{b_k^3} = \frac{c_1 n}{b_k^2}.$$

The total number of steps needed to get below b_{k_0} is then

$$\sum_{k=1}^{k_0} \frac{c_1 n}{b_k^2} = \frac{c_2 n}{b_{k_0}^2}.$$

Taking $b_{k_0} = n^{\alpha-1}$, we need

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Phase 2: fluctuations

The fluctuations of nS_t are similar to those of simple unbiased random walk.

Need additional $n^{2\alpha}$ steps to hit zero.

Total time to hit zero (with high probability) is

$$O(n^{3-2\alpha}) + O(n^{2\alpha})$$

The choice $\alpha = 3/4$ makes the powers equal, and gives hitting time $O(n^{3/2})$.

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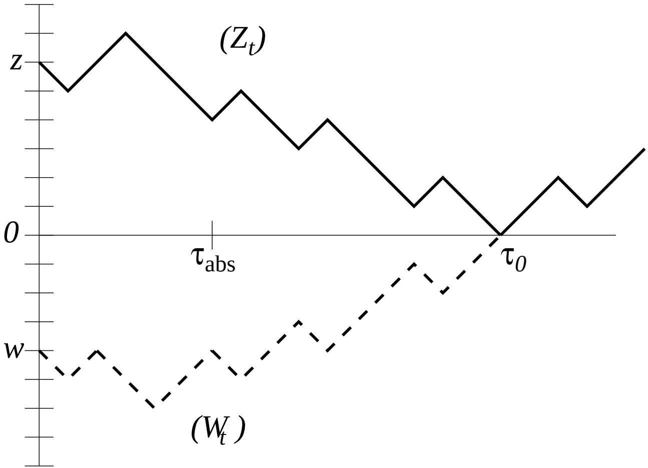
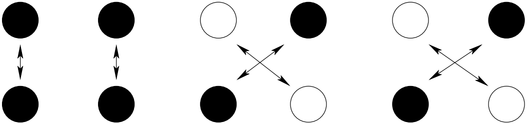
Reflection coupling



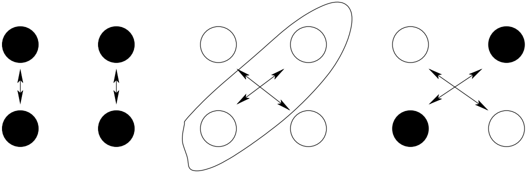
Wait until absolute values meet (they could jump over each other, but that happens with fixed probability), then reflect.

Coupling from same magnetization

Match vertices so that agreements are matched together. Update matched vertices together.



When different vertices are updated together to a different spin, the number of disagreements decreases by two.



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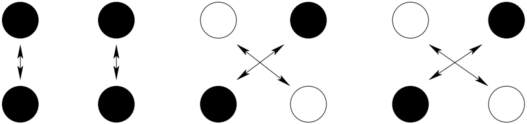

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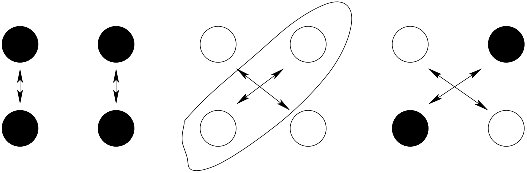

$$E[D_{t+1} - D_t \mid \mathcal{F}_t] \leq \frac{-c}{n} D_t,$$

and so

$$E[D_t] \leq \left(1 - \frac{c}{n}\right)^t \leq e^{-ct/n}.$$

Taking $t = \gamma n \log n$ for γ large makes this small.

Lower bound

Under stationary measure,

$$n^{1/4} S_n \quad \text{converges weakly to non-trivial distribution.}$$

So,

$$P(S_n > A n^{-1/4}) \to 0 \quad \text{as } A \to \infty.$$

(Simon and Griffiths 1973)

If $t = \gamma n^{3/2}$, then $E(S_t) \approx c(\gamma) n^{1/4}$, where $c(\gamma) \to \infty$ as $\gamma \to 0$.

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Conjecture

Consider the d-dimensional torus $(\mathbb{Z}/n\mathbb{Z})^d$. Let β_c be the critical temperature.

For $\beta = \beta_c$, the mixing time is polynomial in n.
Stronger: $t_{\text{mix}} = O(|V_n|^{3/2})$ for $d > d_c$.