Extremality of Gibbs Measure for Colorings on Trees

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- Broadcasting on Trees, Extremality and Reconstruction.
- Reconstruction non-solvability for colorings when $k > \frac{2\Delta}{\ln \Delta}$.

Broadcast Process on a Tree

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$$P(i,j) = \mathrm{P}[\sigma_u = j | \sigma_v = i]$$

  - Uniform distribution on proper colorings.

$$P(i,j) = \frac{\delta_{i \neq j}}{k-1}, \quad \pi \text{ uniform on } [k]$$

- Broadcast process gives the *free Gibbs measure* on colorings.
- For any finite $U \subseteq T$, coloring η of $T \setminus U$.
  Let $\sigma \in [k]^{V(U)}$ be such that $\sigma \cup \eta$ is proper,

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Gibbs Measures, Uniqueness and Extremality

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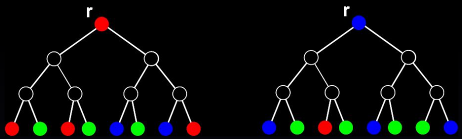
- For which k is μ extremal?
  (Not a convex combination of other Gibbs measures.)

Extremality and Reconstruction

[Georgii '88]
Extremality of μ is equivalent to decay of correlation between the root and the leaves in the limit.

- $\mu^i = \mu(\cdot|\sigma_r = i)$
- σ_ℓ the coloring at level ℓ.



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Equivalently,

$$\forall i \in [k] \quad \lim_{\ell\to\infty} \mathrm{E}[|\mu(\sigma_r = i|\sigma_\ell) - \mu(\sigma_r = i)|] \to 0$$

Motivation

- [Mossel '04], [Daskalakis-Mossel-Roch '06]
  Evolution of DNA, phylogenetic reconstruction.
- [Von Neumann '56], [Pippenger '88], [Evans-Schulman '93]
  Information theory, noisy computation.
- [Montanari-Gerschenfeld '07]
  Random CSP's. Sufficient conditions for equivalence of reconstruction on random graphs and the tree reconstruction. Equivalence for Potts model when $\beta < \infty$.

- [Hayes-Vera-Vigoda '07]
  $O^*(n^4)$ mixing time of Glauber dynamics for planar graphs when $k > C\Delta/\ln\Delta$.
- [Martinelli-Sinclair-Weitz '03] Exponential decay of corellation on the tree implies $O(n \ln n)$ mixing time for Glauber dynamics when uniqueness holds.
- [Berger-Kenyon-Mossel-Peres '05]
  $O(n)$ relaxation time for Glauber dynamics implies extremality.

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  Extremality for $k > \frac{(2+\varepsilon)\Delta}{\ln \Delta}$ for $\Delta \geq \Delta_\varepsilon$.

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- [Sly '08]
  Non-reconstruction for $\Delta \leq k(\ln k + \ln\ln k + 1 + o(1))$.
  Reconstruction for $\Delta \geq k(\ln k + \ln\ln k + 1 - \ln 2 - o(1))$.

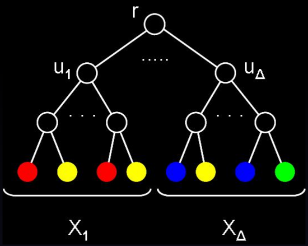
Theorem (B-Vera-Vigoda)

Let $\varepsilon > 0$, and $\Delta > \Delta_\varepsilon$. If $k > \frac{(2+20\varepsilon)\Delta}{\ln \Delta}$, there is a constant $\alpha = \alpha_\varepsilon$ s.t. $\forall c \in [k]$

$$\mathrm{P}[|\mu(\sigma_r = c|\sigma_\ell) - \mu(\sigma_r = c)| \ > \ \Delta^{-\alpha\ell}] < \Delta^{-2^{\alpha\ell}}$$

Translation Invariance

- $T_v \subseteq T$, tree rooted at v.
- $\mu_v(\cdot)$, the free Gibbs measure on T_v.
- $\mu(\cdot) = \mu_{u_i}(\cdot)$.



- If $X = (X_1, \cdots, X_\Delta)$ where $X \sim \sigma_\ell$, then $X_i \sim \sigma_{\ell-1}$.

$X = (X_1, \cdots, X_\Delta)$ coloring of the vertices at level ℓ.

$$\mu(\sigma_r = c|X) = \frac{\prod_i (1 - \mu(\sigma_{u_i} = c|X_i))}{\sum_{d \in [k]} \prod_i (1 - \mu(\sigma_{u_i} = d|X_i))}$$

Recall, non-reconstruction

$$\forall c \in [k], \quad \lim_{\ell \to \infty} \mathrm{E}[|\mu(\sigma_r = c|\sigma_\ell) - \frac{1}{k}|] \to 0$$

We show that whp for $X \sim \sigma_\ell$, the recurrence converges.

- With probability $1 - \Delta^{-2^\ell}$, a coloring X is “good”.

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- Changing one leaf in a (random) coloring X, changes $\mu(\sigma_r = c|X)$ by at most

$$\left(\frac{1}{\Delta^{\frac{1}{2}+\frac{\varepsilon}{4}}}\right)^{\ell}.$$

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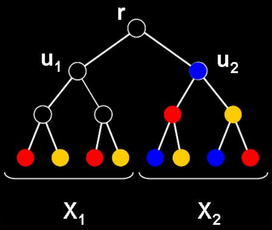
- Trade off against Δ^ℓ possible differences. Concentration of $\mu(\sigma_r = c|X)$ around its expectation $\frac{1}{k}$.

A leaf coloring X is good if

$$\forall c, \quad \mu(\sigma_r = c|X) \leq \varepsilon.$$



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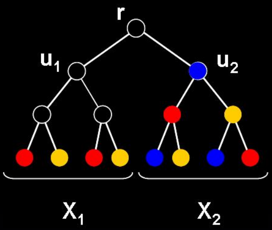

Lemma

Let $k > \frac{(2+20\varepsilon)\Delta}{\ln \Delta}$ *and* $\Delta > \Delta_\varepsilon$. *If at most 2* X_i *are bad, then*

$$\forall c, \quad \mu(\sigma_r = c|X) \le \Delta^{-(\frac{1}{2}+\varepsilon)} \le \varepsilon.$$

A Recursive condition for X to be Good.

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- Suppose 2 bad X_i.
  Each u_i has $\leq \frac{1}{\varepsilon}$ bad colors since $\sum_d \mu(u_i = d|X_i) = 1$.

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$$\forall i, \quad \mu(\sigma_{u_i} = d | X_i) \leq \varepsilon.$$

- $|S| \geq k - \frac{2}{\varepsilon} \geq (1-\varepsilon)k$. (Assuming $\Delta > \Delta_\varepsilon$.)

$$\mu(\sigma_r = c|X) = \frac{\prod_i (1 - \mu(\sigma_{u_i} = c|X_i))}{\sum_{d \in [k]} \prod_i (1 - \mu(\sigma_{u_i} = d|X_i))}$$

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$$\leq \frac{1}{\sum_{d\in S} \prod_i (1 - \mu(\sigma_{u_i} = d|X_i))} \approx \frac{1}{k\exp(\frac{\Delta}{k})} \leq \frac{1}{\Delta^{\frac{1}{2}+\varepsilon}}$$

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$$\leq \frac{1}{(1-\varepsilon)k} \exp\left(\frac{1+\varepsilon}{(1-\varepsilon)k} \sum_i \sum_{d\in S} \mu(u_i = d|X_i) \right)$$

...Proof

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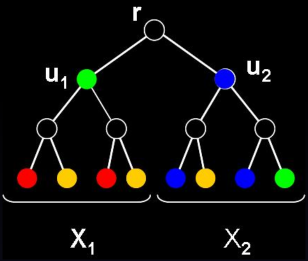
Corollary

If at most 2 X_i are bad, X is good.

- $\mathcal{G}_i$: event that X_i is good. The $\mathcal{G}_i$ are independent.
- Condition on a proper coloring of u_i.

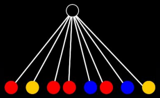


$$
\begin{aligned}
&\mu(\mathcal{G}_1 | u_1, u_2, \mathcal{G}_2) \\
&= \frac{N_\ell(\mathcal{G}_1, u_1, u_2, \mathcal{G}_2)}{N_\ell(u_1, u_2, \mathcal{G}_2)} \\
&= \frac{N_{\ell-1}(\mathcal{G}_1, u_1) N_{\ell-1}(\mathcal{G}_2, u_2)}{N_{\ell-1}(u_1) N_{\ell-1}(\mathcal{G}_2, u_2)} \\
&= \mu(\mathcal{G}_1 | u_1) \\
&= \mu(\mathcal{G}_1)
\end{aligned}
$$

Sum over ways to condition on the colors of u_i.

Colorings are Good whp

- Inductively, $\mathrm{P}[\mathcal{G}_i] \geq 1 - \Delta^{-2^{\ell-1}}$.
- X is good w.p. $\geq 1 - \Delta^{-2^{\ell}}$.
- Base case is the star. X is good w.p. $\geq 1 - e^{-\Delta^{\varepsilon/4}}$



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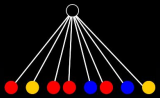

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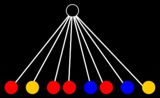

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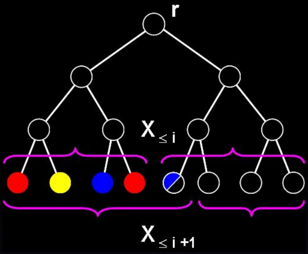
Taking Δ s.t. $(1-\varepsilon)ke^{-\Delta/k} \geq \frac{1}{\varepsilon}$, X is good by definition.

Concentration

$$\mathrm{P}[|\mu(\sigma_r = c|\sigma_\ell) - \mu(\sigma_r = c)| \ > \ \Delta^{-\varepsilon\ell/5}] < \Delta^{-2^{\varepsilon\ell/5}}$$



- Let $X = (X_1, \cdots, X_{\Delta^\ell})$ be a random coloring.

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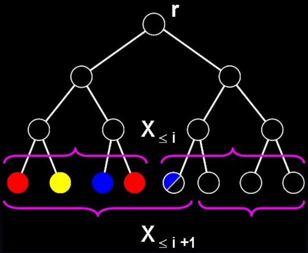

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- $X_{\leq i}$ agrees with X on $X_1, \cdots, X_i$.

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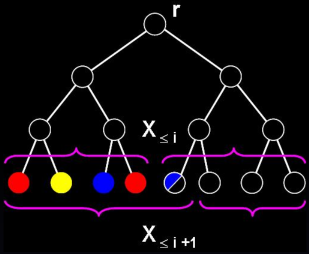

- Let $X = (X_1, \cdots, X_{\Delta^\ell})$ be a random coloring.
- $X_{\leq i}$ agrees with X on $X_1, \cdots, X_i$.
- $Y_i = \mu(\sigma_r = c|X_{\leq i}) = \mathrm{E}[1_{\sigma_r = c}|X_1, \cdots, X_i]$.

Theorem (Chung-Lu)

Let Y be a martingale and let $\mathcal{B}$ be the event that $\exists i$ s.t. $|Y_i - Y_{i+1}| > d_i$. Then,

$$\mathrm{P}[|Y - \mathrm{E}[Y]| \geq \delta] \leq 2\exp\left(-\frac{\delta^2}{2\sum_i (d_i)^2}\right) + \mathrm{P}(\mathcal{B})$$

Concentration

Lemma

If X is an "extra good" coloring, so that for v above level $\varepsilon\ell$

$$\mu(\sigma_v = c|X) \leq \frac{1}{\Delta^{\frac{1}{2}+\varepsilon}}, \qquad \text{then}$$

$$|Y_i - Y_{i+1}| \leq (C\Delta^{-(\frac{1}{2}+\frac{\varepsilon}{4})})^{\ell}.$$

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By Azuma, $\mathbf{P}[|\mu(\sigma_r = c|\sigma_\ell) - \frac{1}{k}| \geq \delta^\ell]$

$$< \quad 2\exp\left(\frac{-\delta^{2\ell}}{2\Delta^\ell\left(C\Delta^{-((\frac{1}{2}+\frac{\varepsilon}{4})\ell)^2}\right)}\right) + \Delta^{-2^{\varepsilon\ell/2}}$$

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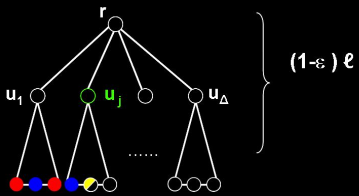
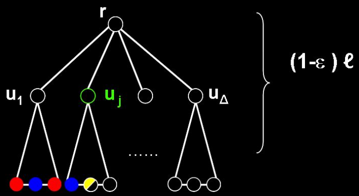


Let $X_{\leq i} = W$ and $X_{\leq i+1} = W'$.

$$Y_i = \mu(\sigma_r = c|W),$$

$$Y_{i+1} = \mu(\sigma_r = c|W').$$



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Bounding the Taylor series,

$$|Y_i - Y_{i+1}| \leq [C \cdot \mu(\sigma_r = c|W)]|\mu(\sigma_{u_j} = c|W_j) - \mu(\sigma_{u_j} = c|W')|$$

Concentration

- W is extra good $\Rightarrow \mu(\sigma_r = c|W) \le \frac{1}{\Delta^{\frac{1}{2}+\varepsilon}}$.

Concentration

- W is extra good $\Rightarrow \mu(\sigma_r = c|W) \leq \frac{1}{\Delta^{\frac{1}{2}+\varepsilon}}$.
- By induction,

  $$|\mu(\sigma_{u_j} = c|W_j) - \mu(\sigma_{u_j} = c|W'_j)| \leq \left(C\frac{1}{\Delta^{\frac{1}{2}+\varepsilon}}\right)^{h(u_j)-\varepsilon\ell}.$$

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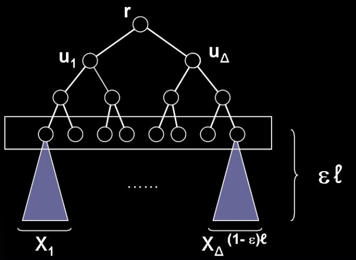
$$|Y_i - Y_{i+1}| \leq \left(C\frac{1}{\Delta^{\frac{1}{2}+\varepsilon}}\right)^{(1-\varepsilon)\ell} \leq \left(C\frac{1}{\Delta^{\frac{1}{2}+\frac{\varepsilon}{4}}}\right)^{\ell} \quad \square$$

Conclusions

- [Sly '08]
  Closing the gap between reconstruction and non-solvability.
- [Martinelli-Sinclair-Weitz '03]
  In uniqueness region, decay of corellations implies rapid mixing on the tree. Rapid mixing below uniqueness for the free boundary?
- [Montanari-Gerschenfeld '07] Reconstruction on $G\left(n, \frac{\Delta}{n}\right)$ and tree reconstruction?

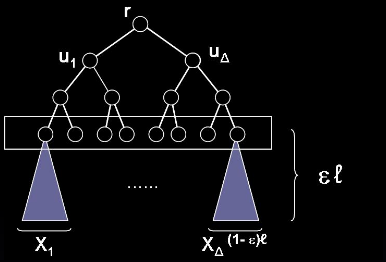
X is extra good if each $X_1, \cdots, X_{\Delta^{(1-\varepsilon)\ell}}$ is good.



$$\mathrm{P}[\text{Some } X_1, \cdots, X_{\Delta^{(1-\varepsilon)\ell}} \text{ is bad}] < \Delta^{(1-\varepsilon)\ell} \exp\left(\Delta^{-2^{\varepsilon\ell}}\right) < \Delta^{-2^{\varepsilon\ell/2}}.$$

Extra Good Colorings

X is extra good if each $X_1, \cdots, X_{\Delta^{(1-\varepsilon)\ell}}$ is good.



$$\mathrm{P}[\text{Some } X_1, \cdots, X_{\Delta^{(1-\varepsilon)\ell}} \text{ is bad}] < \Delta^{(1-\varepsilon)\ell} \exp\left(\Delta^{-2^{\varepsilon\ell}}\right) < \Delta^{-2^{\varepsilon\ell/2}}.$$

- W.p. $1 - 2^{\varepsilon\ell/2}$, X is extra good.
- By the Lemma, $\forall v$ above level $\varepsilon\ell$, $\mu(\sigma_v = c|X_v) \leq \frac{1}{\Delta^{\frac{1}{2}+\varepsilon}}$

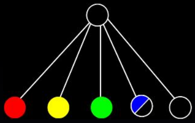
If X is good, so are each of the colorings $X_{\leq i}$.

Proof.

By induction. In the base case, freeing can only make X good.

$|X| =$ number of distinct colors.

$$\mathrm{P}[c] = 0 \text{ or } \frac{1}{k-|X|}$$

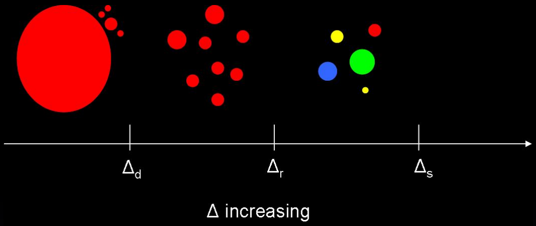


Corollary

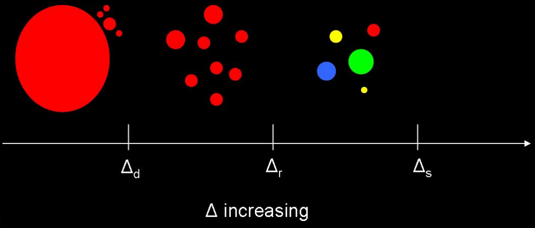
X is extra good $\Rightarrow \forall i,\ X_{\leq i}$ is extra good.

Clustering of the Solution Space for Random CSP's.

Proper colorings of $G(n, \frac{\Delta}{n})$.



Proper colorings of $G(n, \frac{\Delta}{n})$.



Conjectured

- $\Delta_d = k(\ln k + \ln\ln k + \alpha + o(1)),\ \alpha \in [1 - \ln 2, 1]$.
- Δ_d is the same as the reconstruction threshold for trees.
- [Montanari-Gerschenfeld '07] give sufficient conditions, and show equivalence for Potts model when $\beta > 0$.