Graphs classes with given 3-connected components

Marc Noy
Joint work with Omer Giménez and Juanjo Rué

Universitat Politècnica de Catalunya
Barcelona

Number of planar graphs with n vertices

$$ct \cdot n^{-7/2} \gamma^n n! \qquad \gamma \approx 27.23$$

Number of series-parallel graphs

$$ct \cdot n^{-5/2} \gamma^n n! \qquad \gamma \approx 9.07$$

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Efficient algorithm for generating large (10.000 vertices) uniform planar graphs (Fusy)

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$-5/2$	Tree-like
$-7/2$	Map-like

Different structural properties

A general framework

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3. $\mathcal{T} = \emptyset \Rightarrow \mathcal{G}$ = series-parallel
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5. $\mathcal{T}$ = planar triangulations

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$$G(x,y) = \exp(C(x,y))$$

For fixed x, dominant singularity of $T(x,z)$ at $z = r(x)$
$\alpha =$ singular type

$$T(x,z) \sim f(x)\left(1 - \frac{z}{r(x)}\right)^{\alpha}$$

$\alpha = 1/2,\ 3/2$ main singular types

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Theorem

(1) *If $T_z(x,y)$ analytic or singular type $\alpha < 1$ at (x_0, D_0)*
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R, ρ computable positive constants and $\rho < R$

A sample of graph classes

	γ	Exponent	Ref.
Forests $=\mathrm{Ex}(K_3)$	$e \approx 2.72$	$-5/2$	Standard
Outerplanar $=\mathrm{Ex}(K_4, K_{2,3})$	7.320	$-5/2$	BGKN
Series parallel $=\mathrm{Ex}(K_4)$	9.07	$-5/2$	BKGN
$\mathrm{Ex}(W_4)$	10.24	$-5/2$	
$\mathrm{Ex}(K_5 - e)$	12.96	$-5/2$	
$\mathrm{Ex}(K_2 \times K_3)$	14.13	$-5/2$	
Planar	27.226	$-7/2$	GN
Embeddable in a fixed surface	27.226	??	McDiarmid
$\mathrm{Ex}(K_{3,3})$	27.2293	$-7/2$	GGNW
$\mathrm{Ex}(K_{3,3}^{+})$	27.2295	$-7/2$	GGNW
$\mathrm{Ex}(K_5)$	??	??	

GN = Giménez, N

BGKN = Bodirsky, Giménez, N, Kung

GGNW = Gerke, Giménez, N, Weissl

Edge density

Problem: How to estimate $g_{n,\lfloor \mu n \rfloor}$

Fix $y > 0$ suitably depending on μ

$$G(x,y) = \sum g_{n,k} y^k \frac{x^n}{n!}$$

Graphs get weight $y^{\#\text{edges}}$

Only graphs with $\mu n + O(\sqrt{n})$ get non-negligible weight

Critical phenomena

Planar graphs: for all $1 < \mu < 3$

$$g_{n,\lfloor \mu n \rfloor} \sim \frac{c(\mu)}{\sqrt{n}} n^{-7/2} \gamma(\mu)^n n!$$

SP graphs: always $-5/2$ for $1 < \mu < 2$

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For $1 < \mu < \mu_0$ exponent is $-5/2$ $\quad\quad$ $\mu_0 \approx 1.87$
For $\mu_0 < \mu < 3$ exponent is $-7/2$

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Dominant singularity comes from $T(x, z)$ or
from a branch point when solving

$$\log\left(\frac{1 + D(x,y)}{1+y}\right) = \frac{xD(x,y)^2}{1 + xD(x,y)} + \frac{2}{x^2} T_z(x, D(x,y))$$

Random graphs

For all classes under study

1. Number of edges is asympt. normal with linear mean and variance
2. Number of components is asympt. Poisson
3. Largest conn. component is $n - O(1)$

Difference is in size of largest 2- and 3-connected components

Banderier, Flajolet, Schaeffer, Soria (2001)
Random maps, . . . and Airy phenomena

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Critical composition schemes in maps

$$M(z) = C(uH(z)), \qquad H(z) = z(1 + M(z))^2$$

u marks size of 2-connected core

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to planar graphs and

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For series-parallel graphs size of largest 2-connected component is $O(1)$ whp