Mayer expansion of repulsive polymer models

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\[-1 \leq F(p, p') \leq 0\]

PLAN OF THE LECTURE:

1. Mayer expansion of

\[
Z = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{\pi_1 \cdots \pi_N} \prod_{a=1}^{N} w_{\pi_a} \sum_G F_G
\]

Th. \( Z = \exp \left( \sum_{\varepsilon} F_{\varepsilon} n_{\varepsilon} \right) \)

\( \varepsilon = (\pi_1, \ldots, \pi_M) \quad n_{\varepsilon} = \frac{1}{N!} \prod_{a=1}^{N} w_{\pi_a} \)

\( F_{\varepsilon} = \sum_G F_G \)

\( F_G \) connected on \((\pi_1, \ldots, \pi_M)\)

2. K.P. formula for

\[
\alpha_p = \sum_{\varepsilon} F_{\varepsilon} n_{\varepsilon} \quad F_{\varepsilon} = \prod (1 + F(p, p')) - 1
\]

Th. \( \alpha_p = \sum_{p'} F(p, p') w_{p'} \int e^{\alpha t} dt \)
Resummation of $\sum F_{\gamma}$

*soft repulsion case*

$$F_{\gamma} = \sum_{G \text{ connected}} F_G = \sum_{\text{left tree on } (1, \ldots, N)} F_G$$

Resummation

*hard repulsion by intersection*

$$\sum_{\text{clusters}} F_{\gamma} w_{\gamma} = \sum F_{\gamma} w_{\gamma} \text{ 'cactuses' } \quad [\text{all } \gamma \text{ the same sign}]$$

[Drawings of clusters and 'cactuses']

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**NOTATION**

$$z = \sum_{\{\pi_i\}} \frac{1}{N!} \prod_{i=1}^{N} w_{\pi_i}$$

$$= \sum_{N=0}^{8} \frac{1}{N!} \sum_{(r_1, \ldots, r_N)} \prod_{i=j}^{N} (1 + F(r_i, r_j))$$

$$= \sum_{N=0}^{8} \frac{1}{N!} \sum_{(r_1, \ldots, r_N)} \sum_{G \text{ graph on } (1, \ldots, N)} F_{G}$$

$G = (r_1, \ldots, r_N) \quad w_{G} = \frac{1}{N!} \prod_{i=1}^{N} w_{\pi_i}$

$F_{G} = \prod_{(i,j) \in G} F(r_i, r_j)$
Using \[ x^{t_1 \cdots t_k} = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{y_1, \ldots, y_N} (t_1, \ldots, t_k) \]
we get

**Theorem (Hayer)**

\[ Z = \exp \left( \sum_{x} F_x \omega_x \right) \quad F_x = \frac{\sum F_G}{G} \]

\[ \xi = (\xi_1, \ldots, \xi_N) \quad N \geq 1 \quad \text{connected} \]

Example: \[ 1 + x = \exp \left( \sum_{M=1}^{\infty} \frac{1}{M!} \sum (-1)^{M-1} x^M \right) \]

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Kotecky-Preiss-Cannarota-Seiler-Dobrushin---

Put \[ a_n = \sum_{\text{all } \xi} \frac{1}{n!} \sum_{G \text{ connected}} \]

\[ F_{n, \xi} = \prod_i \left( 1 + F_{\xi_i, \xi_i} \right) - 1 \]

= \(-1\) in the hard repulsion case

Standard trick: auxiliary parameter

\[ 0 \leq t \leq 1 \]

**Def:** Consider the "\((t, \Gamma)\) relaxed" model: replace \( F(\Gamma, \Gamma') \) by softer model: replace \( F(\Gamma, \Gamma') \) by softer

\[ t F(\Gamma, \Gamma') \]

Remind \[ 0 \leq 1 + t F(\Gamma, \Gamma') \leq 1 \]

\[ t = 0 \quad \text{indepencency of } \Gamma \]

\[ t = 1 \quad \text{original model} \]
Consider also the "\([t,F]\) relaxed" model where all \(w_p\) are replaced by smaller quantities \((1+tF(p,p'))w_p\) (including \(p'=p\)).

**Theorem**

\[
\frac{\partial}{\partial t} a_{p'}(t,F) = \sum_{p'} F(p,p') a_{p'}(t,F)
\]

**Note.** If

\[
b_p > \sum |w_{p'}| |F(p,p')| L_{p'}
\]

then iterative method of solution converges.

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**Resummation - Soft Repulsion**

Left trees' instead of Penrose trees

**Definition.**

Left tree on an ordered set \((1,2,\ldots,m)\) \(\equiv M\) is defined recursively by decomposing \(M\) into a singleton \(\{1\}\) (the 'chief' of the tree) and a collection of disjoint subtrees

\[
L(m) = \sum \prod L(m_i)
\]

\(!\) same for cycles:

\[
c(m) = \sum \prod c(m_i)
\]
HENCE the number of left trees on $M$, $|M| = m$, is $(m-1)!$
and
$$\log (1-x) = -\sum_{m=1}^{\infty} \frac{(m-1)!}{m!} x^m$$

Put
$$F_{i,j} = \prod_{j \in J} (1 + F(i, j)) - 1$$

for a left tree $T = \{18, \{T_i \} \}$

$$(1-1)^e - 1 = -1$$

**THEOREM**

$$\sum_{G} F_G = \sum_{T} F_T$$

where for any left tree we define

$$F(T) = \prod_{\text{subchief of } T} F(\text{subordinates to } u)$$

$$F_{i,u} = \prod_{v \in U} (1 + F(1, v)) - 1$$
Hence

\[ Z = \exp \left( \sum F_\mathcal{E} \mu_\mathcal{E} \right) \]

\[ \mathcal{E} = (\mathcal{E}_1, \ldots, \mathcal{E}_m) \]

\[ F_\mathcal{E} = \sum_{T \text{ left trees on } (\mathcal{E}_1, \ldots, \mathcal{E}_m)} F(T) \]

\[ \text{same sign} \]

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**HARD REPULSION, CASE**

\[ F(\mathcal{P}, \mathcal{P}') = -1 \quad \text{IFF} \quad \text{supp } \mathcal{P} \cap \text{supp } \mathcal{P}' \neq \emptyset \]

\[ = 0 \]

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Forget left trees.

**SUBSTITUTION OF SPECIES:**

**QUILTED CLUSTERS, bonds of G**

replaced by **BIPARTITE GRAPHS**

\[ (1-1)^4 - 1 = -1 \]

**Auxiliary ordering** \( \mathcal{L} \) **OF THE**

**'LATTICE'** \( \Lambda \) **WHERE POLYMERS LIVE**

\[ \pm \text{first point of } \Lambda \]
PUT BRACKETS INTO THE SUM

\[ \sum_{Q} F_{Q} w_{Q} \]

of quilted clusters \( C \)

ERASE red bonds \( t \leftarrow t \)

\[ \sum_{\{Q_i\}} \prod \left( \sum F_{Q_i} \right)^{-1} \]

No of bonds \( t \leftarrow t \)

TERMS corresponding to components like

\[ t \]

Can be erased

bond \( t \leftarrow t \)

Here is optional does not influence the bracketing

What remains are quilted clusters of the following type
From connected graphs on $(1, 2, \ldots, k)$ to cycles.

Lemma:
$$
\sum_{G} (-1)^{|G|} = (-1)^{(k-1)} \cdot \frac{k-1}{1}!
$$

E.T.C.
This method is close to exact solutions
\[ \det (J - W) = \exp \left( \sum \frac{1}{m} \text{Tr} W^m \right) \]
(closed) walks interpreted as cactuses
-onsager\[ Z = \sum_{\text{cactuses}} \prod_{a} \prod_{a} \prod_{a} \]
cactuses of contours interpreted as walks in dual lattice