Potts Model, O(n) non-linear σ-model and Spanning Forests

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Potts and $\mathrm{O}(n)$ non-linear σ-models

- Potts Model: variables $\sigma_i \in \{0, 1, \ldots, q-1\}$;
  $\exp(-\beta\mathcal{H}(\sigma)) = \exp\left[\sum_{\langle ij\rangle} J_{ij}\delta(\sigma_i, \sigma_j)\right]$
  Symmetry: 'global' permutations in $\mathcal{S}_q$.

- $\mathrm{O}(n)$ non-linear σ-model: variables $\vec{\sigma}_i \in \mathbb{R}^n$;
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  Symmetry: 'global' rotations in $\mathrm{O}(n)$ (continuous!).

- If $\frac{1}{2}\left((\vec{\sigma}_i \cdot \vec{\sigma}_j)^2 - 1\right)$ instead of $(\vec{\sigma}_i \cdot \vec{\sigma}_j - 1)$ above:
  extra 'local' $\mathbb{Z}_2$ symmetry $\vec{\sigma}_i \to \epsilon_i \vec{\sigma}_i$, with $\epsilon_1 = \pm 1$.
  In other words, also because of δ's, the $\vec{\sigma}$'s are in RP^{n-1}.
  $\left[\; \mathrm{RP}^{n-1} := \{\vec{x} \in \mathbb{R}^n \setminus \{0\}\} / \vec{x} \sim \lambda\vec{x} \;\right]$

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Some goals:

- Understanding relations between these two models, and with combinatorial "generating functions" (i.e. countings of graphical objects);
- Understanding analytic continuation in q for Potts Model, and in n for $\mathrm{O}(n)$;
- Understanding asymptotic freedom in a non-perturbative way, for our 'favourite' model: Potts $[q \to 0;\ J/q \text{ fixed}]$ $\equiv \mathrm{O}(n)$ non-lin σ-model $[n \to -1] \equiv$ Spanning Forests, in $D = 2$ Euclidean lattice.

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Analytic continuation is easy for Potts...

[Fortuin-Kasteleyn, 1972. . . but also Tutte, '60s]

$$Z_G = \sum_{\sigma} e^{-\beta \mathcal{H}(\sigma)} = \sum_{\sigma} \prod_{(ij)} \big(1 + v_{ij}\, \delta(\sigma_i, \sigma_j)\big) \qquad \Big[v_{ij} := e^{J_{ij}} - 1 \Big]$$

$$= \sum_{H \subseteq G} \prod_{(ij) \in E(H)} v_{ij} \left(\sum_{\sigma} \prod_{(ij) \in E(H)} \delta(\sigma_i, \sigma_j) \right)$$

$$= \sum_{H \subseteq G} q^{K(H)} \prod_{(ij) \in E(H)} v_{ij} \,. \qquad \left[K(H) = \# \left\{ \begin{matrix} \text{comp.} \\ \text{in } H \end{matrix} \right\} \right]$$

You recognize the (slightly reparametrized and rescaled) multivariate Tutte Polynomial of G, and even better on next slide...

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...and leads to the Random Cluster Model

Recall:

- $L(H)$, the *cyclomatic number*, is the number of linearly-independent cycles in H.
- Euler formula states that $V - K = E - L$.

$$Z_{\mathrm{RC}}(G; \vec{w}; \lambda, \rho) = \sum_{H \subseteq G} \lambda^{K(H)-K(G)} \, \rho^{L(H)} \prod_{(ij) \in E(H)} w_{ij} \qquad \begin{bmatrix} \lambda\rho = q \\ w_{ij} = v_{ij}/\rho \end{bmatrix}$$

Tutte: $w = 1$; $x := Z[$ [edge] $] = 1 + \lambda$ and $y := Z[$ [loop] $] = 1 + \rho$.

Random Cluster Model —($\lambda \to 0$)→ Max.-connected subgraphs —($\lambda, \rho \to 0$)→ Gen. funct. Spanning Trees

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Max.-connected subgraphs (dual if G planar) ⇕ Spanning Forests

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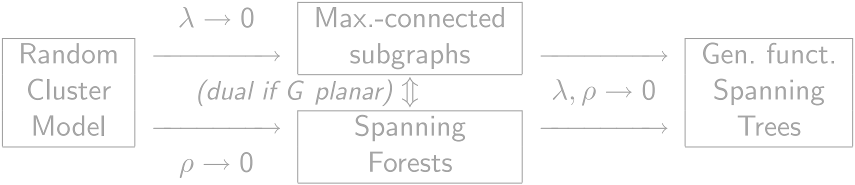

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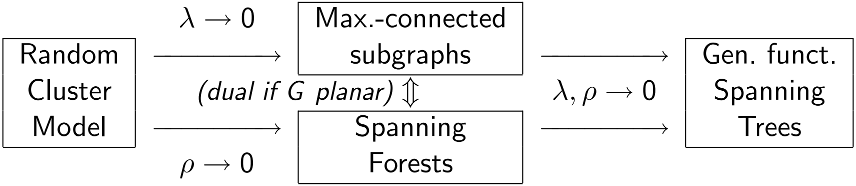
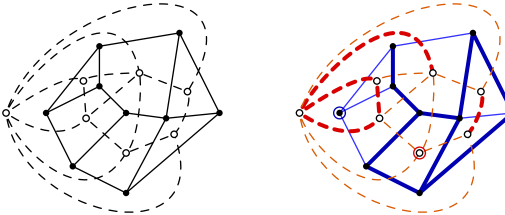


Planar duality

If graph G is connected and planar:

- Spanning Forests and Connected Subgraphs are dual;
- Trees are self-dual, and the intersection of the two.

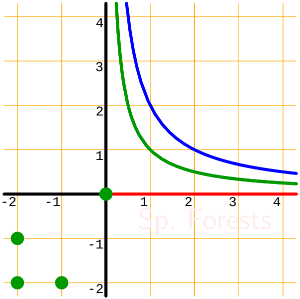
More generally: $E(\widehat{H}) = \widehat{E(H)}^{c}$, and $L(\widehat{H}) = K(H) - 1$,
so duality acts as $\lambda \leftrightarrow \rho$ and $w_{ij} \leftrightarrow 1/w_{ij}$.



Temperley-Lieb Algebra with parameter $\sqrt{\lambda\rho}$ plays a role.

Comput. complexity of Random-Cluster Partition Function

$Z_{\rm RC}(G;\vec{w};\lambda,\rho)$ is 'hard' to calculate (#P) in general, except for some special loci in the (λ,ρ) plane: [Welsh, 1990]



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- Computable in poly-time as a Pfaffian if $\lambda\rho = 2$ (Ising) and G is planar [Kasteleyn; Kač, Ward; 60's]
- Computable in poly-time at exceptional special points $(\lambda,\rho) = (-2,-2)$, $(-2,-1)$, $(-1,-2)$ and $(0,0)$.

$(0,0)$: Spanning Trees, counted by a determinant through Matrix-Tree Theorem [Kirchhoff, 1848 (!)]

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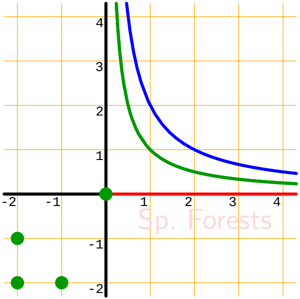

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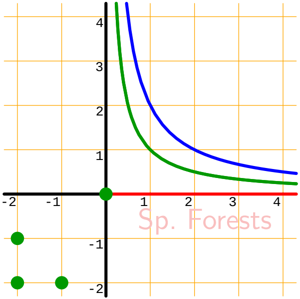

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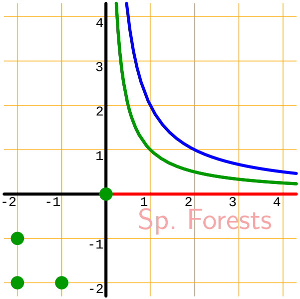

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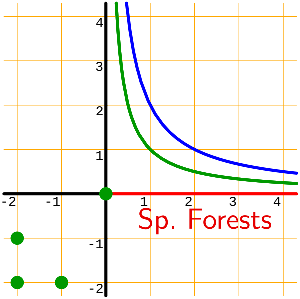

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Matrix-Tree Theorem

$$Z_{\mathrm{RC}}(G;\vec{w};\lambda=\rho=0)=\sum_{\substack{T\subseteq G\\ \text{trees}}}\prod_{(ij)\in E(T)} w_{ij} = \det L(i_0)$$

where i_0 is any vertex of G (the 'root'), $L(i_0)$ is the minor of L with row and col. i_0 removed, and L is the graph Laplacian matrix:

$$L_{ij}=\begin{cases} -w_{ij} & (ij)\in E(G)\\ 0 & (ij)\notin E(G)\\ \sum_{k\sim i} w_{ik} & i=j\end{cases} \qquad\qquad L\sim -\nabla^2$$

From Gaussian Integral formula in complex Grassmann Algebra:

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A digression on Grassmann Calculus

For $i = 1, .., n$, introduce the formal symbols θ_i, with $\theta_i \theta_j = -\theta_j \theta_i$, and symbols $(\int \mathrm{d}\theta_i)$ with formal rule $\int \mathrm{d}\theta_i \theta_i = 1$ and $\int \mathrm{d}\theta_i 1 = 0$. As $\theta_i^2 = 0$, the most general monomial $\prod_i \theta_i^{n_i}$ has $n_i = 0, 1$ (this justifies the name 'fermion'). Remark

$$\int \mathrm{d}\theta_n \cdots \mathrm{d}\theta_1 \prod_i \theta_i^{n_i} = \begin{cases} 1 & n_i = 1 \quad \forall i \\ 0 & \text{otherwise} \end{cases}$$

Special application, for $n \times n$ antisymmetric matrix A,

$$\int \mathrm{d}\theta_n \cdots \mathrm{d}\theta_1 \exp\left(\tfrac{1}{2}\theta A \theta\right) = \mathrm{pf} A = (\det A)^{\frac{1}{2}} .$$

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However, my advice on Grassmann Algebra vs. Exterior Algebra is:

- ✔ "$\int$" and "$\mathrm{d}x$" never far away (just as teached at high school);
- ✔ Multiplets $(\phi; \bar{\psi}, \psi)$ are 'natural'. Symmetry under rotations is an inspiring principle;
- ✔ Generators of symmetry and other things will involve Clifford Algebra (i.e. introduce also ∂_i and $\bar{\partial}_i$). Easy for Grassmann, I don't know for $\wedge$ fans...
- ✘ Extra variables required in the genuinely antisymmetric case (as many bosons as fermions);
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Going to "complex" is good and natural...

Consider the case of $2n$ symbols $\bar\psi_1, \ldots, \bar\psi_n$ and $\psi_1, \ldots, \psi_n$, and $\mathcal{D}(\psi,\bar\psi) := \mathrm{d}\psi_n \mathrm{d}\bar\psi_n \cdots \mathrm{d}\psi_1 \mathrm{d}\bar\psi_1$.
Then, for any matrix A

$$\int \mathcal{D}(\psi,\bar\psi)\ \exp(\bar\psi A \psi) = \det A\,;$$

$$\int \mathcal{D}(\psi,\bar\psi)\, \bar\psi_{i_1}\psi_{j_1} \cdots \bar\psi_{i_k}\psi_{j_k}\ \exp(\bar\psi A \psi) = \det A_{\{i_1,\cdots,i_k\},\{j_1,\ldots,j_k\}}\,.$$

These are the fermionic counterparts of Gaussian Integral and Wick Theorem that we have seen in these days...

An extension of the Matrix-Tree Theorem

In the following we will prove that a similar expression holds for arbitrary λ:

$$Z_{\mathrm{RC}}(G;\vec{w};\lambda,\rho=0)=\int\mathcal{D}_{V(G)}(\psi,\bar{\psi})\exp(\bar{\psi}L\psi)$$
$$\times\exp\left[\lambda\left(\sum_i\bar{\psi}_i\psi_i+\sum_{(ij)}w_{ij}\,\bar{\psi}_i\psi_i\bar{\psi}_j\psi_j\right)\right]$$

$$=\int\mathcal{D}_V(\psi,\bar{\psi})\exp\left[\lambda\sum_i\bar{\psi}_i\psi_i+\sum_{(ij)}w_{ij}\Big((\bar{\psi}_i-\bar{\psi}_j)(\psi_i-\psi_j)-\lambda\bar{\psi}_i\psi_i\bar{\psi}_j\psi_j\Big)\right]$$

Non-Gaussian integral, as expected from intrinsic hardness of the counting problem. However consequences can be drawn from such an expression.

Analytic continuation is hard for $\mathrm{O}(n)$ models...

Dimensional reduction tools can be useful?
Generalize $\mathrm{O}(n)$ to $\mathrm{OSP}(n|2m)$ models:

$$\vec{\sigma} = (\phi^{(a)})_{a=1,\dots,n} \qquad |\vec{\sigma}|^2 = \sum_{a=1}^{n} (\phi^{(a)})^2$$

$$\Downarrow$$

$$\vec{\sigma} = (\underbrace{\phi^{(a)}}_{B}; \underbrace{\bar{\psi}^{(b)}, \psi^{(b)}}_{F})_{\substack{a=1,\dots,n \\ b=1,\dots,m}} \qquad |\vec{\sigma}|^2 = \sum_{a=1}^{n} (\phi^{(a)})^2 + 2\lambda \sum_{a=1}^{m} \bar{\psi}^{(b)} \psi^{(b)}$$

For $n \in \mathbb{N}^+$ and $m \in \mathbb{N}$, analytic continuation should depend on $n - 2m$ only. [Parisi, Sourlas, 1979; Cardy, 1983]

Simplest non-trivial choice: $\mathrm{OSP}(1|2)$, i.e. $\vec{\sigma} = (\phi; \bar{\psi}, \psi)$.

OSP(1|2) – Spanning-Forest correspondence

Theorem: the $\mathrm{OSP}(1|2)$ non-linear σ-model partition function is related to the Random Cluster partition function at $\rho = 0$

$$Z_{\mathrm{OSP}(1|2)}(G; -\vec{w}/\lambda) = Z_{\mathrm{RC}}(G; \vec{w}; \lambda, \rho = 0)$$

at a perturbative level. For the $\mathrm{RP}^{0|2}$ model, the relation is non-perturbative.

...Let's prove it...

From the δ's, for each i we have $\phi_i^2 + 2\lambda\bar{\psi}_i\psi_i = 1$.

$\vec{\sigma}_i = \epsilon_i(\sqrt{1-2\lambda\bar{\psi}_i\psi_i}; \bar{\psi}_i, \psi_i) = \epsilon_i(1-\lambda\bar{\psi}_i\psi_i; \bar{\psi}_i, \psi_i)$, $\quad [\epsilon_i = \pm 1]$

Forget about ϵ's (say, all $+1$). [this why 'perturbative'...]

A Jacobian in the resolution of the δ's gives

$$\prod_i \frac{1}{\sqrt{1-2\lambda\bar{\psi}_i\psi_i}} = \exp\left(\lambda \sum_i \bar{\psi}_i\psi_i\right)$$

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Theorem: the $\mathrm{OSP}(1|2)$ non-linear σ-model partition function is related to the Random Cluster partition function at $\rho = 0$

$$Z_{\mathrm{OSP}(1|2)}(G; -\vec{w}/\lambda) = Z_{\mathrm{RC}}(G; \vec{w}; \lambda, \rho = 0)$$

at a perturbative level. For the $\mathrm{RP}^{0|2}$ model, the relation is non-perturbative.

...Let's prove it...

From the δ's, for each i we have $\phi_i^2 + 2\lambda\bar{\psi}_i\psi_i = 1$.

$\vec{\sigma}_i = \epsilon_i(\sqrt{1 - 2\lambda\bar{\psi}_i\psi_i}; \bar{\psi}_i, \psi_i) = \epsilon_i(1 - \lambda\bar{\psi}_i\psi_i; \bar{\psi}_i, \psi_i)$, $\quad \left[\epsilon_i = \pm 1\right]$

Forget about ϵ's (say, all $+1$). [this why 'perturbative'...]

A Jacobian in the resolution of the δ's gives

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The action, in both cases

$$\mathrm{OSP}(1|2): \qquad \mathcal{S} = -\sum_{(ij)} \frac{w_{ij}}{\lambda}\big(1 - \vec{\sigma}_i \cdot \vec{\sigma}_j\big)$$

$$\mathrm{RP}^{0|2}: \qquad \mathcal{S} = -\sum_{(ij)} \frac{w_{ij}}{2\lambda}\big(1 - (\vec{\sigma}_i \cdot \vec{\sigma}_j)^2\big)$$

gives the peculiar expression

$$\mathcal{S} = \sum_{(ij)} w_{ij} f_{ij}^{(\lambda)} \qquad f_{ij}^{(\lambda)} := (\bar{\psi}_i - \bar{\psi}_j)(\psi_i - \psi_j) - \lambda \bar{\psi}_i \psi_i \bar{\psi}_j \psi_j$$

and we are left to prove our "generalized Matrix-Tree theorem":

$$\int \mathcal{D}(\psi, \bar{\psi})\, e^{\lambda \bar{\psi}\psi + \sum_{(ij)} w_{ij} f_{ij}^{(\lambda)}} = Z_{\mathrm{RC}}(G; \vec{w}; \lambda, \rho = 0)$$

- Define $\tau_A := \prod_{i \in A} \bar{\psi}_i \psi_i$. Generalize f_{ij} to f_A, with $A \subseteq V(G)$:

$$f_A = \lambda(1 - |A|)\tau_A + \sum_{i \in A} \tau_{A \smallsetminus i} - \sum_{(i \neq j) \in A} \bar{\psi}_i \psi_j \tau_{A \smallsetminus \{i,j\}}$$

- Check the algebraic lemma, for $A \cap B \neq \varnothing$:

$$f_A f_B = \begin{cases} f_{A \cup B} & |A \cap B| = 1 \\ 0 & |A \cap B| \geq 2 \end{cases} \qquad (\textit{corollary: } f_{ij}^2 = 0)$$

- Expand the action: $\exp\left(\sum_{(ij)} w_{ij} f_{ij}\right) = \sum_{E' \subseteq E(G)} \prod_{(ij) \in E'} w_{ij} f_{ij}$
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So, our fermionic integral has already been reduced to a sum over spanning forests, and factors w_{ij} are appropriate. We still have to prove that the remaining fermionic integral of each summand gives exactly $\lambda^{K(F)}$.

Of course, the integral factorizes on various $V(T_\alpha)$, and we can concentrate on a single component, with $V(T_\alpha) = U$:

$$\int \mathcal{D}(\psi,\bar\psi) \prod_i (\overset{\spadesuit}{1} + \underbrace{\lambda\bar\psi_i\psi_i}_{\clubsuit}) \Big[\overbrace{\lambda(1-|U|)\tau_U}^{\spadesuit} + \sum_i \underbrace{\tau_{U\setminus i}}_{\clubsuit} - \sum_{(i\neq j)} \bar\psi_i\psi_j \tau_{U\setminus\{i,j\}} \Big]$$

Term $\spadesuit$ contributes $\lambda(1-|U|)$. The terms $\clubsuit_i$ contribute λ each. So we get a factor $\lambda(1-|U| + \sum_{i\in U} 1) = \lambda$, as claimed. □

Conclusions in the “continuum limit”

$$Z_{\mathrm{OSP}(1|2)} = \int \mathcal{D}(\psi, \bar{\psi})\, e^{\lambda \bar{\psi}\psi + \bar{\psi}\nabla^2\psi + \frac{\lambda}{2}\bar{\psi}\psi\nabla^2\bar{\psi}\psi} = Z_{\mathrm{RC}}(\lambda, \rho = 0)$$

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$$Z' \begin{bmatrix} \text{massless} \\ \text{fermion} \end{bmatrix} = \int \mathcal{D}(\psi, \bar{\psi})\, \bar{\psi}_0 \psi_0\, e^{\bar{\psi}\nabla^2\psi} = Z_{\mathrm{RC}}(\lambda = 0, \rho = 0)$$

Of course, the theory at $\lambda = 0$ (Spanning Trees) is critical. In $D = 2$, it is a $c = -2$ logarithmic CFT.

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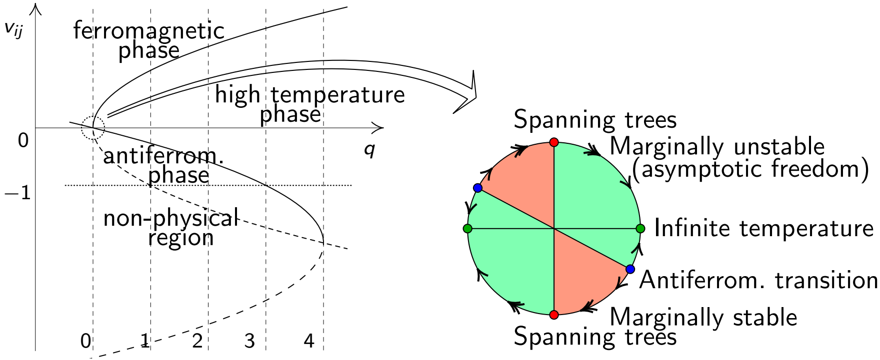
E.g., our present understanding for Potts on the square lattice (combined with Baxter solution):

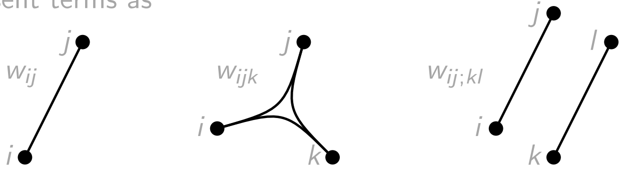


More on the algebraic side

The set of $\{f_{ij}^{(\lambda)}\}_{1\leq i<j\leq n}$ generates all functions of scalar products $\{\vec{\sigma}_i \cdot \vec{\sigma}_j\}$ for n unit vectors in $\mathrm{RP}^{0|2}$, as an algebra of polynomials. So the most general function $\mathcal{S}(\bar{\psi}, \psi)$ invariant under $\mathrm{OSP}(1|2)$ global rotation is of the form

$$\mathcal{S}(\bar{\psi}, \psi) = \sum_{(ij)} w_{ij} f_{ij} + \sum_{(ijk)} w_{ijk} f_{ijk} + \cdots + \sum_{(ij;kl)} w_{ij;kl} f_{ij} f_{kl} + \cdots$$

Represent terms as



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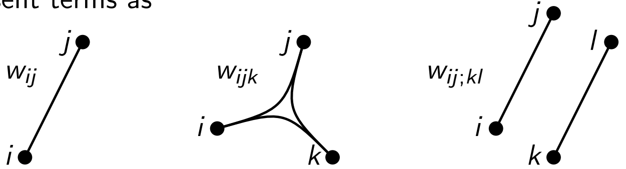


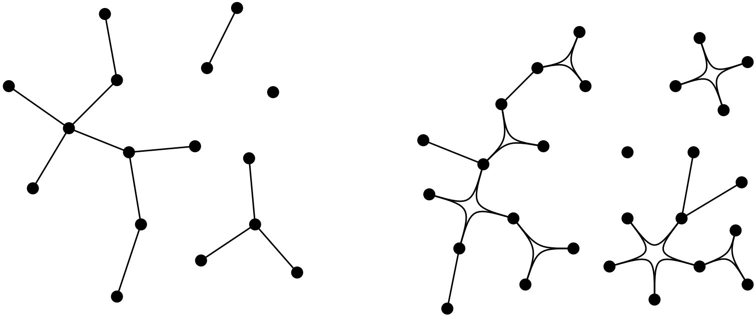
then

$$\int \mathcal{D}(\psi, \bar{\psi})\, e^{\lambda \bar{\psi}\psi + \mathcal{S}(\bar{\psi},\psi)} = \sum_{\substack{F \subseteq G \\ \text{hyperforests}}} \lambda^{K(F)}\, P(w; F)$$

with G a hypergraph with edges $(i_1 \cdots i_k)$ corresponding to k-uples such that some coefficient w is non-zero, and $P(w;F)$ is a polynomial in the w's whose k-uples appear as hyper-edges in F.

Even for the most general $\mathrm{OSP}(1|2)$-invariant action, restriction to cycle-free sub-(hyper)graphs, i.e. forests, appears as an algebraic consequence of symmetry, and even in the Grassmann sub-algebra of f_{ij}'s before integration.

...a forest and a hyperforest...



Linear-space dimension of the polynomial algebra of f_{ij}'s

As $f_i = 1$ and $f_\varnothing = \lambda$, the most general monomial in the polynomial algebra generated by f_{ij}'s is labeled by a partition $\mathcal{C} = (C_1, \ldots, C_k)$ of $[n]$:

$$\mathcal{C} \in \Pi(n) : \qquad f_\mathcal{C} := f_{C_1} \cdots f_{C_k}$$

They must be a redundant basis for the linear space of global $\mathrm{OSP}(1|2)$-invariant functions, as $|\Pi(n)| = B_n \sim n!$, while the whole Grassmann Algebra has dimension only 4^n...

- Which dimension has the linear space?
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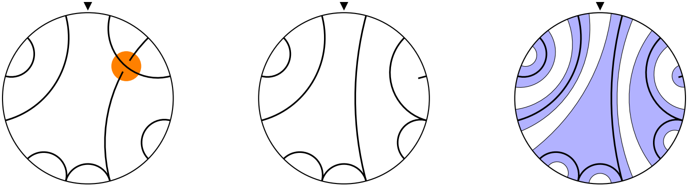
a few answers...

❶ The dimension of the linear space is $C_n = \frac{1}{n+1}\binom{2n}{n} \sim 4^n n^{-3/2}$, the n-th Catalan number;

❷ A basis is $\mathrm{NC}(n)$, the non-crossing partitions. $\mathcal{C} \in \mathrm{NC}(n)$ iff for all A, B distinct blocks of $\mathcal{C}$, and all a, $c \in A$ and b, $d \in B$, it is never $a < b < c < d$.



❸ A single 4-point relation generates the kernel:

$$R_{abcd} = \lambda f_{abcd} + f_{ab} f_{cd} + f_{ac} f_{bd} + f_{ad} f_{bc} - f_{abc} - f_{abd} - f_{acd} - f_{bcd} = 0$$

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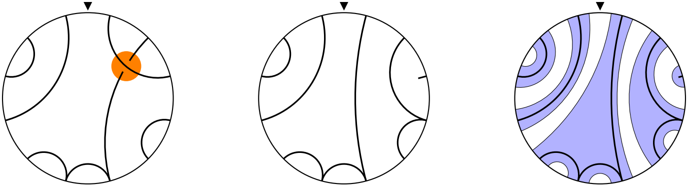

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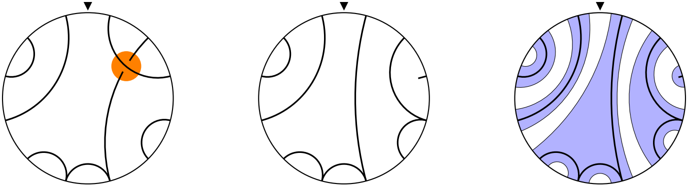

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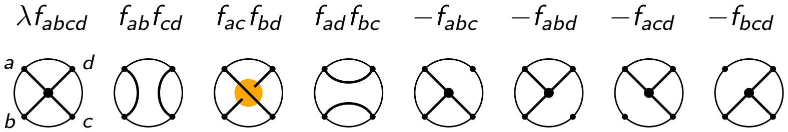

Can be used to recursively write a $f_{\mathcal{C}}$ with $\mathcal{C}$ crossing as a linear combination of $f_{\mathcal{C}'}$'s, with all $\mathcal{C}'$ non-crossing.

Consider Clifford Algebra. Other OSP(1|2)-invariant objects are:

$$p_i := \partial_i \bar{\partial}_i (1 + \lambda \bar{\psi}_i \psi_i) = \int \mathrm{d}\psi_i \mathrm{d}\bar{\psi}_i e^{\lambda \bar{\psi}_i \psi_i}$$

Some algebra:

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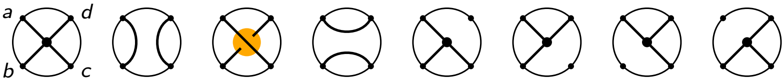

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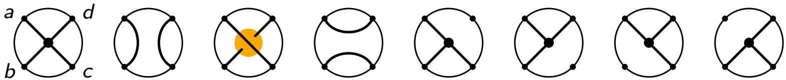
Clifford Algebra and $R^b_{ac} = 0$

With p_i's we get a three-point relation in Clifford Algebra: $R^b_{ac} = 0$.
It is an easy check that $R^b_{ac} f_{bd} = R_{abcd}$.

Compare the terms appearing in R_{abcd} and in R^b_{ac}:

λf_{abcd} $\quad f_{ab} f_{cd}$ $\quad f_{ac} f_{bd}$ $\quad f_{ad} f_{bc}$ $\quad -f_{abc}$ $\quad -f_{abd}$ $\quad -f_{acd}$ $\quad -f_{bcd}$



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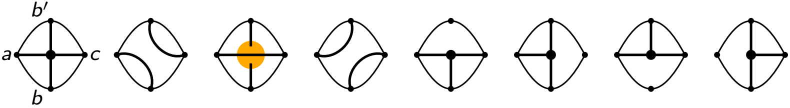


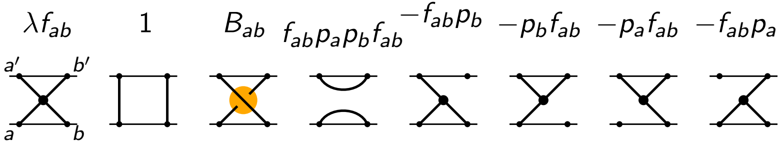
Exchange operator and $R^{ab} = 0$

Another interesting $\mathrm{OSP}(1|2)$-invariant in Clifford Algebra is the "exchange" operator

$$B_{ab} := \left(1 - (\bar\psi_a - \bar\psi_b)(\bar\partial_a - \bar\partial_b)\right)\left(1 - (\psi_a - \psi_b)(\partial_a - \partial_b)\right)$$

$$B_{ab} P(\bar\psi_a, \psi_a, \bar\partial_a, \partial_a, \bar\psi_b, \cdots) = P(\bar\psi_b, \psi_b, \bar\partial_b, \partial_b, \bar\psi_a, \cdots) B_{ab}$$

With B_{ab} we can build a two-point relation $R^{ab} = 0$:



and $R^{bc} f_{ab} f_{cd} = R_{abcd}$.

Comments on R_{abcd}, R^b_{ac} and R^{ab}

The three relations $R_{abcd} = 0$, $R^b_{ac} = 0$ and $R^{ab} = 0$ are different forms of the "fundamental" $\mathrm{OSP}(1|2)$ relation, which relates the only 4-point crossing partition to the other seven 2-block non-crossing ones.

They all involve eight fermions, and have eight terms, four positive and four negative.

A version of $R_{abcd} = 0$ for $\lambda = 0$ (thus with seven terms) was also in [Kenyon, Wilson, 2006].

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Recognizing even/odd Temperley-Lieb

We have seen some algebraic rules for f_{ij}'s and p_i's:

$$f_{i\,i+1}^2 = 0\,; \qquad [f_{i\,i+1}, f_{j\,j+1}] = 0\,; \qquad f_{i\,i\pm1}^2 p_i f_{i\,i\pm1} = f_{i\,i\pm1}\,;$$
$$p_i^2 = \lambda p_i\,; \qquad [p_i, p_j] = 0\,; \qquad p_i f_{i\,i\pm1} p_i = p_i\,;$$
$$[p_i, f_{j\,j+1}] = 0 \qquad \text{if } j \neq i, i-1.$$

. . . look similar to Temperley-Lieb Algebra [T.,L., 1971],

$$e_i^2 = \lambda e_i\,; \qquad e_i e_{i\pm1} e_i = e_i\,; \qquad [e_i, e_j] = 0 \qquad \text{if } |i-j| \geq 2.$$

by identifying $e_{2i} = p_i$ and $e_{2i+1} = f_{i\,i+1}$, but $e_i^2 = \lambda_{\mathrm{parity}(i)}$ with $\lambda_{\mathrm{even}} = \lambda$ and $\lambda_{\mathrm{odd}} = 0$.

...comments on Temperley-Lieb

Indeed, T-L describes the transfer matrix of the Random Cluster Model, on planar graphs, at $\lambda = \rho = \sqrt{q}$, and allow to integrate the model, say on the square lattice, on Baxter parabola.

Instead, this algebra describes the line $\lambda > 0$, $\rho = 0$ corresponding to spanning forests.

As a result of $\rho = 0$, we do not need to deal with $L(H)$, and through $R_{abcd} = 0$ we can build a transfer matrix on $\mathrm{NC}(n)$ also for non-planar graphs.

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Conclusions

- We put in correspondence the $\mathrm{OSP}(1|2)$ non-linear σ-model with Spanning Forests, i.e. Potts Model for $q \to 0$ and $v_{ij}/q = w_{ij}$ fixed.
- Even the most general $\mathrm{OSP}(1|2)$-invariant action admits a combinatorial expansion in terms of sub-hyperforests only (no cycles in subgraphs). The symmetry is a precious guideline when building proofs.
- Study of linear independence in the symmetric subalgebra led to a 'fundamental' relation $R_{abcd} = 0$, generalizing the one for spanning trees, i.e. free-fermion theory.
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Things left apart

- Combinatorial interpretation of fermionic observables. Probabilistic understanding of Ward identities.
- Raise to a $\mathrm{OSP}(1|2m)$–Spanning-Forest relation. For higher m, can access more probabilistic observables.
- You can add a "vector field", and count unicyclics with topological weights proportional to the circuitation.
- Relation between Spanning Forests and Abelian Sandpile Model, through Dhar work and a Biggs-Merino theorem.
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