Solving Bogoliubov's recursion in renormalisation using a simple algebraic identity

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A Brief History of "Everything"

... in this talk...

commutative setting

• F. Spitzer	(1956)	:	Spitzer's Identity
• M. Schützenberger	(1958)	:	half-shuffles, chronological algebras
• G. Baxter	(1960)	:	(Rota-)Baxter Identity
• F. Atkinson	(1963)	:	factorization theorems
• G.-C. Rota	(1969/1972)	:	Bohnenblust-Spitzer
• P. Cartier - D. Foata	(1969)	:	random variables
• P. Cartier	(1972)	:	quasi-shuffle product

non-commutative setting

• W. Magnus	(1954)	:	Magnus expansion
• F. Fer	(1958)	:	Fer expansion
• R. M. Wilcox	(1967)	:	Magnus-BCH, Zassenhaus-Fer
• B. Mielnik - J. Plebański	(1970)	:	continuous BCH
• I. Gelfand - D. Krob - A. Lascoux - B. Leclerc - V. Retakh - J.-Y. Thibon	(1995)	:	NCSF
• J.-L. Loday	(1998)	:	dendriform algebras

motivation-application

• N. N. Bogoliubov - O. S. Parasiuk	(1957)	:	pert. renormalization
• D. Kreimer - D. Broadhurst - A. Connes	(2000)	:	Birkhoff factorization for Feynman rules

Goal: to extend functional identities for Rota–Baxter algebras first discovered in fluctuation theory (Spitzer, Baxter, Rota...) to the noncommutative setting, and to use them in perturbative renormalization, i.e. solving Bogoliubov's counterterm recursion

Observation particular type of non-associative algebra plays a crucial role, i.e. pre-Lie algebra

Setting: Connes–Kreimer's Hopf algebra of Feynman graphs, i.e. Feynman graphs are organized into a (commutative, graded, connected) Hopf algebra

$$H = \bigoplus_{n \in \mathbf{N}} H_n,\ H_0 = \mathbf{C},\ \pi : H \otimes H \longrightarrow H,$$

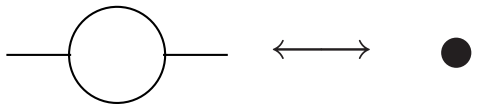
$$\Delta : H \longrightarrow H \otimes H,\ \eta : H \to \mathbb{C}$$

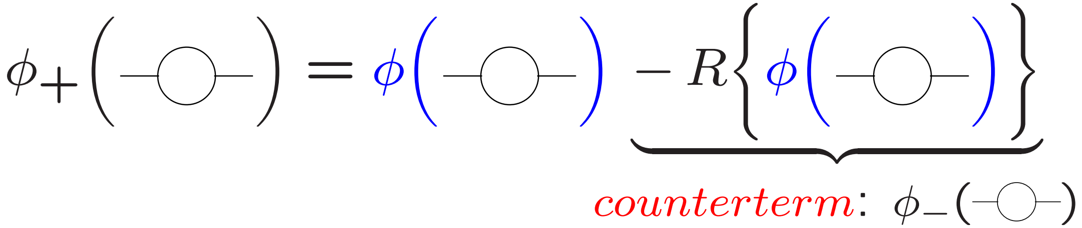
$$f, g : H \to A, \qquad f \star g = m_A \circ (f \otimes g) \circ \Delta$$

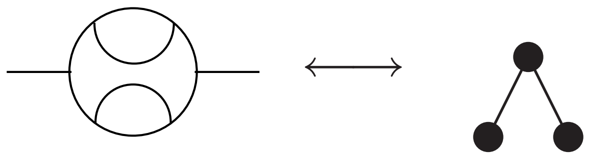
Renormalization: Bogoliubov's recursive subtraction

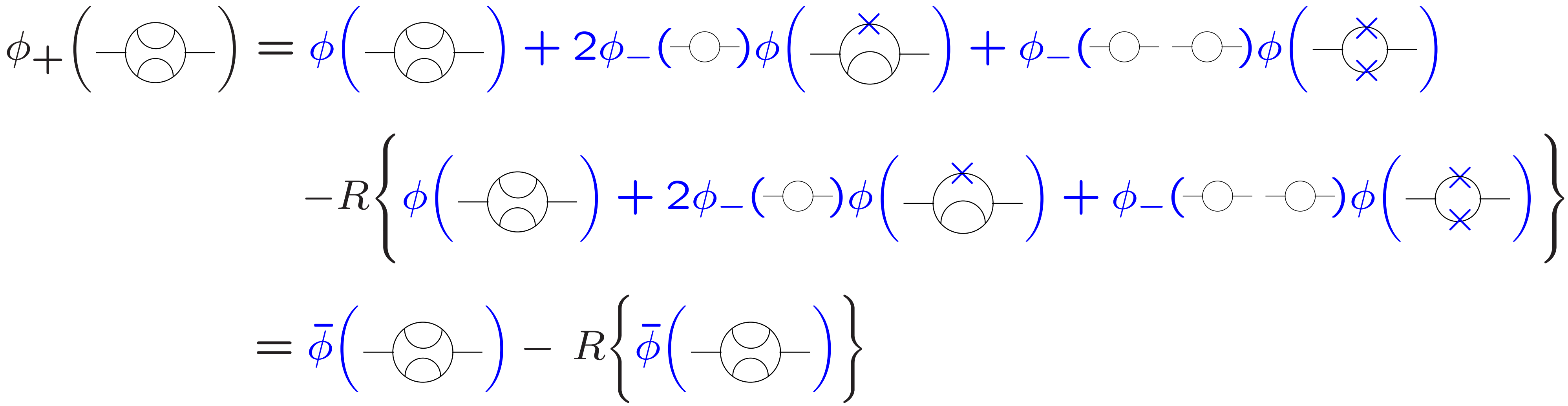
Feynman rules: $\phi : H \to A$

renormalization scheme: $R : A \to A$

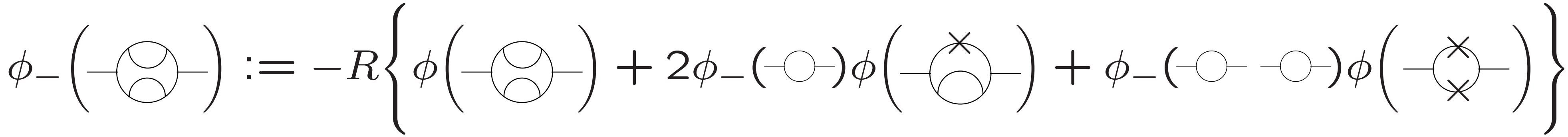




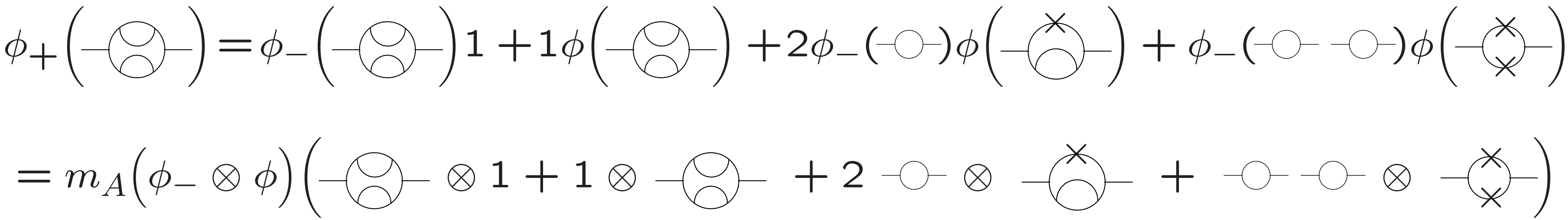




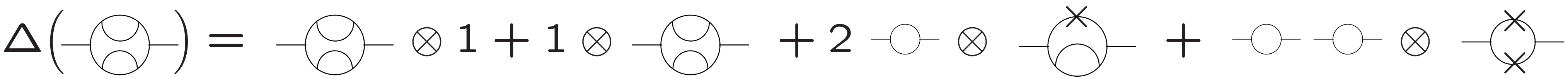
- Counterterm: $\phi_- : H \to A$

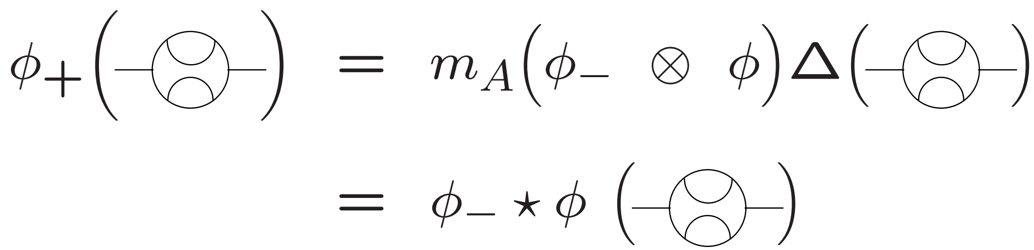


- Renormalized Feynman amplitude: $\phi_+ : H \to A$



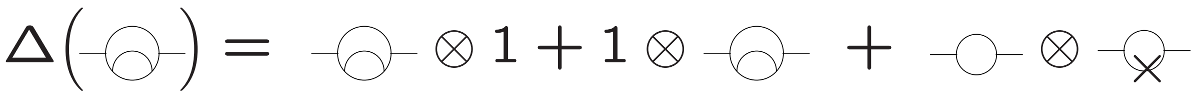
- Coproduct: $\Delta : H \to H \otimes H$





Theorem: Hopf algebra of Feynman graphs:
$H := \left(m, \Delta, \epsilon, \eta, S\right)$ unital, graded, connected, associative, commutative, coassociative, non-cocommutative, Hopf algebra.

- Coproduct: $\Delta : H \to H \otimes H$



- Feynman rules: $\phi : H \to A$

$$\phi(t_1 t_2) = \phi(t_1)\phi(t_2)$$

- Renormalization scheme map:

linear operator $R : A \to A$

A is a commutative unital algebra

- A-valued linear functionals: $H \xrightarrow{\text{Hom}(H,A)} (A, R)$

$$f \star g := m_A(f \otimes g)\Delta$$

$$H \xrightarrow{\Delta} H \otimes H \xrightarrow{f \otimes g} A \otimes A \xrightarrow{m_A} A$$

- Renormalization:

$$\phi_\pm : H \to A \qquad \begin{cases} \phi_- = \mathbf{1} - R\Big(\phi_- \star (\phi - \mathbf{1})\Big) \\ \phi_+ = \mathbf{1} + (id - R)\Big(\phi_- \star (\phi - \mathbf{1})\Big) \end{cases}$$

$$\begin{aligned} \phi_+ &= \mathbf{1} - R\Big(\phi_- \star (\phi - \mathbf{1})\Big) + \phi_- \star (\phi - \mathbf{1}) \\ &= \phi_- - \phi_- \star \mathbf{1} + \phi_- \star \phi \\ &= \phi_- - \phi_- + \phi_- \star \phi \\ &= \phi_- \star \phi \end{aligned}$$

- Factorization: $\phi \in G(A)$ $\quad \phi_\pm \overset{?}{\in} G(A)$

$$\phi_+ = \phi_- \star \phi \quad \longrightarrow \quad \phi = \phi_-^{-1} \star \phi_+$$

$$\phi_- = \mathbf{1} - R\Big(\phi_- \star (\phi - \mathbf{1})\Big) \qquad\qquad \bar{\phi} := \phi_- \star (\phi - \mathbf{1})$$

$$\phi_-(t_1 t_2) \overset{?}{=} \phi_-(t_1)\phi_-(t_2)$$

- "multiplicativity constraint" $R : A \to A$

$$R(x)R(y) + R(xy) = R(R(x)y + xR(y))$$

Example: Let $A = \mathbb{C}[\varepsilon^{-1}, \varepsilon]]$

$$R\Big(\sum_{n=-k}^{\infty} a_n \varepsilon^n\Big) := \sum_{n=-k}^{-1} a_n \varepsilon^n.$$

Non-commutative associative unital $\mathbb{K}$-algebra $(A, \cdot)$

$Z \in A$, $\quad a := Z - 1$

linear map $R : A \to A$

$$R(x) \cdot R(y) = R(R(x) \cdot y + x \cdot R(y)) \quad - R(x \cdot y)$$

★ $X = 1 - R\Big(X \cdot a\Big)$

$$X = 1 - R(a) + R(R(a) \cdot a) - R(R(R(a) \cdot a) \cdot a) + R(R(R(R(a) \cdot a) \cdot a) \cdot a) - \dots$$

★ $Y^{-1} = 1 + (id - R)(X \cdot a)$ such that

$$(1 + a) = Z = X^{-1} \cdot Y^{-1}$$

Observation: $R(x) \cdot R(y) = R(R(x) \cdot y + x \cdot R(y)) - R(x \cdot y)$

1. $\tilde{R} := id - R$

$$\tilde{R}(x) \cdot \tilde{R}(y) = \tilde{R}(\tilde{R}(x) \cdot y + x \cdot \tilde{R}(y)) - \tilde{R}(x \cdot y)$$

2.

$$\left.\begin{aligned} X &= 1 - R(X \cdot a) \\ Y &= 1 - \tilde{R}(a \cdot Y) \end{aligned}\right\} \rightarrow \quad X \cdot (1 + a) \cdot Y = 1$$

$$(1 + a) = X^{-1} \cdot Y^{-1}$$

$Y^{-1} = 1 + \tilde{R}(X \cdot a)$

$X^{-1} = 1 + R(a \cdot Y)$

Integration by parts (commutative setting)

Riemann integral: $\mathcal{F} = \mathbf{Cont}(\mathbb{R})$ $\quad \begin{array}{l} I : \mathcal{F} \to \mathcal{F} \\ I(f)(x) := \int_0^x f(t)dt. \end{array}$

Let $F(x) := I(f)(x)$, $G(x) := I(g)(x)$.

Integration by parts rule: $\quad I(FG')(x) = F(x)G(x) - I(F'G)(x)$

$$I(f)I(g) = I\Big(I(f)\, g\Big) + I\Big(f\, I(g)\Big)$$

$$f' = af, \quad f(0) = 1 \qquad \longrightarrow \qquad f = 1 + I(af)$$

$$f = 1 + \sum_{n=0}^{\infty} I\Big(\underbrace{aI(\cdots I(aI}_{n\text{-times}}(a))\cdots)\Big) \;=\; \exp(I(a))$$

$$\boxed{(I(f))^n = n!\; I\Big(\underbrace{fI(f\cdots I(fI}_{n\text{-times}}(f))\cdots)\Big)}$$

non-commutative setting:

$$Y' = AY, \;\; Y(0) = 1 \qquad\qquad Y = 1 + I(AY)$$

W. Magnus: [1954] Magnus expansion to solve matrix diff. eqn.

$$Y(t) = \exp\Big(\Omega(A)(t)\Big), \quad \Omega(A)(0) = 0$$

$$\boxed{\dot{\Omega}(A) = \sum_{m\geq 0} \frac{B_m}{m!}\; ad^{(m)}_{\Omega(A)}[A]}.$$

Bernoulli numbers: $\sum_{n\geq 0} \frac{B_m}{m!} z^m = \frac{z}{\exp(z)-1} = 1 - \frac{1}{2}z + \frac{1}{12}z^2 - \frac{1}{720}z^4 + \cdots$

$$\Omega(A) = \sum_{n>0} \Omega_n(A)(t) = \ln\Big(1 + \sum_{m=0}^{\infty} I\Big(AI(\cdots I(AI(A))\cdots)\Big)(t)\Big)$$

$$\Omega(A)(t) = \int_0^t A(s_1)ds_1 - \frac{1}{2}\int_0^t \Big[\Omega(A)(s_1), A(s_1)\Big]ds_1 + \frac{1}{12}\int_0^t \Big[\Omega(A)(s_1), [\Omega(A)(s_1), A(s_1)]\Big]ds_1 + \cdots$$

$$\Omega(A)(t) = \underbrace{\int_0^t A(s_1)ds_1}_{\Omega_1(A)(t)} \underbrace{-\frac{1}{2}\int_0^t \Big[\int_0^{s_1} A(s_2)ds_2, A(s_1)\Big]ds_1}_{\Omega_2(A)(t)}$$

$$+\frac{1}{4}\int_0^t \Big[\int_0^{s_1} \Big[\int_0^{s_2} A(s_3)ds_3, A(s_2)\Big]ds_2, A(s_1)\Big]ds_1$$

$$+\frac{1}{12}\int_0^t \Big[\int_0^{s_1} A(s_3)ds_3, \Big[\int_0^{s_1} A(s_2)ds_2, A(s_1)\Big]\Big]ds_1 + \cdots$$

$$\Omega(A)(t) = \sum_{\substack{n>0 \\ \sigma\in S_n}} \frac{(-1)^{d(\sigma)}}{n^2\binom{n-1}{d(\sigma)}} \int_0^t \int_0^{t_1} \ldots \int_0^{t_{n-1}} \Big[A(t_{\sigma(n)}), \Big[\cdots [A(t_{\sigma(2)}), A(t_{\sigma(1)})]\cdots\Big]\Big]\, dt_n \ldots dt_1$$

Weighted integration-by-parts (commutative setting)

G. Baxter: [1960] replace the Riemann integral map I by an operator R –on an associative commutative algebra A– satisfying

$$R(x)R(y) = R\Big(R(x)y + xR(y)\Big) + \theta R\Big(xy\Big)$$

with weight parameter $\theta \in \mathbb{K}$; and look for solutions of

$$Y = 1 + R(Ya).$$

Frank Spitzer: [1956] Spitzer's classical identity

$$Y = 1 + \sum_{n=1}^{\infty} \underbrace{R\Big(R(\cdots R(R}_{n\text{-times}}(a)a)a\cdots)a\Big) = \exp\left(R\Big(\frac{\log(1+\theta a)}{\theta}\Big)\right)$$

In the limit $\theta \to 0$ we get back the classical case since

$$\frac{1}{\theta}\log(1+\theta a) = -\sum_{n>0} \theta^{n-1}\frac{(-a)^n}{n} \xrightarrow{\theta\to 0} a$$

Spitzer's classical identity in low degrees

$$
\begin{aligned}
Y &= 1 + R(a) + R\Big(R(a)a\Big) + R\Big(R(R(a)a)a\Big) + \dots \\
&= \sum_{n\geq 0} \frac{1}{n!}\Bigg(R\Big(-\sum_{m>0} \theta^{m-1}\frac{(-a)^m}{m}\Big)\Bigg)^n
\end{aligned}
$$

degree 2:

$$-\theta\frac{1}{2}R(a^2) + \frac{1}{2}R(a)R(a) = R(R(a)a)$$

degree 3:

$$\theta^2\frac{1}{3}R(a^3) - \theta\frac{1}{2}R(a^2)R(a) + \frac{1}{3!}R(a)^3 = R(R(R(a)a)a)$$

weight $\theta = 0$: $R(x)R(x) = R(xR(x)) + R(R(x)x)$

$$n!\, \underbrace{R\Big(R(\cdots R(R}_{n\text{-times}}(a)a)a\cdots a)\Big) = (R(a))^n$$

Bohnenblust–Spitzer formula: (1950s, 1972)

$$\sum_{\sigma\in S_n} R(a_{\sigma(1)}R(\cdots R(a_{\sigma(n)})\cdots)) = \sum_{\mathcal{T}}(-\theta)^{n-|\mathcal{T}|}\prod_{T\in\mathcal{T}}(|T|-1)!\, R\Big(\prod_{j\in T} a_j\Big).$$

Here $\mathcal{T}$ runs through all unordered set partitions of $\{1,\cdots,n\}$.

Example $n = 2, 3$:

$$R(a_1R(a_2)) + R(a_2R(a_1)) = -\theta R(a_1a_2) + R(a_1)R(a_2)$$

$$3!R(R(R(a)a)a) = \theta^2 2R(a^3) - \theta 3R(a^2)R(a) + R(a)^3$$

Rota–Baxter Algebra

Definition: $\theta \in \mathbb{K}$, (A, R), $R : A \to A$

$$R(x)R(y) = R\Big(R(x)y + xR(y)\Big) + \theta R(xy)$$

- The map $\tilde{R} := -\theta\text{id} - R$ is a Rota–Baxter map.
- $R(A)$ and $\tilde{R}(A)$ are subalgebras in A.

Examples:

Projectors: Let $A = \mathbb{C}[\varepsilon^{-1}, \varepsilon]]$, $\sum_{n=-k}^{\infty} a_n \varepsilon^n$, $\theta = -1$

$$R\Big(\sum_{n=-k}^{\infty} a_n \varepsilon^n\Big) := \sum_{n=-k}^{-1} a_n \varepsilon^n.$$

q-difference equation: $\sigma_q f(x) := f(qx)$

$$\partial_q f(x) := (id - \sigma_q) f(x) = f(x) - f(qx)$$

$$\widehat{P}_q[f] := \sum_{n \geq 0} \sigma_q^n f$$

$$\widehat{P}_q[f]\widehat{P}_q[g] = \widehat{P}_q\Big[\widehat{P}_q[f]\, g\Big] + \widehat{P}_q\Big[f\, \widehat{P}_q[g]\Big] - \widehat{P}_q\Big[fg\Big]$$

Summation:

$$S[f](x) := \sum_{n > 0} f(x+n)$$

$$S[f]S[g] = S\Big[S[f]\, g\Big] + S\Big[f\, S[g]\Big] + S\Big[fg\Big]$$

Double Rota–Baxter Algebra structure

Proposition: (A, R) is a Rota–Baxter algebra. Define:

$$x *_R y := R(x)y + xR(y) + \theta xy$$

The vector space underlying A, equipped with the product $*_R$ is again a Rota–Baxter algebra of the same type. (A_R, R) is called the **double Rota–Baxter algebra**.

Lemma: R, $\tilde{R}$ are algebra morphism $A_R \to A$:

$$R\Big(x *_R y\Big) = R(x)\, R(y)$$

$$\tilde{R}\Big(x *_R y\Big) = -\tilde{R}(x)\, \tilde{R}(y)$$

$$R(x)R(y) = R\Big(R(x)y + xR(y) + \theta xy\Big)$$

Rota–Baxter Pre-Lie Algebra structure

Recall: that a left **pre-Lie algebra** P is a vector space, together with a bilinear pre-Lie product $\bullet : P \otimes P \to P$, satisfying the left pre-Lie relation

$$(a \bullet b) \bullet c - a \bullet (b \bullet c) = (b \bullet a) \bullet c - b \bullet (a \bullet c), \qquad a, b, c \in P,$$

The commutator $[a, b] := a \bullet b - b \bullet a$ for $a, b \in P$ satisfies the *Jacobi* identity. Hence L_P is a Lie algebra.

Proposition: (A, R) is an associative Rota–Baxter algebra. Define:

$$x \bullet_R y := [R(x), y] - \theta yx = R(x)y - yR(x) - \theta yx$$

The vector space underlying A, equipped with the product $\bullet_R$ is a **Rota–Baxter pre-Lie algebra**.

Magnus expansion: weight $\theta = 0$, $Y = 1 + I(AY)$

$$\begin{aligned}
\dot{\Omega} &= \sum_{m\geq 0} \frac{B_m}{m!}\ ad^{(m)}_{I(\dot{\Omega}(A))}[A] \\
&= A - \frac{1}{2}[I(\dot{\Omega}), A] + \frac{1}{12}\Big[I(\dot{\Omega}), [I(\dot{\Omega}), A]\Big] + \sum_{m\geq 4} \frac{B_m}{m!}\ ad^{(m)}_{I(\dot{\Omega}(A))}(A) \\
&= A - \frac{1}{2}\dot{\Omega} \bullet_I A + \frac{1}{12}\dot{\Omega} \bullet_I (\dot{\Omega} \bullet_I A) + \sum_{m\geq 4} \frac{B_m}{m!}\ ad^{(m)}_{I(\dot{\Omega})}(A) \\
&= \sum_{m\geq 0} \frac{B_m}{m!} L^{(m)}_{\bullet_I}[\dot{\Omega}](A)
\end{aligned}$$

$$L_{\bullet_I}[a](b) \ := \ a \bullet_I b = I(a)b - bI(a)$$

Pre-Lie Magnus expansion

Theorem: Let (A, R) be a Rota–Baxter algebra. Let $\Omega' := \Omega'(\lambda a)$, $a \in A$, be the element of $\lambda A[[\lambda]]$ such that

$$Y' = \exp^{*_R}(\Omega')$$

where Y' is the solution of $Y' = 1 + \lambda R(Y')a$. This element obeys the following recursive equation:

$$\Omega'(\lambda a) = \frac{L_{\bullet_R}[\Omega']}{\exp(L_{\bullet_R}[\Omega']) - 1}(\lambda a) = \sum_{m \geq 0} \frac{B_m}{m!} \, L^{(m)}_{\bullet_R}[\Omega'](\lambda a)$$

where the B_l's are the Bernoulli numbers.

$$\begin{aligned} x *_R y &= R(x)y + xR(y) + \theta xy \\ x \bullet_R y &= R(x)y - yR(x) - \theta yx = [R(x), y] - \theta yx \end{aligned}$$

Remark: **Bogoliubov's preparation map** $\bar{\phi} = \phi_- \star (\phi - \mathbf{1})$

$$\phi_- = \mathbf{1} - R(\bar{\phi}) \qquad\qquad \bar{\phi} = \exp^{*R}(\Omega')$$

Remark: **pre-Lie Fer expansion**

Theorem: Let (A, R) be a Rota–Baxter algebra. Let $U'_0 := \lambda a$, and $U'_n := U'_n(a)$, $n \in \mathbb{N}$, $a \in A$, be elements in $\lambda A[[\lambda]]$, such that

$$Y' = \overrightarrow{\prod_{n \geq 0}}^{*R} \exp^{*R}(U'_n)$$

where Y' is a solution of $Y' = 1 + \lambda R(Y')a$. Then these elements U'_n obey the following recursive equation:

$$U'_{n+1} := \sum_{l>0} \frac{(-1)^l l}{(l+1)!} \; L^{(l)}_{\bullet_R}[U'_n](U'_n) \qquad n \geq 0.$$

$$(Ra)^{[n]} := R((Ra)^{[n-1]}a)$$

$$(Ra)^{\{n\}} := R(a(Ra)^{\{n-1\}})$$

$$X = 1 + tR(Xa) \qquad X = \sum_{n\geq 0} t^n (Ra)^{[n]}$$

$$Y = 1 + t\tilde{R}(aY) \qquad Y = \sum_{n\geq 0} t^n (\tilde{R}a)^{\{n\}}$$

$$(1-a) = X^{-1}Y^{-1}$$

$$Y^{-1} = 1 - \tilde{R}(Xa) \qquad X^{-1} = 1 - R(aY)$$

We have a cocommutative Hopf algebra structure: $(Ra)^{[n]}$s form a sequence of divided powers

$$\Delta\Big((Ra)^{[n]}\Big) = \sum_{0\leq m\leq n} (Ra)^{[m]} \otimes (Ra)^{[n-m]};$$

The *free noncommutative Spitzer algebra* $\mathcal{S}$ on one generator.

The series $X = \sum_{n\geq 0}(Ra)^{[n]}$ is a group-like element in $\mathcal{S}$.

Proposition: The action of the antipode S on the Spitzer algebra $\mathcal{S}$, is given by

$$S\Big((Ra)^{[n]}\Big) = -R\Big(a(\tilde{R}a)^{\{n-1\}}\Big).$$

$$\begin{aligned} D(X) = S \star N(X) &= S(X)N(X) \\ &= X^{-1}N(X) = X^{-1}\sum_{n=1}^{\infty} n(Ra)^{[n]} \end{aligned}$$

Theorem: The action of the Dynkin map on the generators $(Ra)^{[n]}$ of the Spitzer algebra $\mathcal{S}$ is given by

$$D((Ra)^{[n]}) = C^{(n)}(a)$$

where $C^{(n)}(a) := R\Big(\cdots\Big((a \bullet_R a) \bullet_R a\Big)\cdots \bullet_R a\Big) =: R(c^{(n)}(a))$.

Corollary: We have the following identity in the Spitzer algebra $\mathcal{S}$

$$(Ra)^{[n]} = \sum_{\substack{i_1+\cdots+i_k=n,\\ i_1,\ldots,i_k>0}} \frac{C^{(i_1)}(a)\cdots C^{(i_k)}(a)}{i_1(i_1+i_2)\cdots(i_1+\cdots+i_k)}.$$

$$X = \sum_{n=0}^{\infty}(Ra)^{[n]} \;=\; \sum_{n=0}^{\infty}\sum_{\substack{i_1+\cdots+i_k=n,\\ i_1,\ldots,i_k>0}} \frac{C^{(i_1)}(a)\cdots C^{(i_k)}(a)}{i_1(i_1+i_2)\cdots(i_1+\cdots+i_k)}$$

$$= \; 1 + R\Big(\sum_{n=1}^{\infty}\sum_{\substack{i_1+\cdots+i_k=n,\\ i_1,\ldots,i_k>0}} \frac{c^{(i_1)}(a) *_R \cdots *_R c^{(i_k)}(a)}{i_1(i_1+i_2)\cdots(i_1+\cdots+i_k)}\Big)$$

$$\underbrace{R\Big(R(\cdots R(R}_{n\text{-times}}(a)a)a\cdots)a\Big) = R\Big(\sum_{\substack{i_1+\cdots+i_k=n,\\ i_1,\ldots,i_k>0}} \frac{c^{(i_1)}(a) *_R \cdots *_R c^{(i_k)}(a)}{i_1(i_1+i_2)\cdots(i_1+\cdots+i_k)}\Big)$$

Non-commutative Bohnenblust–Spitzer Identity

Theorem: Let (A, R) be an associative Rota–Baxter algebra. For $a_i \in A$, $i = 1, \ldots, n$, we have:

$$\sum_{\sigma\in S_n} R\Big(R\Big(\cdots(Ra_{\sigma(1)})a_{\sigma(2)}\cdots\Big)a_{\sigma(n)}\Big) = \sum_{\sigma\in S_n} R\Big(a_{\sigma(1)} \diamond_1 a_{\sigma(2)} \diamond_2 \cdots \diamond_n a_{\sigma(n)}\Big),$$

$$a_{\sigma(i)} \diamond_i a_{\sigma(i+1)} = \begin{cases} a_{\sigma(i)} *_R a_{\sigma(i+1)}, & \max\,(\sigma(j)|j \leq i) < \sigma(i+1) \\ a_{\sigma(i)} \bullet_R a_{\sigma(i+1)}, & \text{otherwise;} \end{cases}$$

consecutive $\bullet_R$ products should be performed from left to right, and always before the $*_R$ product.

$$X = \sum_{n=0}^{\infty} t^n (Ra)^{[n]} = \mathsf{exp}\Big(R(\Omega'(a)\Big)$$

$$\Omega'(\lambda a) = \sum_{m \geq 0} \frac{B_m}{m!} \; L_{\bullet_R}^{(m)}[\Omega'](\lambda a)$$

$$N((Ra)^{[n]}) = n(Ra)^{[n]} \longleftrightarrow t\frac{d}{dt}t^n(Ra)^{[n]}$$

$$D(X) = X^{-1}t\frac{d}{dt}X = \sum_{n>0} t^n C^{(n)}(a) \quad \longrightarrow \quad \frac{d}{dt}X(t) = X(t)\,\psi(t)$$

$$\psi(t) := \sum_{n>0} t^{n-1} C^{(n)}(a)$$

Which is solved in terms of Magnus' expansion

$$X(t) = \exp\Big(\Omega(\psi)(t)\Big) \qquad\qquad \Omega(\psi)(t) = \sum_{n>0} \Omega_n(\psi)t^n$$

Strichartz (and others) found a closed solution for $\Omega(\psi)(t)$ in terms of the *continuous BCH* formula

$$\Omega(\psi)(t) = \sum_{\substack{n>0 \\ \sigma\in S_n}} \frac{(-1)^{d(\sigma)}}{n^2\binom{n-1}{d(\sigma)}} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \Big[[...[\psi(t_{\sigma(1)}), \psi(t_{\sigma(2)})]...], \psi(t_{\sigma(n)})\Big].$$

Recall that an index $i \in \{1, \cdots, n-1\}$ is called a descent of the permutation $\sigma \in S_n$ if $\sigma(i) > \sigma(i+1)$. $d(\sigma)$ *#descents* in $\sigma \in S_n$.

$$\phi_- = \mathbf{1} - R\Big(\phi_- \star (\underbrace{\phi - \mathbf{1}}_{=:a})\Big) \qquad\qquad \phi_- \;=\; \sum_{n\geq 0} (-1)^n (Ra)^{[n]}$$

$$\psi := \sum_{n=1}^{\infty} R\Big((\cdots((a \bullet_R a) \bullet_R a)\cdots) \bullet_R a\Big)$$

THANK YOU!!