Forest-Root Formulas in Statistical Physics

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An interpolation formula

\[ f(0) = - \int_{0}^{\infty} f'(t) dt \]

We’d better assume \( f \) goes to zero at infinity. Note the minus sign!
Generalization: Chevron interpolation

Consider a 2-particle version with

\[ f(t_1, t_2, t_{12}) \]

Interpret \( t_1 \) and \( t_2 \) as the position of 2 particles on the half line and let \( t_{12} = |t_1 - t_2| \). Use subscripts 1, 2, or 12 to indicate partial derivatives:

\[ f(0) = - \int_0^\infty ds (f_1(s, s, 0) + f_2(s, s, 0)) \]
Apply the $N = 1$ formula to $f_1$, integrating wrt $t_2 - t_1 > 0$, $t_2 - t_1 = t_{12}$

And to $f_2$, integrating wrt $t_1 - t_2 > 0$, $t_1 - t_2 = t_{12}$

\[
f(0) = \int_0^\infty dt_1 \int_0^\infty dt_2 (f_{1,2} + f_{1,12}) + (1 \leftrightarrow 2)
\]

since $\frac{dt_{12}}{dt_2} = 1$ for $t_2 > t_1$, and $\frac{dt_{12}}{dt_1} = 1$ for $t_1 > t_2$.

The two $f_{1,2}$ terms combine to form

\[
\int_{\mathbb{R}_+^2} dt_1 dt_2 f_{1,2}
\]

Graphically, there are 3 terms (rooted forests):

- $f_{1,2} \quad \bigcirc \quad \bigcirc$
- $f_{1,12} \quad \bigcirc \quad \bigcirc$
- $f_{2,12} \quad \bigcirc \quad \bigcirc$

\[
X------------------------O
\]
General case

Let $f(t)$ be a smooth function of $\{t_i\}, \{t_{ij}\}$ which tends to zero when any of the $t_i$ tends to infinity.

We allow “one-body” and “two-body” variables only.

Claim:

$$f(0) = \sum_{(F,R)} \int_{\mathbb{R}^N_+} \prod_{r \in R} \left[ -dt_r \right] \prod_{ji \in F} \left[ -d(t_j - t_i) \right] f^{(F,R)}(t)$$

Here $F$ is a forest on $\{1, \ldots, N\}$, each tree has a root.

$R$ is the set of roots. $(F,R)$ denotes derivatives wrt edge variables $t_{ij}$ with $ij$ in $F$ and root variables $t_i$ with $i$ in $R$. 
Inductive proof of Forest-Root formula

Begin with diagonal interpolation as before:

\[ f(0) = - \int_0^\infty ds \sum_{k=1}^{N} f_k(s, \ldots, s, 0, \ldots, 0) \]

Proceed along the wings of the chevron, which now represent the \( N - 1 \) variables not yet differentiated.

Apply F-R formula, case \( N - 1 \), to each of the \( N \) terms

Each root derivative gives 2 terms as in previous example. Trees grow from below or sprout anew:

\[ \text{o becomes } \text{o-----------------X} + \text{o } \text{X} \]
Application to HC gases

Need 2-body interactions.

Full disclosure: AR 95 book has a formula which is related to our F-R formula but the margin is too small to demonstrate it.
Theorem II.2

\[ \exp \left( \sum_{l \in \mathcal{P}_n} u_l \right) = \sum_{\mathcal{F} = \{ \mathcal{T}_1, \ldots, \mathcal{T}_\tau \} \atop \text{u-forest}} \left( \prod_{\nu = 1}^{\tau} \int_0^1 dw_{l,\nu} \right) \left( \prod_{\nu = 1}^{\tau} u_{l,\nu} \right) \exp \left( \sum_{l \in \mathcal{P}_n} w_{l}^{\mathcal{F}}(w) . u_l \right), \quad (II.2) \]

where the summation extends over all possible lengths \( \tau \) of \( \mathcal{F} \), including \( \tau = 0 \) hence the empty forest. To each link of \( \mathcal{F} \) is attached a variable of integration \( w_l \). We define the \( w_{l}^{\mathcal{F}} \) as follows.

\( w_{\{ij\}}^{\mathcal{F}}(w) = 0 \) if \( i \) and \( j \) are not connected by the \( \mathcal{F} \). If \( i \) and \( j \) fall in the support \( C \) of the same tree \( \mathcal{T} \) of \( \mathcal{F} \) then

\[ w_{\{ij\}}^{\mathcal{F}}(w) = \begin{cases} 0 & \text{if } |l_{\mathcal{F}}(i) - l_{\mathcal{F}}(j)| \geq 2 \ (i \text{ and } j \text{ in distant layers}) \\ 1 & \text{if } l_{\mathcal{F}}(i) = l_{\mathcal{F}}(j) \ (i \text{ and } j \text{ in the same layer}) \\ w_{\{ii'\}}^{\mathcal{F}}(w) & \text{if } l_{\mathcal{F}}(i) - 1 = l_{\mathcal{F}}(j) = l_{\mathcal{F}}(i'), \text{ and } \{ii'\} \in \mathcal{T}. \ (i \text{ and } j \text{ in neighboring layers, } i' \text{ is then unique}). \end{cases} \]

In particular, if \( \{ij\} \in \mathcal{F} \), then \( w_{\{ij\}}^{\mathcal{F}}(w) = w_{\{ij\}} \).
HC gas partition function

\[ Z_{HC}(z) = \sum_{N=0}^{\infty} \frac{z^N}{N!} \int_{\Lambda^N} \prod_{i=1}^{N} dx_i \prod_{1 \leq i < j \leq N} U(0, x_{ij}) \]

Here \( x_i \) are spatial variables in \( \mathbb{Z}^D \) or \( \mathbb{R}^D \), \( x_{ij} = x_i - x_j \).

The idea is to pick \( U(t,x) \) so that when all \( t \)’s are zero we get a HC condition \( \Pi_{ij}(1 - I(x_{ij})) \), where \( I(x) \) is the indicator function of a ball about 0. We take for example

\[ U(t, x) = 1 - I(x)\theta(1 - t) \]

So the HC condition abruptly vanishes when \( t > 1 \).
Density of HC gas

\[ \rho_{\text{HC}}(z) = \lim_{\Lambda \to S} \lim_{\epsilon \to 0} \frac{1}{Z_{\text{HC}}(z)} \sum_{N=1}^{\infty} \frac{z^N}{N!} \int \prod_{i=1}^{N} dx_i f(0) \]

Where \( S = \mathbb{R}^d \) or \( \mathbb{Z}^d \) and

\[ f(t) = g(t_1/\epsilon) \prod_{i=2}^{N} g(\epsilon t_i) \prod_{1 \leq i < j \leq N} U(t_{ij}, x_{ij}) \]

\( g \) is a smooth bump function, \( g(0) = 1 \). Apply F-R formula:

\[ f(0) = \sum_{(F,R)} \int_{\mathbb{R}_+^N} \prod_{r \in R} [-dt_r] \prod_{ji \in F} [-d(t_j - t_i)] f^{(F,R)}(t) \]
An Extra Dimension

Apply the (F,R) derivatives to \( f \). Each \( j_i \) in \( F \) hits a \( U \):

\[
\frac{d}{dt}U(t,x) = V(t,x) = I(x)\delta(t-1)
\]

Each \( i \) in \( R \) differentiates a \( g \). For an \( n \)-vertex tree, \(-(g(\varepsilon t)^n)'\) is a spread out probability measure. Free floating trees cancel the partition function. Only one tree remains, the one pinned at 0. The result is:

\[
\sum_{N=1}^{\infty} (-1)^{N-1} \frac{z^N}{(N-1)!} \sum_T \int_{(\mathbb{R}^+ \times S)^{N-1}} \prod_{j_i \in T} [dy_{j_i}V(y_{j_i})] \prod_{j_i \notin T} U(y_{j_i})
\]

Note that \( x \) and \( t \) variables have been combined, \( y=(t,x) \) lives in \( \mathbb{R}^p \times \mathbb{R}^+ \). The trees now live in an extra dimension!
Upward links denote factors of $V$ which link the site below to an element of the ball about that site. Horizontal links not in the tree have hard core exclusion from factors of $U$. 
Directed Branched Polymers

We have proven the following identity, which connects the density of the hard core gas in $D$ dimensions with the generating function of directed branched polymers in $D + 1$ dimensions:

$$\rho_{HC}(z) = -Z_{DBP}(-z)$$
Critical Exponents

In $D = 1$, the nearest neighbor example is a dimer model which has a computable pressure:

$$p(z) = \ln \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 + 4z} \right).$$

This has a square-root singularity at $z_c = -1/4$. The two-dimensional DBP generating function is thus

$$Z_{DBP}(z) = -z \frac{d}{dz} p(-z) = \sum_{N=1}^{\infty} \frac{[2N - 1]!!2^{N-1}z^N}{N!}$$

which has an inverse square-root singularity. The coefficient of $z^N$ counts DBPs and it behaves like

$$d_N \sim z_c^{-N} N^{-\theta} \quad \text{with} \quad \theta = \frac{1}{2}.$$
Other examples

In $D = 2$ the density of the hard hexagon model behaves as $(z-z_c)^{1-\alpha}$ with $\alpha = 7/6$. This implies $\theta = 5/6$ for a three-dimensional DBP model based on it.

Continuous examples have similar behavior.
Related Models

Directed Animals: There are exact results in two dimensions by Dhar, Sumedha, Bousquet-Melou, etc. coming from a representation as HC gas dynamics (80’s). Results on exponents similar to those for DBP.

Di Francesco & Guitter 02: Lorentzian semi-random lattices (2d) are related to directed animals and also to D=1 HC gases.
A 2d Forest-Root Formula

As before we interpolate a function $f(t)$ but now

$$t_{ij} = |w_i - w_j|^2, \quad t_i = |w_i|^2 \text{ with } w_i \in \mathbb{C}.$$ 

$w_i$ can be thought of as the position of a particle in the plane instead of the half-line. The new formula is

$$f(0) = \sum_{(F,R)} \int_{\mathbb{C}^N} f^{(F,R)}(t) \left( \frac{d^2 w}{-\pi} \right)^N$$

Case $N = 1$:

$$f(0) = \int_{\mathbb{C}} f'(t) \frac{d^2 w}{-\pi} = -\int_0^\infty f'(t) dt$$
Differential Forms

We need Grassman variables to get anywhere with this identity. It is natural to introduce them as differential forms as in Brydges’s lecture.

\[ \tau_i = w_i \bar{w}_i + \frac{d w_i \wedge d \bar{w}_i}{2\pi i} \]

\[ \tau_{ij} = w_{ij} \bar{w}_{ij} + \frac{d w_{ij} \wedge d \bar{w}_{ij}}{2\pi i} \]

Then \( f(\tau) \) is defined by its Taylor series.

Suppose we generate a loop of 2-forms on edges \( ij \). Since \( w_{ij} = w_i - w_j \), there is a relation and the product of 2-forms is 0. Therefore only forests appear in the Taylor series.
Each tree of the forest has a root which corresponds to a 2-form

\[ \frac{dw_i \wedge d\bar{w}_i}{2\pi i} \]

This is needed to keep the total form degree equal to 2N and thereby allow the term to be nonvanishing.
F-R formula in form language

When expanded out, the equation

$$\int_{C^N} f(\mathcal{T}) = f(0)$$

turns into the Forest-Root formula:

$$f(0) = \sum_{(F,R)} \int_{C^N} f^{(F,R)}(t) \left( \frac{d^2 w}{-\pi} \right)^N$$

The constants are explained by the change of variable:

$$dw_i \wedge d\bar{w}_i = -2idx_i \wedge dy_i$$
Feynman parameters

There is a "cosmological" proof of the F-R formula, using supersymmetry, but here I will give a more earthly argument which goes back to Brydges-Wright 88. Compute the Laplace transform of $f$:

$$\hat{f}(a) = \int_0^\infty dt f(t) e^{a \cdot t}$$

Here $a = (a_i), (a_{ij})$ is dual to $t$, that is

$$a \cdot t = \Sigma_{ij} a_{ij} t_{ij} + \Sigma_i a_i t_i.$$

Note that $t_i = |x_i|^2$, $t_{ij} = |x_i - x_j|^2$. One could also use $p$'s instead of $x$'s which would make it a momentum space formula, and then the $a$'s are Feynman parameters.
Linearity

Assume $f$ decays exponentially in each $t_i$. Then the $a$’s have positive real part in the inverse Laplace transform

$$f(t) = \int \prod_i \frac{da_i}{2\pi i} \prod_{ij} \frac{da_{ij}}{2\pi i} \hat{f}(a) e^{-a \cdot t}$$

By linearity, the F-R formula reduces to the case $f(t) = e^{-a \cdot t}$.
Quadratic form

Note that

\[ a \cdot t = \langle w, A\bar{w} \rangle + \left\langle \frac{dw}{\sqrt{2\pi i}}, A\frac{d\bar{w}}{\sqrt{2\pi i}} \right\rangle \]

where

\[
A_{ij} = -a_{ij}, \quad i \neq j, \\
A_{ii} = a_i + \sum_{j \neq i} a_{ij}.
\]

Get this by expanding into four terms the two-form

\[ a_{ij} (dw_i - dw_j) \wedge (d\bar{w}_i - d\bar{w}_j) \]

and combining coefficients for each monomial \( dw_i \wedge d\bar{w}_j \)
Determinants cancel

\[ \int e^{-a \cdot t} = \int e^{-\langle w, A\bar{w} \rangle} \exp \left[ -\left\langle \frac{dw}{\sqrt{2\pi i}}, A \frac{d\bar{w}}{\sqrt{2\pi i}} \right\rangle \right] \]

\[ = \int e^{-\langle w, A\bar{w} \rangle} \det A \prod_{i=1}^{N} \left[ -\frac{dw_i \wedge d\bar{w}_i}{2\pi i} \right] . \]

The “Fermionic” determinant is actually a Jacobian. It cancels the “Bosonic” determinant that arises from doing the Gaussian integral over \( w \). This proves the F-R formula for exponentials.

\[ \pi^{-N} \det A \int e^{-\langle w, A\bar{w} \rangle} d^N x \, d^N y = 1 \]
Matrix-Tree theorem

We have learned that

\[ 1 = \int e^{-a \cdot t} = \sum_{(F,R)} \int e^{-a \cdot t(F,R)(t)} \left( \frac{d^2 w}{-\pi} \right)^N \]

Perform the derivatives to obtain

\[ 1 = \sum_{(F,R)} \prod_{i \in R} a_i \prod_{ij \in F} a_{ij} \int e^{-\langle w, A\bar{w} \rangle} \left( \frac{d^2 w}{-\pi} \right)^N \]

\[ \det A = \sum_{(F,R)} \prod_{i \in R} a_i \prod_{ij \in F} a_{ij} \]

, so
and one can express the a’s in terms of the matrix entries $A_{ij}$:

$$a_i = \sum_{j=1}^{N} A_{ij}, \quad a_{ij} = -A_{ij}$$

which gives the usual form of the matrix tree theorem. (If one puts $a_i = 0$ one gets a sum over trees only.) Grassman variable proofs of matrix-tree theorems appear in AA04 and CJSSS04.
More dimensional reduction

A new interpolation formula should lead to new results for the HC gas. We restrict attention to the continuum, so let $x_i$ be in $\mathbb{R}^D$.

Use the following HC weight for hard disks:

$$U(t_{ij}) = \theta(t_{ij} - 1)$$

Then

$$Z_{HC}(z) = \sum_{N=0}^{\infty} \frac{z^N}{N!} \int_{\Lambda^N} dx_1 \cdots dx_N \prod_{1 \leq i < j \leq N} U(|x_i - x_j|^2)$$
Two extra dimensions

The integrand is

\[ f(0) = \prod_{1 \leq i < j \leq N} U(|x_{ij}|^2) \]

Extend to nonzero t:

\[ f(t) = \prod_{1 \leq i < j \leq N} U(|x_{ij}|^2 + t_{ij}) \times \text{(large t cutoff)} \]

With \( y_{ij} = (x_{ij}, w_{ij}) \) in \( \mathbb{R}^{D+2} \), U now represents a hard core condition in \( D + 2 \) dimensions.
Branched Polymers in dimension $D + 2$

Apply the Forest-Root Formula:

$$f(0) = \sum_{(F,R)} \int_{\mathbb{C}^N} f^{(F,R)}(t) \left( \frac{d^2 w}{-\pi} \right)^N$$

Each $d/dt_{ij}$, when applied to $U_{ij}$, becomes $\frac{1}{2}$ surface measure for the integration over $y_{ij} = (x_{ij}, w_{ij})$ in $\mathbb{R}^{D+2}$. The spheres are stuck together according to the forest $F$. 
Dimensional Reduction

Thinking of the trees as connected Mayer graphs, they become independent of each other as the large $t$ cutoff is removed. Then we can compute the logarithm and the pressure as

$$\lim_{\Lambda \to \mathbb{R}^D} \frac{1}{|\Lambda|} \log Z_{HC}(z) = -2\pi Z_{BP} \left(-\frac{z}{2\pi}\right)$$

where

$$Z_{BP}(z) = \sum_{N=1}^{\infty} \frac{z^N}{N!} \sum_T \int dy_2 \cdots dy_N \prod_{ij \in T} [2U_{ij}'] \prod_{ij \notin T} U_{ij}$$

is the generating function for branched polymers in dim $D + 2$. Here

$$2U_{ij}' = 2U'(\|y_i - y_j\|^2) = \delta(\|y_i - y_j\| - 1)$$
Critical Exponents

The hard disk pressure is computable in $D = 0$ or $1$, so we learn the exact form of the BP generating function in dimensions $2$:

$$Z_{BP}(z) = -\frac{1}{2\pi} \log(1 - 2\pi z)$$

and $3$:

$$Z_{BP}(z) = \frac{1}{2\pi} T(2\pi z) = \sum_{N=1}^{\infty} \frac{N^{N-1}}{2\pi N!} (2\pi z)^N$$

From this we obtain the volume exponents:

$\theta = 1$ in $D + 2 = 2$

$\theta = 3/2$ in $D + 2 = 3$
Remarks

Due to the alternating nature of the Mayer expansion, the first singularity of the pressure is at negative activity. It is one of several systems classified with the Lee-Yang edge singularity (Ising in imaginary field).

We confirm the Parisi-Sourlas predicted relation between $\theta$ and the Lee-Yang edge exponent.

Kenyon-Winkler 07: More invariances for the hard sphere BP model, allow varying radii in dim 2.

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