A Rosetta Stone: Combinatorics, Physics, and Probability

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Outline

1. Enumerative combinatorics
2. Equilibrium particle systems
3. Trees versus enriched trees
Colored sets


Fix $\mathcal{P}$, a finite set. This is the palette of colors.

If $U$ is a finite set of labels, then $a : U \rightarrow \mathcal{P}$ is a colored set.

Two colored sets $a$ and $a'$ are isomorphic if there is a bijection $f : U \rightarrow U'$ with $a' \circ f = a$.

If $a : U \rightarrow \mathcal{P}$ is a colored set, then the corresponding occupation number function (multi-index) is defined for $p \in \mathcal{P}$ by

$$N_a(p) = \#\{j \mid a(j) = p\}.$$
Species of structures

A species $F$ assigns to each colored set a corresponding weighted set. Example: Suppose that there are weights depend on the colors. Thus for each pair of colors $p, q$ we have $t(p, q) = t(q, p)$.

Let $G$ be the species of graphs. For each colored set $a : U \to \mathcal{P}$ the set $G[a]$ consists of all (simple) graphs $G$ with vertex set $U$. The weight of each graph is

$$\text{wt}(G) = \prod_{\{i, j\} \in E(G)} t(a(i), a(j)).$$
Exponential generating functions

Let $w_p$ be a variable for each color $p$. The exponential generating function of the species $F$ is

$$F(w) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{a:[n] \to \mathcal{P}} f(N_a) \prod_{j \in [n]} w_{a(j)}.$$ 

Here

$$f(N_a) = \sum_{G \in F[a]} \text{wt}(G).$$

Example: For graphs

$$G(w) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{a:[n] \to \mathcal{P}} \prod_{\{i,j\}} (1 + t(a(i), a(j))) \prod_{j \in [n]} w_{a(j)}.$$
The exponential generating function for graphs

We can also write

\[ F(w) = \sum_N \frac{1}{N!} f(N)w^N. \]

Here \( N! = \prod P N(p)! \) and \( w_N = \prod P w_p^{N(p)}. \)

Example: For graphs

\[ G(w) = \sum_N \frac{1}{N!} (1 + t)^{Pair(N)} w^N, \]

where \( Pair(N)({p, q}) = N(p)N(q) \) for \( p \neq q \) and \( Pair(N)({p, p}) = \binom{N(p)}{2}. \)
Graph examples

- $G$ graphs $G(w) = \sum N \frac{1}{N!} (1 + t)^{\text{Pair}(N)} w^N$.
- $C$ connected graphs $G(w) = \exp(C(w))$.
- $G_p^\bullet$ rooted graphs $G_p^\bullet(w) = w_p \frac{\partial}{\partial w_p} G(w)$.
- $C_p^\bullet$ rooted connected graphs $C_p^\bullet(w) = w_p \frac{\partial}{\partial w_p} C(w)$.
- $T$ trees
- $T_p^\bullet$ rooted trees $T_p^\bullet(w) = w_p \frac{\partial}{\partial w_p} T(w)$. 
Operations on species

Good operations: $a : U \rightarrow \mathcal{P}$.

- Convolution product: $(F \ast G)[a] = \bigsqcup_{(V, W)} F[a_V] \times G[a_W]$, where $V \cup W = U$, $V \cap W = \emptyset$.

- Composition: $(F \circ G)[a] = \bigsqcup_{c: \Gamma \rightarrow \mathcal{P}} F[c] \times \prod_{V \in \Gamma} G_{c(V)}[a_V]$, where $\Gamma$ is a partition of $U$ and $c : \Gamma \rightarrow \mathcal{P}$ is a colored partition.

- Point (root): $F_p \bullet [a] = \bigsqcup_{j:a(j)=p} F(a)$.

Bad operation:

- Cartesian product: $(F \times G)[a] = F[a] \times G[a]$. 
Graph equations

Species

- \( G = E \circ C. \)
- \( G_p^\bullet = C_p^\bullet \ast G. \)
- \( T_p^\bullet = E_{1p} \ast (E \circ E_1^p \circ T^\bullet). \)

In words

- Graph = partition, Connected graphs.
- Rooted Graph = Rooted Connected Graph on subset, Graph on complement.
- Rooted Tree = Root, partition of complement, Rooted Trees, Edge to each new root.

Exponential generating functions

- \( G(w) = \exp(C(w)). \)
- \( G_p^\bullet(w) = C_p^\bullet(w)G(w). \)
- \( T_p^\bullet(w) = w_p \exp(\sum_q t(p, q)T_q^\bullet(w)). \)
More graph equations

Species

$G_p^\bullet = E_1 p \star (G \times P^p)$.

$C_p^\bullet = E_1 p \star (E \circ (C \times P^p_+))$.

In words

- Rooted Graph $=$ Root, (Graph on complement, Edges to subset of complement).
- Rooted Connected Graph $=$ Root, partition of complement, (Connected Graphs, Edges to non-empty subsets).

Exponential generating functions

$G_p^\bullet(w) = w_p \cdot G((1 + t_p)w)$.

$C_p^\bullet(w) = w_p \cdot \exp(C((1 + t_p)w) - C(w))$.

Closed form equation for rooted connected graphs

$C_p^\bullet(w) = w_p \cdot \exp(\sum q \cdot t(p, q) \int_0^1 C_q((1 + s t_p)w) \, ds)$. 
The equation for rooted trees with root of color $p$ is

$$w_p \exp \left( \sum_{q} t(p, q) T_q^\bullet (w) \right) = T_p^\bullet (w).$$

It is a fixed point equation

$$w \exp(Tz) = z.$$

Take $w \geq 0$ and $T \geq 0$. Then this is the fixed point equation

$$\phi(z) = z$$

for an increasing function. When does it have a finite solution $z \geq 0$?
Fixed point equations

Let $\phi$ be an increasing function from a complete lattice to itself. If $S = \{x \mid \phi(x) \leq x\}$, then $z = \inf S$ satisfies $\phi(z) = z$. Here the complete lattice is $[0, +\infty]^P$ and $\phi(z) = w \exp(Tz)$. There is always a fixed point. Is it finite? The Kotecký-Preiss condition is that there is a finite vector $x \geq 0$ such that

$$\phi(x) \leq x.$$ 

This condition is necessary and sufficient for a finite fixed point.
Solution by iteration

Let $\phi$ be an increasing function from a complete lattice to itself. Suppose $\phi$ satisfies monotone convergence.

Define $z^{(0)} = 0$ (least element) and

$$z^{(n+1)} = \phi(z^{(n)}).$$

By induction $z^{(n)} \leq z^{(n+1)}$ is increasing. Let $z$ be the least fixed point.

By induction $z^{(n)} \leq z$. Let $z' = \sup_n z^{(n)}$. Then $z' \leq z$. On the other hand, by monotone convergence

$$z' = \sup_n z^{(n+1)} = \sup_n \phi(z^{(n)}) = \phi(\sup_n z^{(n)}) = \phi(z').$$

So $z'$ is a fixed point. Thus $z \leq z'$. It follows that $z' = z$.

Conclusion: $z' = \sup_n z^{(n)}$ is the least fixed point $z$. 
Other forms of the condition

Say we have a fixed point equation

\[ z = \phi(z) = wF(z). \]

We want a fixed point that gives \( z \) as a function of \( w \).
We can write this as

\[ w = \frac{z}{F(z)}. \]

So we must invert this function. Lagrange inversion might help?
Kotecký-Preiss becomes the existence of a finite \( x \) with

\[ w \leq \frac{x}{F(x)}. \]

This is not illuminating, since there is no intermediate value theorem for many dimensions.
Lattice gas physical interpretation

Here is a probability model for a lattice gas.

- The set of colors $\mathcal{P}$ is a set of locations.
- A label set $U$ is a set of particles.
- A colored set $a : U \rightarrow \mathcal{P}$ is a particle configuration.
- The weight $0 \leq 1 + t(p, q) \leq 1$ is a repulsive interaction between a particle at location $p$ and a particle at location $q$.
- Particle configuration $a$ has Boltzmann factor

$$g(N_a) = \prod_{\{i,j\}} (1 + t(a(i), a(j))) = (1 + t)^{\text{Pair}(N)}.$$

- The variable $w_p \geq 0$ is a prior weight (activity) for having a particle at point $p$. 
The probability model

The discrete probability density for particle configuration $a$ is

$$\frac{1}{G(w)} \frac{1}{n!} g(N_a) \prod_j w_{a(j)}.$$ 

Here

$$G(w) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{a:[n] \rightarrow \mathcal{P}} g(N_a) \prod_{j \in [n]} w_{a(j)}.$$ 

is the grand partition function.

The discrete probability density for occupation number $N$ is

$$\frac{1}{G(w)} \frac{1}{N!} g(N) w^N.$$ 

Here

$$G(w) = \sum_N \frac{1}{N!} g(N) w^N.$$ 

is the grand partition function.
Particles at a location

$C(w)$ is the pressure.
The expected number of particles at location $p$ is

$$\frac{1}{G(w)} \sum_{N} \frac{1}{N!} N(p)g(N)w^{N} = \frac{1}{G(w)} G_{p}(w).$$

This is just

$$\frac{1}{G(w)} G_{p}(w) = C_{p}(w).$$
Cluster expansion theorem. Let $-1 \leq t(p, q) \leq 0$ and $w_p \geq 0$. Suppose the Kotecký-Preiss condition. Then

$$0 \leq C_p^\bullet(w) \leq -C_p^\bullet(-w) \leq T_p^\bullet(w) < +\infty.$$ 

where the weights for the rooted trees with root of color $p$ is computed with weights $0 \leq |t(p, q)| \leq 1$. 

Connected graphs versus trees
Proof sketch

Recall

\[ C_p^\bullet(w) = w_p \exp\left(\sum_q t(p, q) \int_0^1 C_q^\bullet((1 + st_p)w) \, ds\right). \]

Let \( \tilde{C}_p^\bullet(w) = -C^\bullet(-w) \). Then

\[ \tilde{C}_p^\bullet(w) = w_p \exp\left(\sum_q |t(p, q)| \int_0^1 \tilde{C}_q^\bullet((1 + st_p)w) \, ds\right). \]

The right hand size is increasing. Also, since \( t_p \leq 0 \) we have

\[ \int_0^1 \tilde{C}_q^\bullet((1 + st_p)w) \, ds \leq \int_0^1 \tilde{C}_q^\bullet(w) \, ds = \tilde{C}_q^\bullet(w). \]

So one can bound \( \tilde{C}_q^\bullet(w) \) by the tree fixed point \( T_q^\bullet(w) \).
The Fernández-Procacccci result

Let $g(N) = (1 + t)^{\text{Pair}}(N)$. Let

$$F_p(x) = \sum_{N} \frac{1}{N!} |t_p|^N g(N) x^N$$

Fernández-Procacci result. Take $-1 \leq t(p, q) \leq 0$, and take the activities $w_q \geq 0$. Suppose that there exists a finite vector $x \geq 0$ such that

$$w_p F_p(x) \leq x_p.$$

Then the power series expansion of the expected number $C_p^\bullet(w)$ of particles at location $p$ converges absolutely for the given $w$ and has absolute value bounded by $x$.

Notice that since $g(N) \leq 1$ the Kotecký-Preiss condition

$$w_p \sum_{N} \frac{1}{N!} (|t_p|x)^N = w_p \exp(|t_p| \cdot x) \leq x_p$$

implies the Fernández-Procacci condition.
The connected graph identity

This proof uses a connected graph identity of the Brydges-Kennedy-Abdesselam-Rivasseau type. Write $t_{ij} = t(a(i), a(j))$. The following enriched tree bound is an immediate corollary. Continue to suppose that $-1 \leq t_{ij} \leq 0$. Then

\[
\left| \sum_{G_c} \prod_{\{i,j\} \in G_c} t_{ij} \right| \leq \sum_{\tau} c(\tau) g(\tau).
\]

Here the tree weight is

\[
c(\tau) = \prod_{i \neq r} |t_{i\tau(i)}|.
\]

The fiber weight is

\[
g(\tau) = \prod_{\{i,j\} \mid i \downarrow j \mod \tau} (1 + t_{ij})
\]

with $i \downarrow j \mod \tau$ meaning that $\tau(i) = \tau(j)$. This is the weight from edges $\{i, j\}$ that belong to the fiber over a vertex.
The Fernández-Procacci fixed point

Let $F_p(x) = \sum_N \frac{1}{N!} |t_p|^N g(N)x^N$. The exponential generating function $\tilde{T}^\bullet(w)$ for enriched rooted trees satisfies the fixed point equation

$$\tilde{T}^\bullet_p(w) = w_p F_p(\tilde{T}^\bullet(w)).$$

This is of the form

$$z = \phi(z) = wF(z).$$

The Fernández-Procacci condition is that this equation has a finite fixed point.

Since by the enriched tree bound

$$0 \leq C^\bullet_p(w) \leq \tilde{T}^\bullet_p(w)$$

we then have convergence of the cluster expansion.
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