A Rosetta Stone: Combinatorics, Physics, and Probability

William G. Faris

Isaac Newton Institute, April 11, 2008

Outline

1. Enumerative combinatorics
2. Equilibrium particle systems
3. Trees versus enriched trees

Colored sets

F. Bergeron, G. Labelle, and P. Leroux, Combinatorial Species and Tree-like Structures, Cambridge, 1998.

Fix $\mathcal{P}$, a finite set. This is the palette of colors.

If U is a finite set of labels, then $a : U \to \mathcal{P}$ is a colored set.

Two colored sets a and a' are isomorphic if there is a bijection $f : U \to U'$ with $a' \circ f = a$.

If $a : U \to \mathcal{P}$ is a colored set, then the corresponding occupation number function (multi-index) is defined for $p \in \mathcal{P}$ by

$$N_a(p) = \#\{j \mid a(j) = p\}.$$

Species of structures

A species F assigns to each colored set a corresponding weighted set.
Example: Suppose that there are weights depend on the colors. Thus for each pair of colors p, q we have $t(p, q) = t(q, p)$.
Let G be the species of graphs. For each colored set $a : U \to \mathcal{P}$ the set $G[a]$ consists of all (simple) graphs G with vertex set U. The weight of each graph is

$$\mathrm{wt}(G) = \prod_{\{i,j\} \in E(G)} t(a(i), a(j)).$$

Exponential generating functions

Let w_p be a variable for each color p. The exponential generating function of the species F is

$$F(w) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{a:[n]\to\mathcal{P}} f(N_a) \prod_{j\in[n]} w_{a(j)}.$$

Here

$$f(N_a) = \sum_{G\in F[a]} \mathrm{wt}(G).$$

Example: For graphs

$$G(w) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{a:[n]\to\mathcal{P}} \prod_{\{i,j\}} (1 + t(a(i), a(j))) \prod_{j\in[n]} w_{a(j)}.$$

The exponential generating function for graphs

We can also write

$$F(w) = \sum_N \frac{1}{N!} f(N) w^N.$$

Here $N! = \prod_p N(p)!$ and $w_N = \prod_p w_p^{N(p)}$.
Example: For graphs

$$G(w) = \sum_N \frac{1}{N!} (1+t)^{\mathrm{Pair}(N)} w^N,$$

where $\mathrm{Pair}(N)(\{p,q\}) = N(p)N(q)$ for $p \neq q$ and $\mathrm{Pair}(N)(\{p,p\}) = \binom{N(p)}{2}$.

Graph examples

- G graphs $G(w) = \sum_N \frac{1}{N!}(1+t)^{\mathrm{Pair}(N)} w^N$.
- C connected graphs $G(w) = \exp(C(w))$.
- $G_p^\bullet$ rooted graphs $G_p^\bullet(w) = w_p \frac{\partial}{\partial w_p} G(w)$.
- $C_p^\bullet$ rooted connected graphs $C_p^\bullet(w) = w_p \frac{\partial}{\partial w_p} C(w)$.
- T trees
- $T_p^\bullet$ rooted trees $T_p^\bullet(w) = w_p \frac{\partial}{\partial w_p} T(w)$.

Operations on species

Good operations: $a : U \to \mathcal{P}$.

- Convolution product: $(F * G)[a] = \bigsqcup_{\langle V,W \rangle} F[a_V] \times G[a_W]$, where $V \cup W = U$, $V \cap W = \emptyset$.
- Composition: $(F \circ G)[a] = \bigsqcup_{c:\Gamma \to \mathcal{P}} F[c] \times \prod_{V \in \Gamma} G_{c(V)}[a_V]$, where Γ is a partition of U and $c : \Gamma \to \mathcal{P}$ is a colored partition.
- Point (root): $F_p^{\bullet}[a] = \bigsqcup_{j:a(j)=p} F(a)$.

Bad operation:

- Cartesian product: $(F \times G)[a] = F[a] \times G[a]$.

Graph equations

Species

- $G = E \circ C$.
- $G_p^\bullet = C_p^\bullet * G$.
- $T_p^\bullet = E_{1p} * (E \circ E_1^p \circ T^\bullet)$.

In words

- Graph = partition, Connected graphs.
- Rooted Graph = Rooted Connected Graph on subset, Graph on complement.
- Rooted Tree = Root, partition of complement, Rooted Trees, Edge to each new root.

Exponential generating functions

- $G(w) = \exp(C(w))$.
- $G_p^\bullet(w) = C_p^\bullet(w) G(w)$.
- $T_p^\bullet(w) = w_p \exp(\sum_q t(p,q) T_q^\bullet(w))$.

More graph equations

Species

- $G_p^\bullet = E_{1p} * (G \times P^p)$.
- $C_p^\bullet = E_{1p} * (E \circ (C \times P_+^p))$.

In words

- Rooted Graph = Root, (Graph on complement, Edges to subset of complement).
- Rooted Connected Graph = Root, partition of complement, (Connected Graphs, Edges to non-empty subsets).

Exponential generating functions

- $G_p^\bullet(w) = w_p G((1 + t_p)w)$.
- $C_p^\bullet(w) = w_p \exp(C((1 + t_p)w) - C(w))$.

Closed form equation for rooted connected graphs

- $C_p^\bullet(w) = w_p \exp(\sum_q t(p, q) \int_0^1 C_q^\bullet((1 + st_p)w)\, ds)$.

The Kotecký-Preiss condition

The equation for rooted trees with root of color p is

$$w_p \exp(\sum_q t(p,q) T_q^\bullet(w)) = T_p^\bullet(w).$$

It is a fixed point equation

$$w \exp(Tz) = z.$$

Take $w \geq 0$ and $T \geq 0$. Then this is the fixed point equation

$$\phi(z) = z$$

for an increasing function. When does it have a finite solution $z \geq 0$?

Fixed point equations

Let ϕ be an increasing function from a complete lattice to itself. If $S = \{x \mid \phi(x) \leq x\}$, then $z = \inf S$ satisfies $\phi(z) = z$.
Here the complete lattice is $[0, +\infty]^{\mathcal{P}}$ and $\phi(z) = w \exp(Tz)$. There is always a fixed point. Is it finite?
The Kotecký-Preiss condition is that there is there is a finite vector $x \geq 0$ such that

$$\phi(x) \leq x.$$

This condition is necessary and sufficient for a finite fixed point.

Solution by iteration

Let ϕ be an increasing function from a complete lattice to itself. Suppose ϕ satisfies monotone convergence
Define $z^{(0)} = 0$ (least element) and

$$z^{(n+1)} = \phi(z^{(n)}).$$

By induction $z^{(n)} \leq z^{(n+1)}$ is increasing. Let z be the least fixed point. By induction $z^{(n)} \leq z$. Let $z' = \sup_n z^{(n)}$. Then $z' \leq z$. On the other hand, by monotone convergence

$$z' = \sup_n z^{(n+1)} = \sup_n \phi(z^{(n)}) = \phi(\sup_n z^{(n)}) = \phi(z').$$

So z' is a fixed point. Thus $z \leq z'$. It follows that $z' = z$.
Conclusion: $z' = \sup_n z^{(n)}$ is the least fixed point z.

Other forms of the condition

Say we have a fixed point equation

$$z = \phi(z) = wF(z).$$

We want a fixed point that gives z as a function of w.
We can write this as

$$w = \frac{z}{F(z)}.$$

So we must invert this function. Lagrange inversion might help?
Kotecký-Preiss becomes the existence of a finite x with

$$w \leq \frac{x}{F(x)}.$$

This is not illuminating, since there is no intermediate value theorem for many dimensions.

Lattice gas physical interpretation

Here is a probability model for a lattice gas.

- The set of colors $\mathcal{P}$ is a set of locations.
- A label set U is a set of particles.
- A colored set $a : U \to \mathcal{P}$ is a particle configuration.

- The weight $0 \leq 1 + t(p, q) \leq 1$ is a repulsive interaction between a particle at location p and a particle at location q.
- Particle configuration a has Boltzmann factor

$$g(N_a) = \prod_{\{i,j\}} (1 + t(a(i), a(j))) = (1 + t)^{\mathrm{Pair}(N)}.$$

- The variable $w_p \geq 0$ is a prior weight (activity) for having a particle at point p.

The probability model

The discrete probability density for particle configuration a is

$$\frac{1}{G(w)} \frac{1}{n!} g(N_a) \prod_j w_{a(j)}.$$

Here

$$G(w) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{a:[n]\to\mathcal{P}} g(N_a) \prod_{j\in[n]} w_{a(j)}.$$

is the grand partition function.
The discrete probability density for occupation number N is

$$\frac{1}{G(w)} \frac{1}{N!} g(N) w^N.$$

Here

$$G(w) = \sum_N \frac{1}{N!} g(N) w^N.$$

is the grand partition function.

Particles at a location

$C(w)$ is the pressure.
The expected number of particles at location p is

$$\frac{1}{G(w)} \sum_N \frac{1}{N!} N(p) g(N) w^N = \frac{1}{G(w)} G_p^\bullet(w).$$

This is just

$$\frac{1}{G(w)} G_p^\bullet(w) = C_p^\bullet(w).$$

Connected graphs versus trees

Cluster expansion theorem. Let $-1 \leq t(p,q) \leq 0$ and $w_p \geq 0$. Suppose the Kotecký-Preiss condition. Then

$$0 \leq C_p^\bullet(w) \leq -C_p^\bullet(-w) \leq T_p^\bullet(w) < +\infty.$$

where the weights for the rooted trees with root of color p is computed with weights $0 \leq |t(p,q)| \leq 1$.

Proof sketch

Recall

$$C_p^\bullet(w) = w_p \exp(\sum_q t(p,q) \int_0^1 C_q^\bullet((1+st_p)w)\, ds).$$

Let $\check{C}_p^\bullet(w) = -C^\bullet(-w)$. Then

$$\check{C}_p^\bullet(w) = w_p \exp(\sum_q |t(p,q)| \int_0^1 \check{C}_q^\bullet((1+st_p)w)\, ds).$$

The right hand size is increasing. Also, since $t_p \leq 0$ we have

$$\int_0^1 \check{C}_q^\bullet((1+st_p)w)\, ds \leq \int_0^1 \check{C}_q^\bullet(w)\, ds = \check{C}_q^\bullet(w).$$

So one can bound $\check{C}_q^\bullet(w)$ by the tree fixed point $T_q^\bullet(w)$.

The Fernández-Procaccci result

Let $g(N) = (1+t)^{\mathrm{Pair}}(N)$. Let

$$F_p(x) = \sum_N \frac{1}{N!} |t_p|^N g(N) x^N$$

Fernández-Procacci result. Take $-1 \leq t(p,q) \leq 0$, and take the activities $w_q \geq 0$. Suppose that there exists a finite vector $x \geq 0$ such that

$$w_p F_p(x) \leq x_p.$$

Then the power series expansion of the expected number $C_p^\bullet(w)$ of particles at location p converges absolutely for the given w and has absolute value bounded by x.
Notice that since $g(N) \leq 1$ the Kotecký-Preiss condition

$$w_p \sum_N \frac{1}{N!} (|t_p| x)^N = w_p \exp(|t_p| \cdot x) \leq x_p$$

implies the Fernández-Procacci condition.

The connected graph identity

This proof uses a connected graph identity of the Brydges-Kennedy-Abdesselam-Rivasseau type.
Write $t_{ij} = t(a(i), a(j))$.
The following *enriched tree bound* is an immediate corollary. Continue to suppose that $-1 \leq t_{ij} \leq 0$. Then

$$|\sum_{G_c} \prod_{\{i,j\} \in G_c} t_{ij}| \leq \sum_{\tau} c(\tau) g(\tau).$$

Here the tree weight is

$$c(\tau) = \prod_{i \neq r} |t_{i\tau(i)}|.$$

The fiber weight is

$$g(\tau) = \prod_{\{i,j\} | i \downarrow j \bmod \tau} (1 + t_{ij})$$

with $i \downarrow j \bmod \tau$ meaning that $\tau(i) = \tau(j)$. This is the weight from edges $\{i, j\}$ that belong to the fiber over a vertex.

The Fernández-Procacci fixed point

Let $F_p(x) = \sum_N \frac{1}{N!}|t_p|^N g(N) x^N$. The exponential generating function $\tilde{T}^\bullet(w)$ for enriched rooted trees satisfies the fixed point equation

$$\tilde{T}^\bullet_p(w) = w_p F_p(\tilde{T}^\bullet(w)).$$

This is of the form

$$z = \phi(z) = wF(z).$$

The Fernández-Procacci condition is that this equation has a finite fixed point.

Since by the enriched tree bound

$$0 \leq C^\bullet_p(w) \leq \tilde{T}^\bullet_p(w)$$

we then have convergence of the cluster expansion.

A Rosetta Stone

[1] F. Bergeron, G. Labelle, and P. Leroux, Combinatorial Species and Tree-like Structures, Cambridge University Press, Cambridge, 1998.