Vacancy localization in the square dimer model, statistics of geodesics in large quadrangulations

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21 April 2008

Part I

Vacancy localization in the square dimer model

Joint work with M. Bowick[1,2], E. Guitter[1] and M. Jeng[2]
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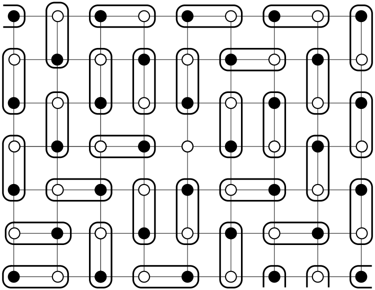
A vacancy in a sea of dimers

The dimer model is a venerable subject in statistical mechanics and combinatorics [Kasteleyn, Fisher, Temperley...].
Here we consider the square lattice, fully covered with dimers except for a few possible vacancies.



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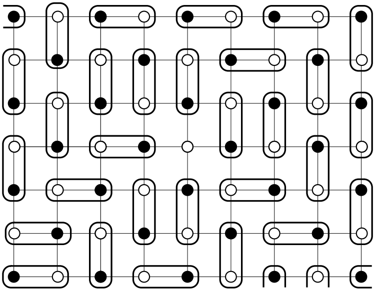

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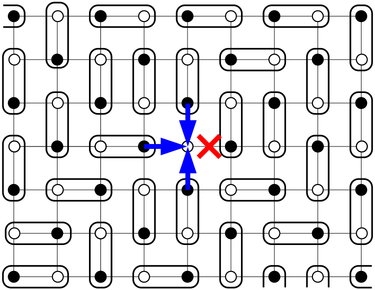

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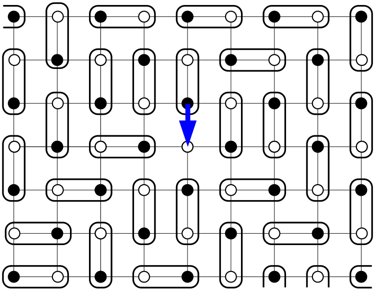

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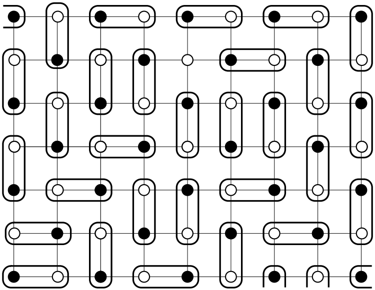

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Questions

Starting from an initial configuration of dimers/vacancy, can we characterize the set of attainable configurations ?

What if the initial configuration is drawn (uniformly) at random ? (toy model for a glassy system)

For simplicity we consider the case of a rectangular grid of finite size (to be taken later to infinity).
Obvious remarks :

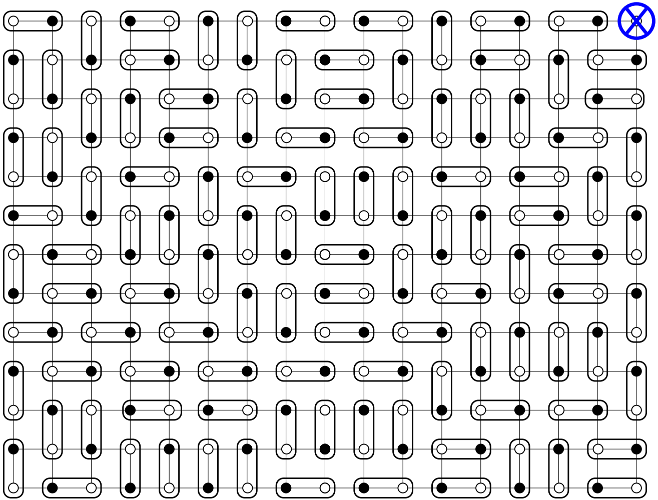
- with one vacancy, the number of sites must be odd : $(2L+1) \times (2M+1)$
- coloring every other site in white and black (with white corners), the vacancy must be on the "white" grid
- the vacancy can reach at most every other site of the white grid ("even/odd" white subgrids, corner : even)

Consider first the case where the vacancy is initially in a corner.
We have the following :

Theorem [Temperley]

There is a bijection between dimer configurations on a $(2L+1) \times (2M+1)$ grid with a given corner removed, and spanning trees on a $(L+1) \times (M+1)$ grid.

Vacancy in a corner : Temperley's bijection revisited



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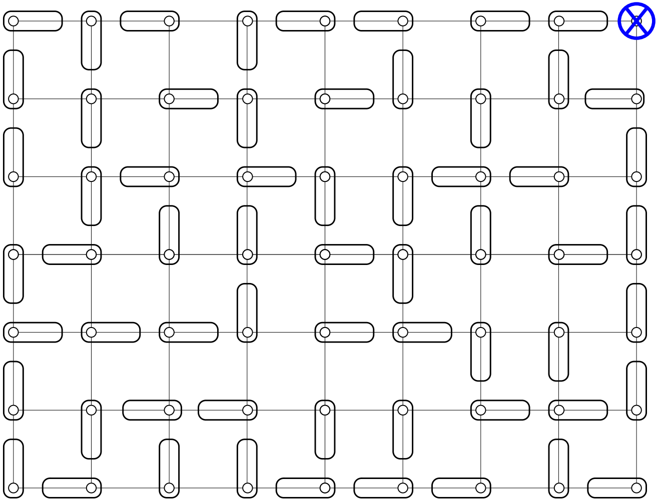

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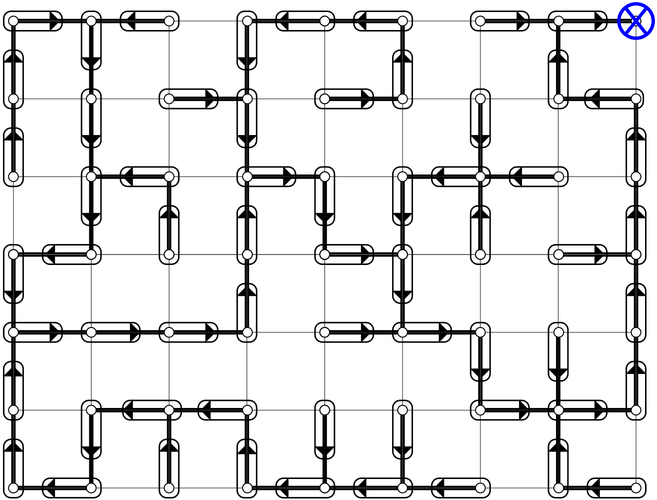

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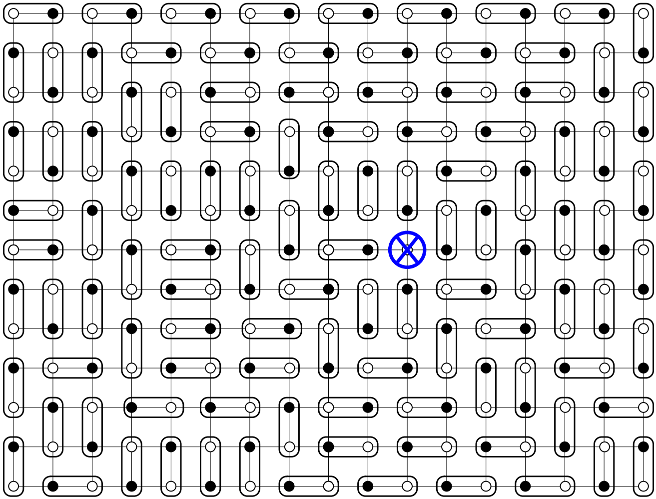
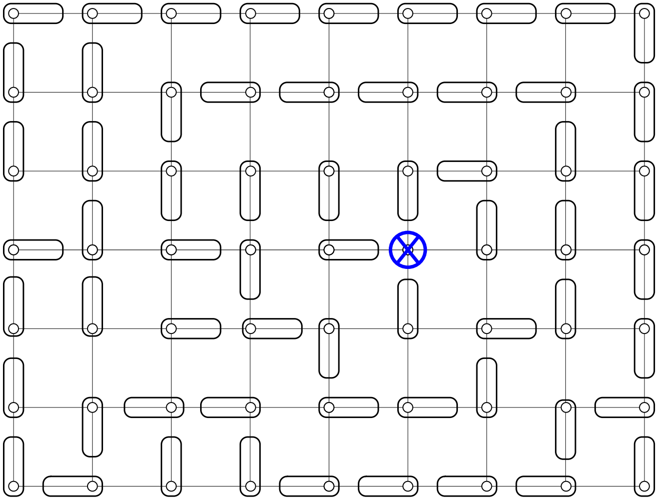
For a given initial configuration with vacancy on the corner (or anywhere on the boundary), the associated spanning tree can be seen as the graph of attainable configurations. The vacancy is fully delocalized.

Vacancy in the bulk : webs

Now for an initial configuration with vacancy in the bulk, the vacancy can be localized.





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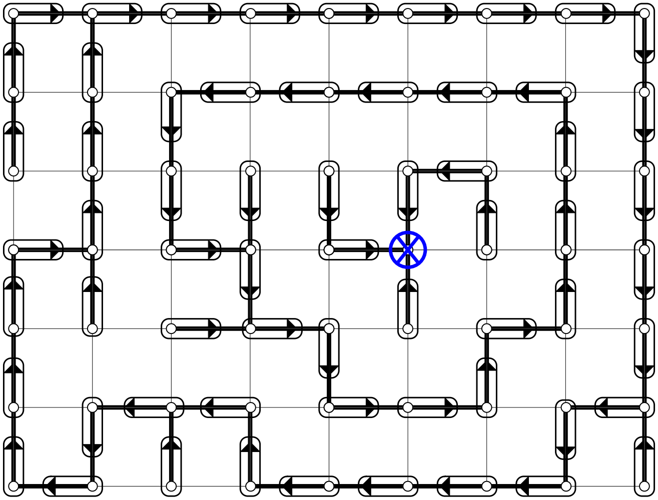

Vacancy in the bulk : webs

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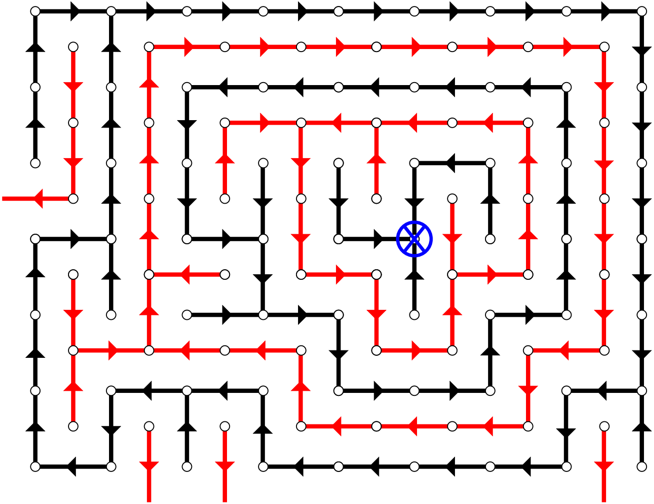
Definition

A web on a grid is a spanning subgraph with a distinguished vertex (the root), and whose connected components consist of :

- a tree containing the root
- graphs containing exactly one cycle, that moreover winds around the root.

A dimer configuration on a $(2L+1) \times (2M+1)$, with vacancy on the even white grid, yields a web on a $(L+1) \times (M+1)$ grid. A web with n cycles has exactly 4^n corresponding dimer configurations.

Vacancy in the bulk : webs



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Findings

- In the thermodynamic limit, s remains finite (localization), with probability distribution $p(s)$.
- However localization is weak in the sense that $p(s) \sim s^{-\delta}$ for $s \to \infty$, with $\delta = 9/8 < 2$

Dynamical properties

Starting from a random initial state, and applying random moves (according to some natural dynamics), how far does the vacancy go ?

- number of distinct sites visited : $\langle \bar{N}(t) \rangle \sim t^{\eta}$, $\eta = 0.54 \pm 0.03$
- mean square displacement : $\langle \bar{r^2}(t) \rangle \sim t^{\theta}$, $\theta = 0.56 \pm 0.01$

This is for the vacancy initially in the bulk. If we start with the vacancy on the boundary, we find different exponents characteristic of diffusion on spanning trees :

- $\eta_0 = 0.61 \pm 0.02$
- $\theta_0 = 0.62 \pm 0.02$

Our methods

- finite-size scaling (via efficient counting)
- numerical simulations (some via perfect algorithms)
- few rigorous/exact results

Counting via matrix-tree theorem

The number of spanning trees rooted at a site i_0 on a graph is $\det(\Delta_{i,j})_{i,j} \neq i_0$ where Δ is the Laplacian matrix :

$$\Delta_{i,j} = \begin{cases} d_i \equiv -\sum_{j \neq i} \Delta_{ij} \text{ for } i = j \\ -1 \text{ for } i,j \text{ neighbours} \\ 0 \text{ otherwise.} \end{cases}$$

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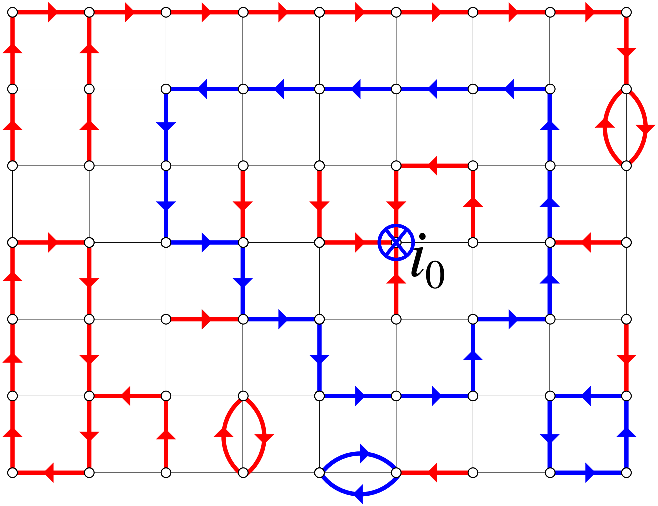

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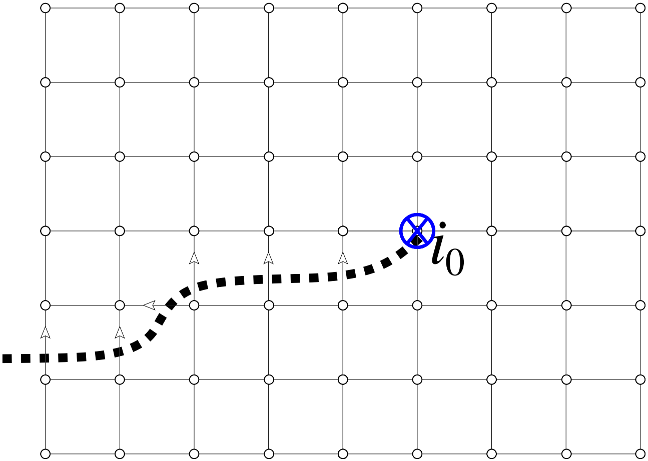

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$\det(\Delta(a)_{i,j})_{i,j} \neq i_0$ counts webs rooted at i_0 and a weight $y = 2 - a - a^{-1}$ per cycle, where :

$$\Delta(a)_{i,j} = \begin{cases} d_i \equiv -\sum_{j \neq i} \Delta_{ij} \text{ for } i = j \\ -1 \text{ for a normal edge } i,j \\ -a \text{ for a seam edge } i \to j \\ -a^{-1} \text{ for a seam edge } j \to j \\ 0 \text{ otherwise.} \end{cases}$$

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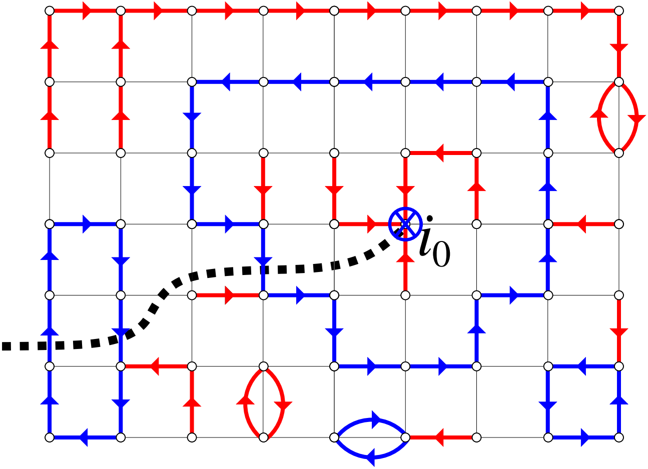

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We consider dimers on a $(4L+1) \times (4L+1)$ grid with the vacancy at the center, corresponding to webs on a $(2L+1) \times (2L+1)$ grid with root at the center. Due to the fourfold rotational symmetry, the determinant factorizes into :

$$Z_L(y) = P_L(\alpha) P_L(-\alpha) P_L(\mathrm{i}\alpha) P_L(-\mathrm{i}\alpha)$$

with $y = 2 - \alpha^4 - \alpha^{-4}$. $P_L(\alpha)$ counts webs symmetric under rotation.

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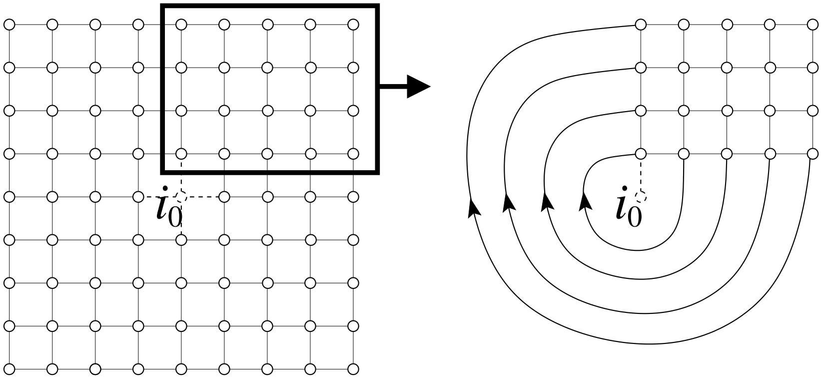

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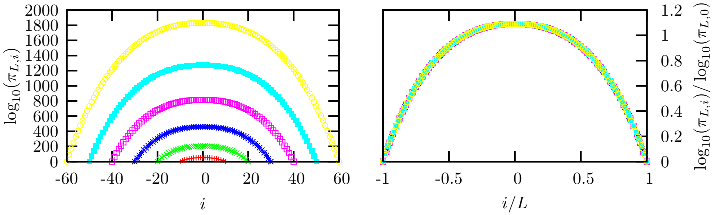
We have computed $P_L(\alpha)$ up to $L = 60$. Amusing conjecture : $P_L(\mathrm{i})$ counts the number of dimer configurations on a cylinder of height $2L$ and width $2L+1$.

Applications



(a) (b)

Asymptotic behaviour

We expect :

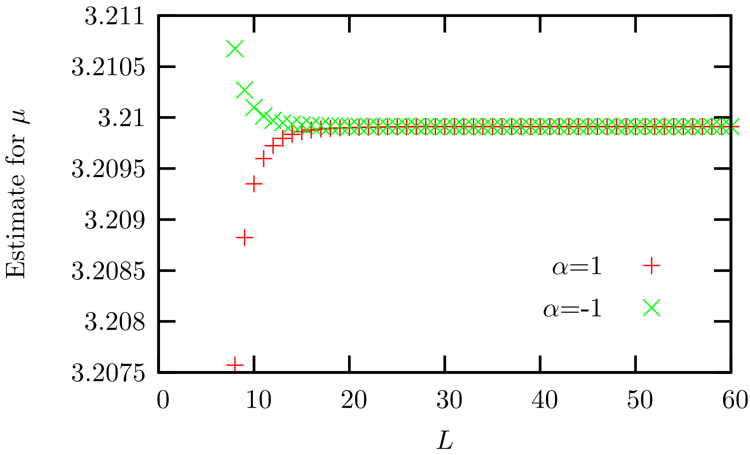
$$P_L(\alpha) \sim \tilde{c}(\alpha) \frac{\mu^{L(L+1)} \lambda^L}{L^{\tilde{\gamma}(\alpha)}}$$

μ and λ are known from spanning trees [Duplantier-David] :

$$\mu = \exp\left(\frac{4G}{\pi}\right) = 3.209912300728158\cdots$$

$$\lambda = \sqrt{2} - 1 \qquad G = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)^2} \text{ (Catalan's constant)}$$

Asymptotic behaviour



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For the critical exponent $\tilde{\gamma}(\alpha)$ we find :

$$\tilde{\gamma}(1) = 0.749999(1)$$
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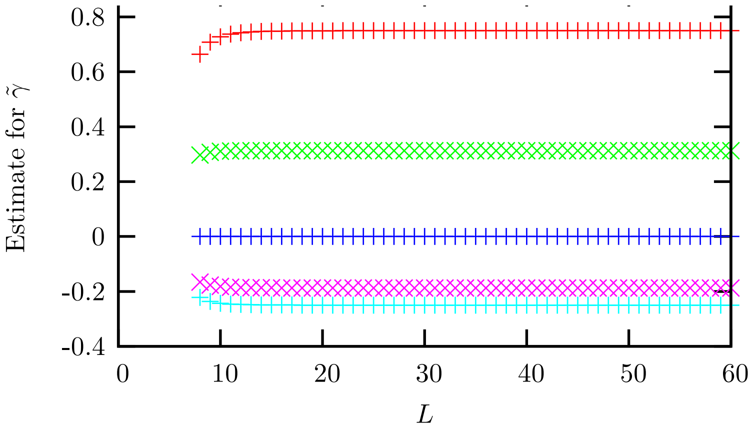

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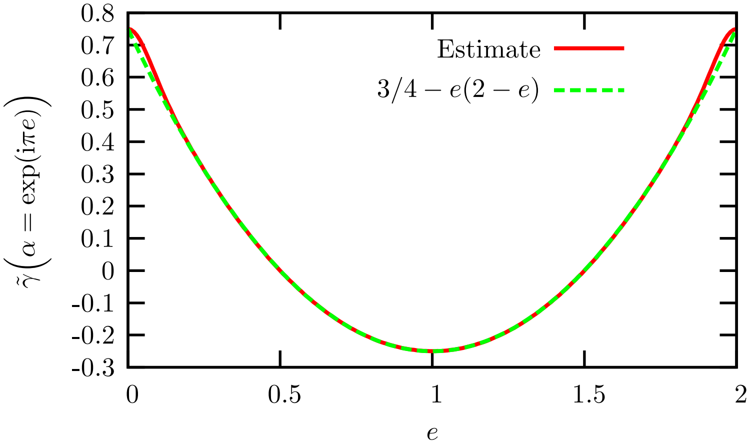
This is consistent with the analytic expression :

$$\tilde{\gamma}(\alpha) = \frac{3}{4} - e(2-e) \qquad \alpha = e^{\mathrm{i}\pi e} \qquad 0 \le e \le 2$$

Asymptotic behaviour



Asymptotic behaviour

Back to Z_L we find :

$$Z_L(y) \sim c(y) \frac{\mu^{(2L+1)^2} \lambda^{4L}}{L^{\gamma(y)}}$$

$$\gamma(y) = \tilde{\gamma}(\alpha) + \tilde{\gamma}(-\alpha) + \tilde{\gamma}(\mathrm{i}\alpha) + \tilde{\gamma}(-\mathrm{i}\alpha) \qquad y = 2 - \alpha^4 - \alpha^{-4}$$

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This indicates that the vacancy remains localized as $L \to \infty$.
Another indication : at $y = 4$ the average number of cycles in a web grows as $\frac{1}{\pi^2} \ln L$.

Size distribution

For an infinite lattice, the "tree component" accessible to the vacancy is almost surely finite. It has a size probability distribution $p(s)$ with $\sum_{s=1}^{\infty} p(s) = 1$.

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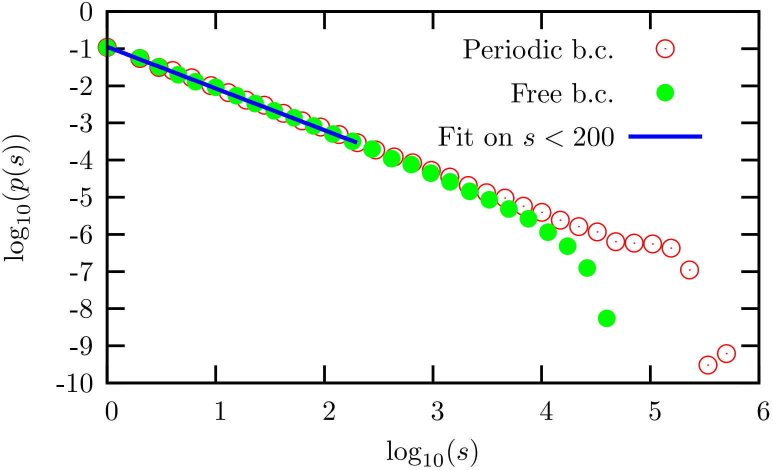
$$p(s) \sim s^{-\delta} \qquad \text{with} \qquad \delta = 1.122 \pm 0.008$$

We conjecture the exact value $\delta = 9/8$, also in agreement with the scaling relation $\gamma(0) - \gamma(4) = 1/4 = 2(1-\delta)$.

Size distribution



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Diffusion exponents

By Monte-Carlo simulations we studied the diffusion on spanning trees and webs (i.e. vacancy moving from the boundary/bulk).

- number of distinct sites visited : $\langle \bar{N}(t) \rangle \sim t^{\eta}$

$$\eta_0 = 0.61 \pm 0.02 \ (\text{trees}) \qquad \eta_4 = 0.54 \pm 0.03 \ (\text{webs})$$

- mean square displacement : $\langle \bar{r^2}(t) \rangle \sim t^{\theta}$

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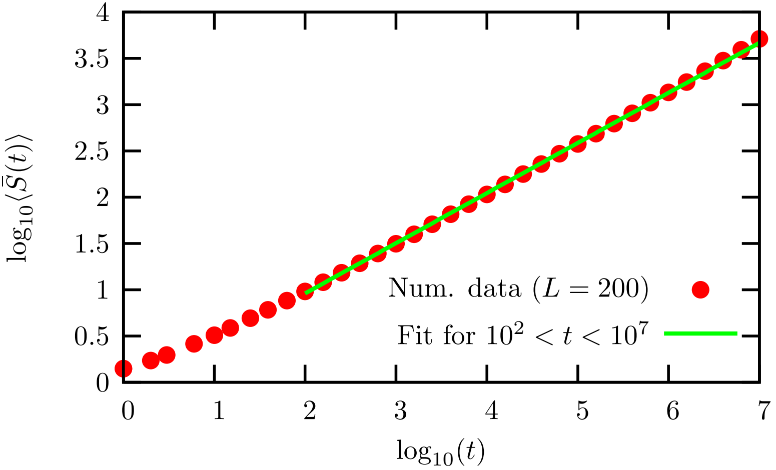

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Predictions : $\eta = \theta$, $\eta_4 = (2 - \delta)\eta_0 = 7/8\eta_0$
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(as seen by a scaling argument : $t^{\eta_4} \sim \int \max(t^{\eta_0}, s) p(s) ds$)
In a less conclusive manner, our results would be consistent with the simple values : $\eta_0 = 5/8$ and $\eta_4 = 35/64$.

Conclusion

- We find a weak localization for the vacancy, leading to a non-trivial diffusive behaviour.
- (Still) many open questions, especially regarding rigorous results (domino tilings of the "holey square").
- Further directions :
  - how about the possible interactions between several vacancies
  - other lattices, e.g. the triangular lattice where a completely different behaviour is expected (strong localization) [see recent numerical results by Jeng et al.]

Part II

Statistics of geodesics in large quadrangulations

Joint work with E. Guitter

Ref: J. Phys. A: Math. Theor. **41** 145001 (2008), arXiv:0712.2160, *IOP Select*

Introduction

We consider random planar quadrangulations as introduced in the talk of J.-F. Le Gall. We consider the “uniform” measure over all quadrangulations with a finite number n of faces (possibly with multiple edges, etc), and will be especially interested in the $n \to \infty$ limit. The general framework : viewing a quadrangulation as a discrete random surface (metric space), what can be said about its statistical properties ?

Introduction

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Our approach is "complementary" to the continuous approach of J.-F. Le Gall : we first delve into the discrete side, derive exact discrete data, then take the continuous limit.

Schaeffer's bijection

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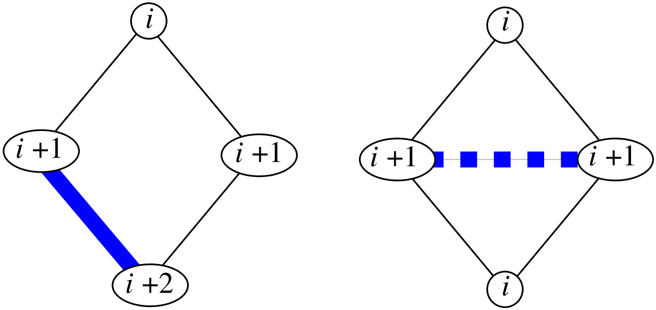
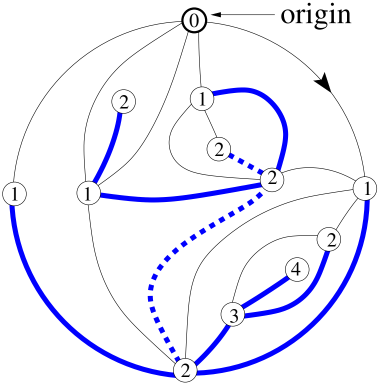


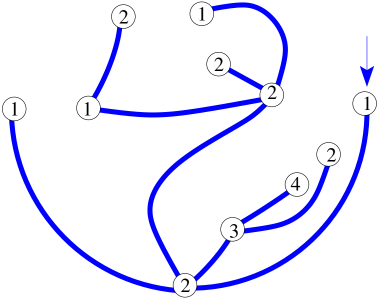
(a) (b)

Schaeffer's bijection

An example :



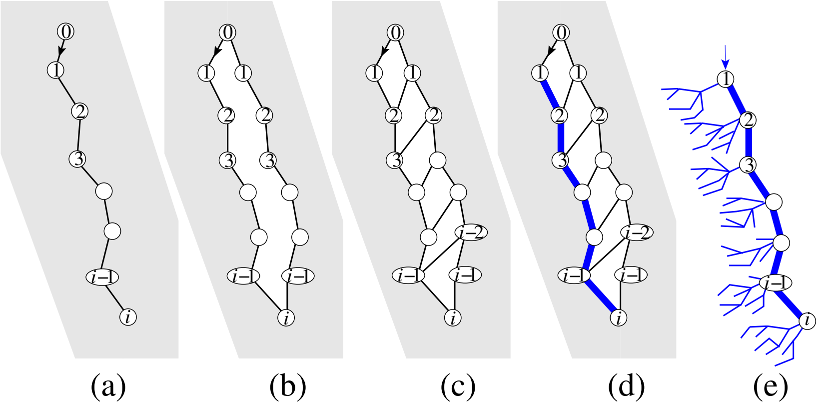
(a)



(b)

Quadrangulations with a marked geodesic

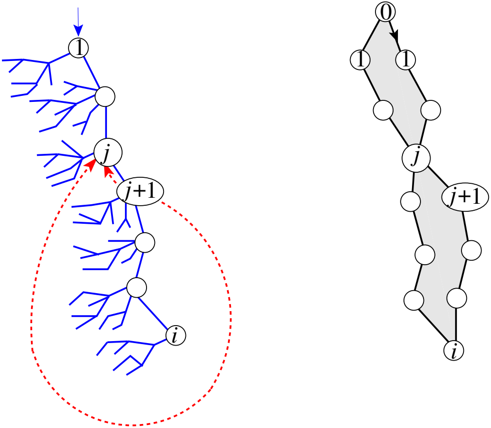
If we mark an entire geodesic instead of an edge, the information about it is “lost” in Schaeffer’s bijection, but the fix is “easy” :



We obtain a spine tree.

Quadrangulations with a geodesic boundary

However not all spine trees are obtained this way. The actual bijection involves quadrangulations with a geodesic boundary.



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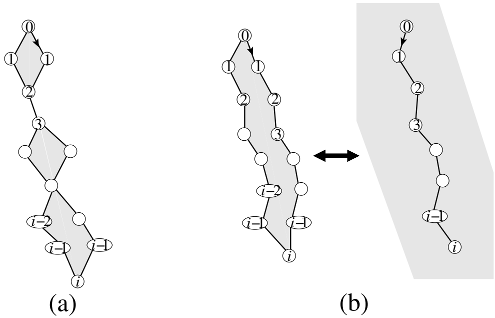
Quadrangulations with a geodesic boundary

However not all spine trees are obtained this way. The actual bijection involves quadrangulations with a geodesic boundary. Quadrangulations with a marked geodesic correspond to irreducible quadrangulations with a geodesic boundary.



Enumeration

Our basic object is the generating function for well-labeled trees :

$$R_i = 1 + gR_i(R_{i-1} + R_i + R_{i+1})$$

where g is a weight per edge(tree)/face(quadrangulation), i is the label of the root vertex. Labels are positive so the equation holds for $i > 0$ only, with convention $R_0 = 0$. NB : “original” well-labeled trees correspond to R_1.

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Theorem [B., Di Francesco, Guitter 2003]

$$R_i = R\frac{(1-x^i)(1-x^{i+3})}{(1-x^{i+1})(1-x^{i+2})}$$

where R, g are power series in g defined by :

$$R = 1 + 3gR^2$$

$$x + \frac{1}{x} + 1 = \frac{1}{gR^2}$$

Enumeration

We deduce the generating function for spine trees of length i (i.e. quadrangulations with a geodesic boundary of length $2i$) :

$$Z_i = \prod_{j=1}^{i} R_j = R^i \frac{(1-x)(1-x^{i+3})}{(1-x^3)(1-x^{i+1})}$$

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Quadrangulations with a marked geodesic are obtained through :

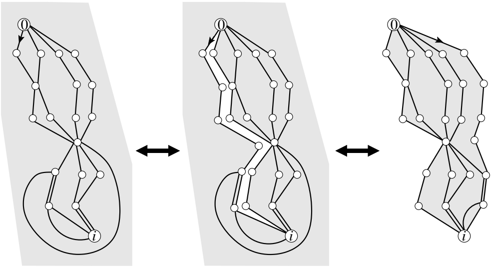
$$U_i = Z_i - \sum_{j=1}^{i-1} U_j Z_{i-j}$$

Confluent geodesics

For “free” we also get quadrangulations with k confluent geodesics.

$$U_i^{(k)} = (Z_i)^k - \sum_{j=1}^{i-1} U_j^{(k)} (Z_{i-j})^k \text{ (weakly avoiding)}$$

$$\tilde{U}_i^{(k)} = (U_i)^k \text{ (strongly avoiding)}$$



(a) (b) (c)

Statistics

Large n asymptotics are obtained by expansion around the critical point $g = 1/12$. For instance :

$$Z_i = A_i + C_i \frac{\xi^2}{n} + \frac{2}{3} \mathrm{i} D_i \frac{\xi^3}{n^{3/2}} + \cdots$$

$$g = \frac{1}{12}\left(1 + \frac{\xi^2}{n}\right) \qquad A_i = \frac{2^i(i+3)}{3(i+1)}$$

$$Z_i|_{g^n} \sim \frac{12^n D_i}{2\sqrt{\pi} n^{5/2}} = \frac{12^n}{\sqrt{\pi} n^{5/2}} \frac{2^i i(i+2)(i+3)(i+4)(3i^2+12i+13)}{840(i+1)}$$

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(i finite)
A natural normalization is to divide by

$$C_{n,i} = \log(R_i/R_{i-1})|_{g^n} \sim \frac{12^n}{\sqrt{\pi} n^{5/2}} \frac{3i^3}{7}, \quad i \to \infty$$

corresponding to quadrangulations with two marked points at distance i.

Results

- $U_i|_{g^n}/C_{n,i} \sim 3 \times 2^i$: average number of geodesics between two points

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  - this is not true for strongly avoiding geodesics : pairs of point linked by distinct geodesics become "rare" in the $i \to \infty$ limit
- results remain mostly true in the scaling limit $i \propto n^{1/4}$

Conclusion

- Our discrete approach enables to study the statistics of geodesics in large quadrangulations in a computational manner.
- Work in progress : an interesting connection with heaps
- Other directions : find other observables, connect with the continuous approach, study more general classes of maps (matter...)