Three-coloring statistical model with ‘domain wall’ boundary conditions

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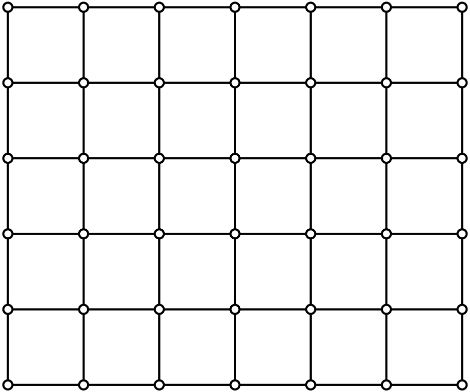

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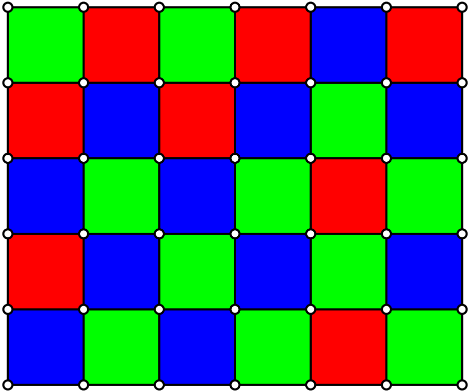

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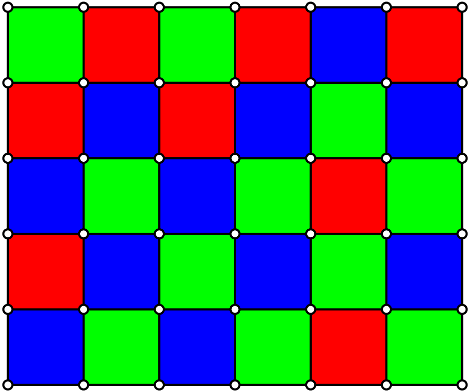

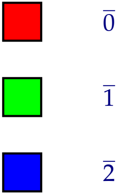

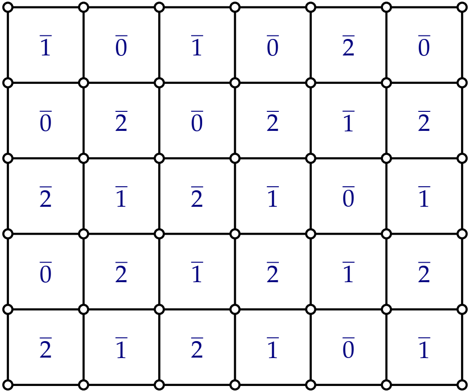
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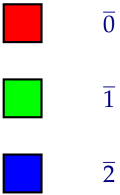
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Three-colorings

Colors

The colors of adjacent faces are different.

Elements of $\mathbb{Z}_3$

The 'colors' of adjacent faces differ by $+\overline{1}$ or $-\overline{1} = +\overline{2}$.

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Combinatorial problems

Enumerations

- Enumerate three-colorings — $C_{n,m}$
- Find refined enumerations — $C_{n,m}(k_{\overline{0}}, k_{\overline{1}}, k_{\overline{2}})$

Generating function

$$Z_{n,m}(z_{\overline{0}}, z_{\overline{1}}, z_{\overline{2}}) = \sum_{\substack{k_{\overline{0}}, k_{\overline{1}}, k_{\overline{2}} \\ k_{\overline{0}} + k_{\overline{1}} + k_{\overline{2}} = nm}} z_{\overline{0}}^{k_{\overline{0}}} z_{\overline{1}}^{k_{\overline{1}}} z_{\overline{2}}^{k_{\overline{2}}} C_{n,m}(k_{\overline{0}}, k_{\overline{1}}, k_{\overline{2}}),$$

Statistical mechanics interpretation

- The numbers $z_{\overline{0}}, z_{\overline{1}}, z_{\overline{2}}$ — the Boltzmann weights of faces
- The product of the Boltzmann weights of the faces — the Boltzmann weight of a state
- $Z_{n,m}(z_{\overline{0}}, z_{\overline{1}}, z_{\overline{2}})$ — the partition function (state sum) of the model

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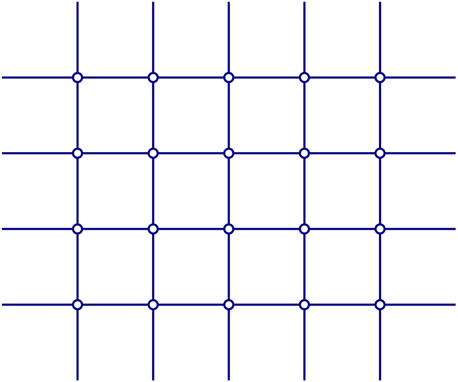

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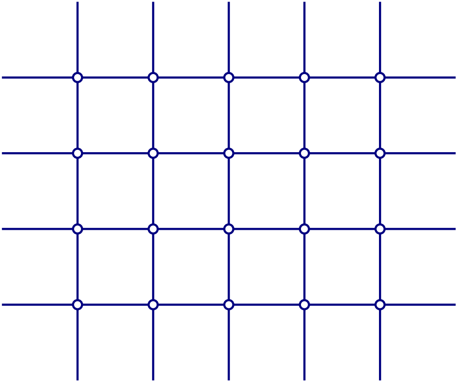

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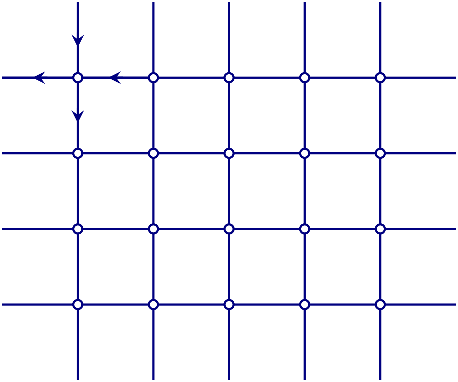

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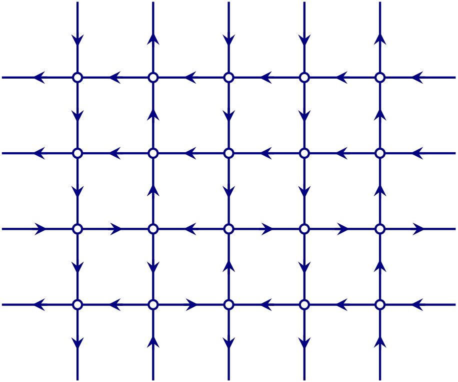

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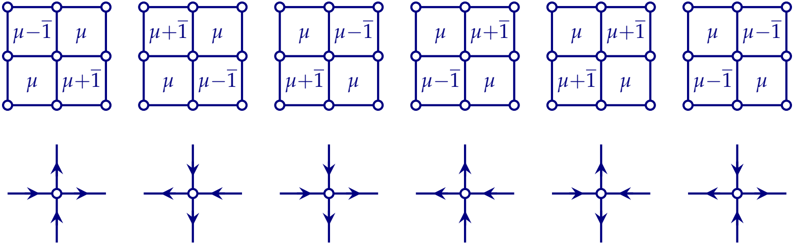
- Consider four edges containing a fixed four-valent vertex of the lattice and four faces containing these edges. The possible color combinations for the four face sets are given below.
- Visit the selected faces moving anticlockwise. If intersecting an edge we see that the color changes by $+\overline{1}$ we place on the edge a pointing in arrow, if the color changes by $-\overline{1}$ we place a pointing out arrow.



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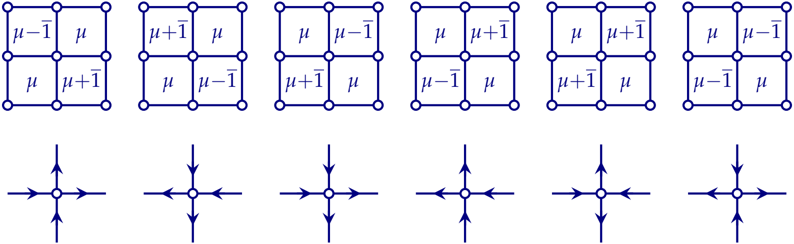
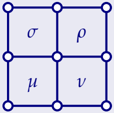


Three-colorings and six vertex model

Weight as a product over vertices



$$(z_\mu z_\nu z_\rho z_\sigma)^{1/4}$$

For the toroidal boundary conditions each face enters four vertex configurations.

In 1970 Baxter found the partition function in the thermodynamic limit.

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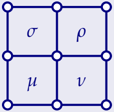

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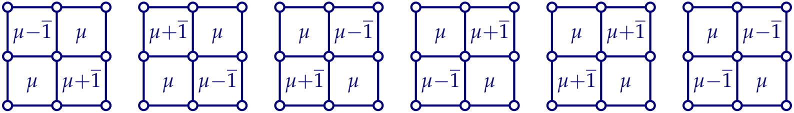


$a_\mu(u)$ $a_\mu(u)$ $b_\mu(u)$ $b_\mu(u)$ $c_\mu(u)$ $c'_\mu(u)$

The spectral parameter u is associated with the central vertex of a four face configuration.

Stroganov's solution of star-triangle relation (1982)

The weights are expressed via standart elliptic θ-functions of nome p:

$$a_\mu(u|\alpha,p) = \zeta_\mu^{3u/4\pi}(\alpha,p)\frac{\theta_1(2\pi/3-u|p)}{\theta_1(2\pi/3|p)},$$

$$b_\mu(u|\alpha,p) = \zeta_\mu^{1/2-3u/4\pi}(\alpha,p)\frac{\theta_1(u|p)}{\theta_1(2\pi/3|p)},$$

$$c_\mu(u|\alpha,p) = \frac{\zeta_{\mu+\bar{1}}^{u/2\pi}(\alpha,p)}{\zeta_\mu^{u/2\pi}(\alpha,p)}\frac{\theta_4(\alpha+2\pi\overline{\mu}/3+u|p)}{\theta_4(\alpha+2\pi\overline{\mu}/3|p)},$$

$$c'_\mu(u|\alpha,p) = \frac{\zeta_{\mu-\bar{1}}^{u/2\pi}(\alpha,p)}{\zeta_\mu^{u/2\pi}(\alpha,p)}\frac{\theta_4(\alpha+2\pi\overline{\mu}/3-u|p)}{\theta_4(\alpha+2\pi\overline{\mu}/3|p)}.$$

Here α is an arbitrary parameter and

$$\zeta_\mu(\alpha,p) = \frac{\theta_4(\alpha+2\pi(\overline{\mu}-1)/3|p)\theta_4(\alpha+2\pi(\overline{\mu}+1)/3|p)}{\theta_4^2(\alpha+2\pi\overline{\mu}/3|p)}.$$

Three-colorings and six vertex model

Relation to enumerations

For $u = \pi/3$ we have

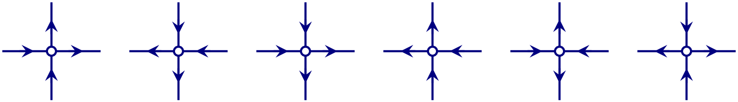
$$a_\mu(\pi/3|\alpha,p) = \zeta_\mu^{1/4}(\alpha,p), \qquad c_\mu(\pi/3|\alpha,p) = \zeta_{\mu+1}^{1/2}(\alpha,p)\zeta_\mu^{1/2}(\alpha,p),$$

$$b_\mu(\pi/3|\alpha,p) = \zeta_\mu^{1/4}(\alpha,p), \qquad c'_\mu(\pi/3|\alpha,p) = \zeta_{\mu-1}^{1/2}(\alpha,p)\zeta_\mu^{1/2}(\alpha,p),$$

and we come to the three-coloring model by Baxter with

$$z_\mu = \zeta_\mu(\alpha,p).$$

Three-colorings and six vertex model



$a(u)$ $a(u)$ $b(u)$ $b(u)$ $c(u)$ $c(u)$

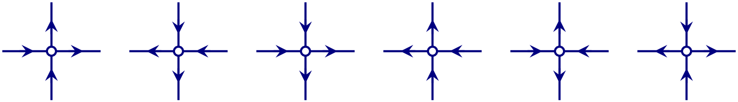
Standard solution of star-triangle relation

$$a(u|\eta) = \frac{\sin(u + \eta/2)}{\sin\eta}, \qquad b(u|\eta) = \frac{\sin(u - \eta/2)}{\sin\eta}, \qquad c(u|\eta) = 1,$$

where η is the crossing parameter.

For $\eta = 2\pi/3$ and the domain wall boundary conditions the partition function of the inhomogeneous six vertex model satisfies some simple functional equations. The partition function of the three-coloring model for appropriate boundary condition satisfies a similar equation.

Three-colorings and six vertex model



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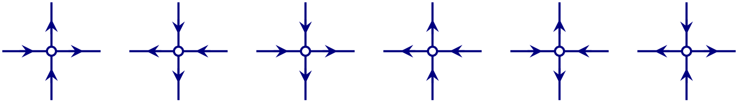

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Proof of functional equation for six vertex model

- Recursion relations (Korepin, 1982)
- Determinant representation of the partition function (Izergin, 1987)
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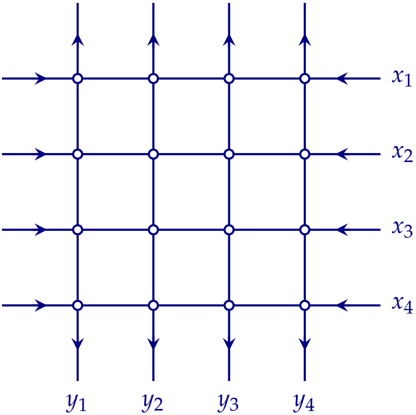
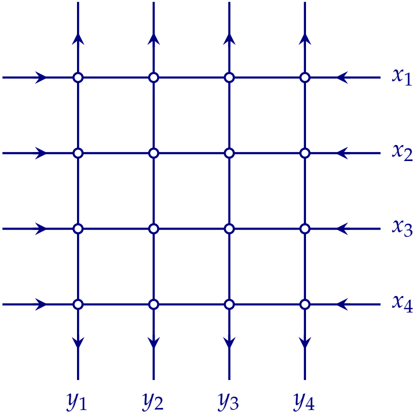
Domain wall boundary conditions



Inhomogeneous model

The spectral parameter associated with a vertex is $x_i - y_j$.



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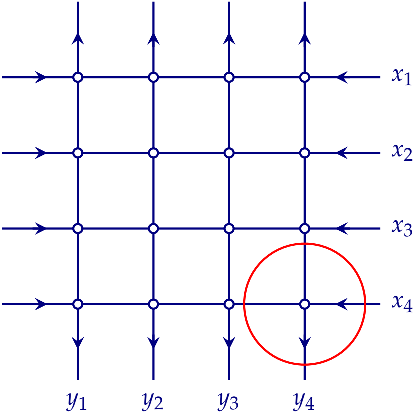

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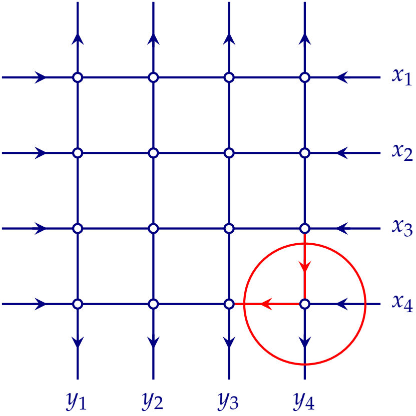
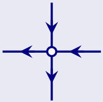
$$x_n - y_n + \eta/2 = 0$$





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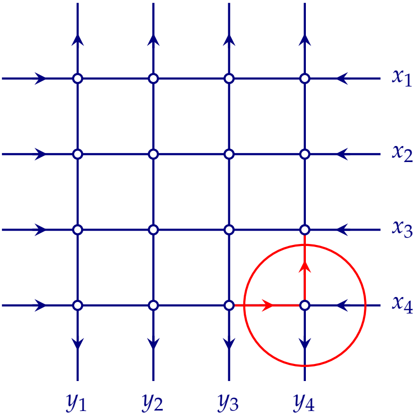

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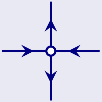

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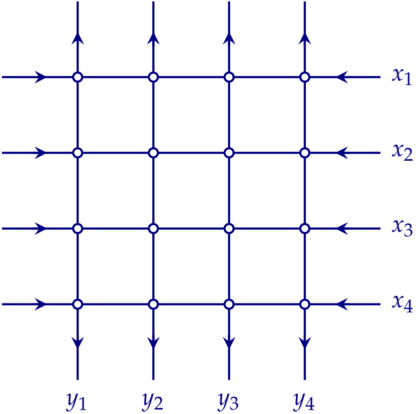

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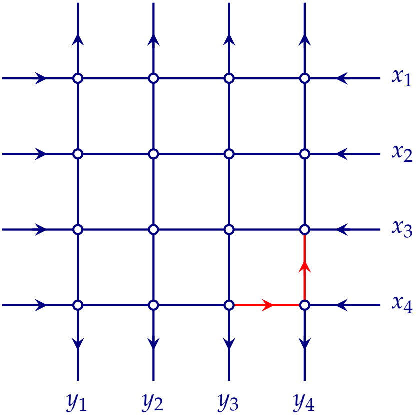

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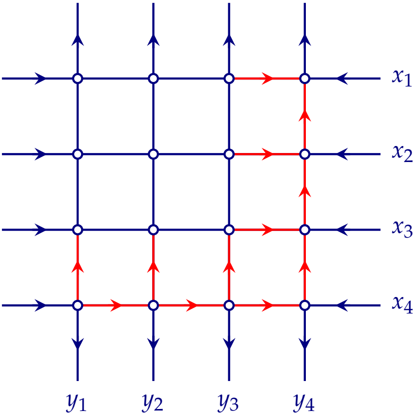

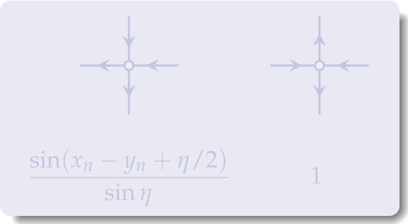

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Recursion relations

$$Z_n(x_1, \ldots, x_{n-1}, x_n; y_1, \ldots, y_{n-1}, y_n)|_{x_n = y_n - \eta/2}$$
$$= \sin^{2-2n}\eta \prod_{i=1}^{n-1} \sin(x_i - y_n - \eta/2) \prod_{i=1}^{n-1} \sin(y_n - y_i - \eta)$$
$$\times Z_{n-1}(x_1, \ldots, x_{n-1}; y_1, \ldots, y_{n-1}).$$

Since the function $Z_n(\{x\};\{y\})$ is symmetric in the variables x_i and y_i we have n^2 recursion relations

$$Z_n(x_1, \ldots, x_k, \ldots, x_n; y_1, \ldots, y_\ell, \ldots, y_n)|_{x_k = y_\ell - \eta/2}$$
$$= \sin^{2-2n}\eta \prod_{\substack{i=1 \\ i \neq k}}^{n} \sin(x_i - y_\ell - \eta/2) \prod_{\substack{i=1 \\ i \neq \ell}}^{n} \sin(y_\ell - y_i - \eta)$$
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One can start with the top rightmost vertex. This gives

$$Z_n(x_1,\ldots,x_k,\ldots,x_n;y_1,\ldots,y_\ell,\ldots,y_n)\big|_{x_k=y_\ell+\eta/2}$$
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Assume now that $\eta = 2\pi/3$ and define the functions

$$F_n(\{x\},\{y\}) = \prod_{\substack{i,j=1\\ i<j}}^{n} \sin(x_i - x_j) \prod_{i,j=1}^{n} \sin(x_i - y_j) \prod_{\substack{i,j=1\\ i<j}}^{n} \sin(y_i - y_j) Z_n(\{x\},\{y\}).$$

These functions satisfy $2n^2$ recursion relations

$$F_n(x_1,\ldots,x_k,\ldots,x_n;y_1,\ldots,y_\ell,\ldots,y_n)\big|_{x_k=y_\ell\pm\pi/3}$$
$$= \mp(-1)^n 4^{2-2n} \sin^{3-2n}(2\pi/3) \prod_{\substack{i=1\\ i\neq k}}^{n} \sin[3(y_\ell - x_i)] \prod_{\substack{i=1\\ i\neq \ell}}^{n} \sin[3(y_\ell - y_i)]$$
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Functional equations

Assume now that $\eta = 2\pi/3$ and define the functions

$$F_n(\{x\},\{y\}) = \prod_{\substack{i,j=1\\i<j}}^{n} \sin(x_i - x_j) \prod_{i,j=1}^{n} \sin(x_i - y_j) \prod_{\substack{i,j=1\\i<j}}^{n} \sin(y_i - y_j) Z_n(\{x\},\{y\}).$$

These functions satisfy $2n^2$ recursion relations

$$\begin{aligned} &F_n(x_1,\dots,x_k,\dots,x_n;y_1,\dots,y_\ell,\dots,y_n)\big|_{x_k=y_\ell\pm\pi/3} \\ &\quad = \mp(-1)^n 4^{2-2n} \sin^{3-2n}(2\pi/3) \prod_{\substack{i=1\\i\neq k}}^{n} \sin[3(y_\ell - x_i)] \prod_{\substack{i=1\\i\neq \ell}}^{n} \sin[3(y_\ell - y_i)] \\ &\qquad \times F_{n-1}(x_1,\dots,\widehat{x}_k,\dots,x_n;y_1,\dots,\widehat{y}_\ell,\dots,y_n). \end{aligned}$$

Functional equations

Now define the functions

$$S_{n,k}(x_1, \ldots, x_k, \ldots, x_n; \{y\}) = \sum_{r=0}^{2} F_n(x_1, \ldots, x_k + 2\pi r/3, \ldots, x_n; \{y\}).$$

and prove by induction that $S_{n,k}(\{x\}; \{y\}) = 0$.

Actually it suffices to prove that $S_{n,1}(\{x\}; \{y\}) = 0$.

The base case of induction is $n = 1$. It is not difficult to show that

$$S_{1,1}(x_1; y_1) = \sum_{r=0}^{2} \sin(x_1 - y_1 + 2\pi/3) = 0.$$

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Assume now that $S_{n-1,1}(\{x\};\{y\}) = 0$ for some $n > 1$. The recursion relations give

$$S_{n,1}(x_1,\dots,x_{n-1}x_n;y_1,\dots,y_\ell,\dots,y_n)|_{x_n=y_\ell\pm\pi/3}$$
$$= \mp(-1)^n 4^{2-2n} \sin^{3-2n}(2\pi/3) \prod_{i=1}^{n-1} \sin[3(y_\ell - x_i)] \prod_{\substack{i=1 \\ i\neq\ell}}^{n} \sin[3(y_\ell - y_i)]$$
$$\times S_{n-1,1}(x_1,\dots,x_{n-1};y_1,\dots,\widehat{y_\ell},\dots,y_n) = 0.$$

Functional equations

Counting orders

With respect to the variable x_n the partition function $Z_n(\{x\};\{y\})$ is a trigonometric polynomial of order less or equal to $n-1$, and $S_{n,1}(\{x\};\{y\})$ is a trigonometric polynomial of order less or equal to $3n-2$. It has at most $6n-4$ zeros in the interval $0 \le x_n < 2\pi$.

Counting zeros

- Recursion relations give $2n$ zeros at the points $x_n = y_\ell \pm \pi/3$, $\ell = 1, \dots, n$.
- By construction there are $2n-2$ zeros at the points $x_n = x_\ell$, $\ell = 2, \dots, n-1$ and at the points $x_n = y_\ell$, $\ell = 1, \dots, n$.
- The relation $S_{n,1}(x_1, \dots, x_n + \pi; \{y\}) = (-1)^n S_{n,1}(x_1, \dots, x_n + \pi; \{y\})$ doubles zeros.
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- Thus, $S_{n,1}(\{x\};\{y\})$ has $8n-4$ zeros in the interval $0 \leq x_n < 2\pi$.

Functional equations

Conclusion

$$\sum_{r=0}^{2} F_n(x_1, \dots, x_k + 2\pi r/3, \dots, x_n; \{y\}) = 0,$$

where

$$F_n(\{x\}, \{y\}) = \prod_{\substack{i,j=1 \\ i<j}}^{n} \sin(x_i - x_j) \prod_{i,j=1}^{n} \sin(x_i - y_j) \prod_{\substack{i,j=1 \\ i<j}}^{n} \sin(y_i - y_j) Z_n(\{x\}, \{y\}).$$

Contents

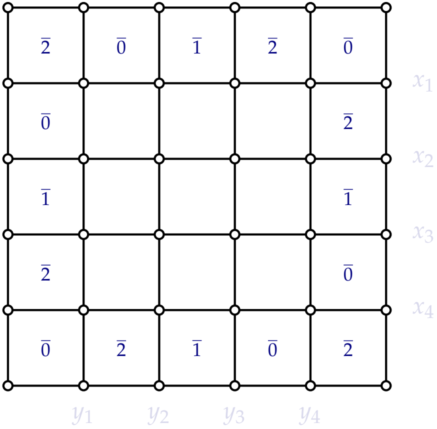
Partial partition functions

The total partition sum can be naturally represented as

$$Z_n(\{x\};\{y\}) = \sum_{\mu\in\mathbb{Z}_3} Z_n^{\mu}(\{x\};\{y\}),$$

where μ is the color of the left topmost vertex of the lattice.

‘Domain wall’ boundary conditions



‘Domain wall’

If one starts with any boundary face and walks anticlockwise along the boundary then the color changes by $+\bar{1}$ from face to face for the vertical boundaries, and by $-\bar{1}$ for the horizontal boundaries.

Inhomogeneous model

The spectral parameter associated with a vertex configuration is equal to $x_i - y_j$.

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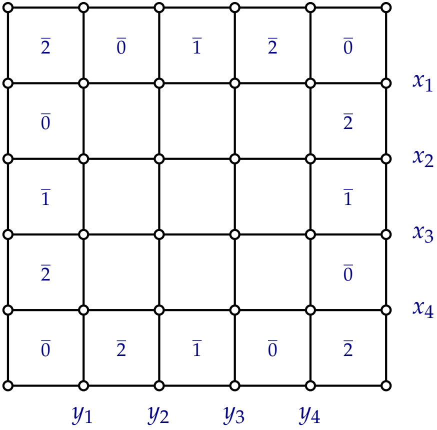

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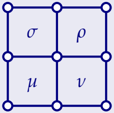
Contents

Gauge transformation

Notation for weights



$$W_{\mu\nu}^{\sigma\rho}(u)$$

Transformation

$$\widetilde{W}_{\mu\nu}^{\sigma\rho}(u) = \Phi_{\mu}^{-1}(u)\Phi_{\nu}(u)\Phi_{\rho}^{-1}(u)\Phi_{\sigma}(u)W_{\mu\nu}^{\sigma\rho}(u)$$

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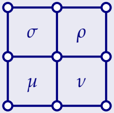
Gauge transformation

Notation for weights



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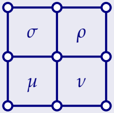

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Gauge transformation

Three-coloring model

$$\Phi_{\mu}(u|\alpha,p) = \zeta_{\mu}^{u/4\pi}(\alpha,p)$$

$$\widetilde{a}_{\mu}(u|\alpha) = \frac{\theta_1(2\pi/3 - u)}{\theta_1(2\pi/3)}, \qquad \widetilde{c}_{\mu}(u|\alpha) = \frac{\theta_4(\alpha + 2\pi\overline{\mu}/3 + u)}{\theta_4(\alpha + 2\pi\overline{\mu}/3)},$$

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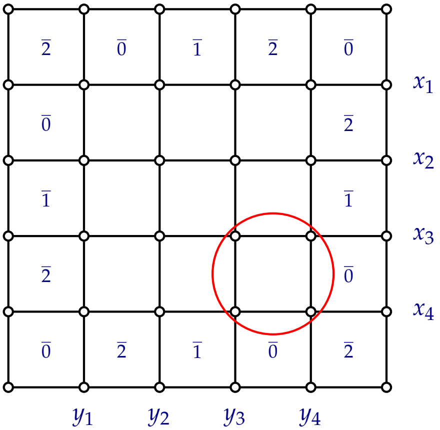
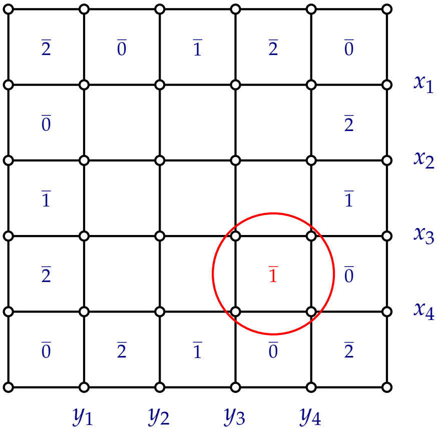
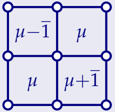


$$\frac{\theta_1(2\pi/3 - u)}{\theta_1(2\pi/3)} \qquad \frac{\theta_4(\alpha + 2\pi\overline{\mu}/3 + u)}{\theta_4(\alpha + 2\pi\overline{\mu}/3)}$$

Recursion relations





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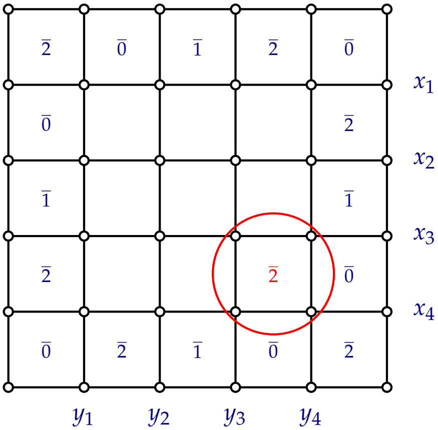

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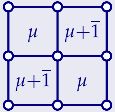

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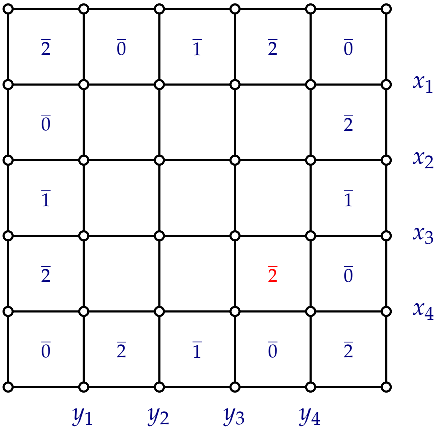

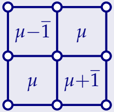

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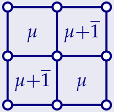

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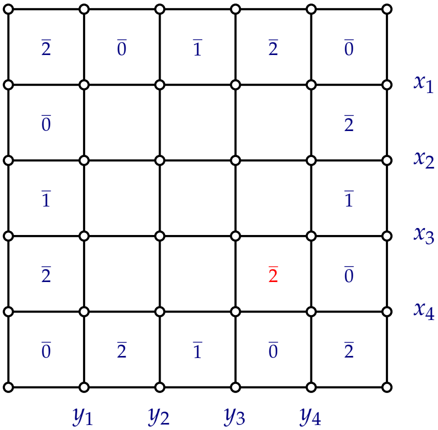

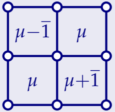

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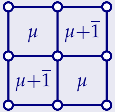

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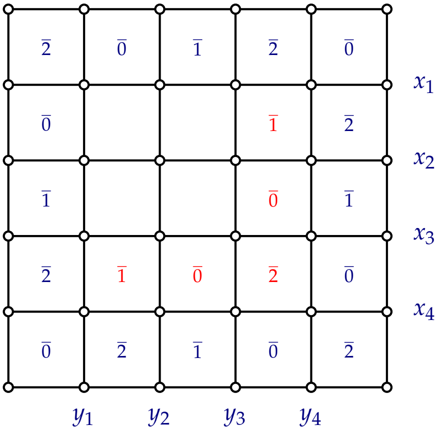

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Total set of relations

$$\widetilde{Z}_n^{\mu}(x_1,\ldots,x_k,\ldots,x_n;y_1,\ldots,y_\ell,\ldots,y_n)\Big|_{x_k=y_\ell+2\pi/3}$$
$$=\frac{\theta_4(\alpha+2\pi(\overline{\mu}+n)/3)}{\theta_4(\alpha+2\pi(\overline{\mu}+n-1)/3)}\,\theta_1^{2-2n}(2\pi/3)\prod_{\substack{i=1\\i\neq k}}^{n}\theta_1(x_i-y_\ell)\prod_{\substack{i=1\\i\neq \ell}}^{n}\theta_1(y_\ell-y_i+2\pi/3)$$
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Functional equations

Define the functions

$$F_n^\mu(x_1,\ldots,x_n;y_1,\ldots,y_n) = \frac{1}{\theta_4(\alpha + 2\pi(m+n)/3)}$$

$$\times \prod_{\substack{i,j=1\\ i<j}}^{n} \theta_1(x_i - x_j) \prod_{i,j=1}^{n} \theta_1(x_i - y_j - \pi/3) \prod_{\substack{i,j=1\\ i<j}}^{n} \theta_1(y_i - y_j)$$

$$\times Z_n^\mu(x_1,\ldots,x_n;y_1,\ldots,y_n),$$

and use the relation

$$\theta_1(u|p)\theta_1(u+\pi/3|p)\theta_1(u+2\pi/3|p) = D(p)\theta_1(3u|p^3),$$

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Functional equations

Recursion relations

$$F_n^\mu(x_1,\ldots,x_k,\ldots,x_n;y_1,\ldots,y_\ell,\ldots,y_n|\alpha,p)\Big|_{x_k=y_\ell+2\pi/3}$$

$$= D^{2n-2}(p)\theta_1^{3-2n}(2\pi/3|p)\prod_{\substack{i=1\\i\neq k}}^{n}\theta_1(3(x_i-y_\ell)|p^3)\prod_{\substack{i=1\\i\neq \ell}}^{n}\theta_1(3(y_\ell-y_i)|p^3)$$

$$\times F_{n-1}^\mu(x_1,\ldots,\widehat{x}_k,\ldots,x_n;y_1,\ldots,\widehat{y}_\ell,\ldots,y_n|\alpha,p),$$

$$F_n^\mu(x_1,\ldots,x_k,\ldots,x_n;y_1,\ldots,y_\ell,\ldots,y_n|\alpha,p)\Big|_{x_k=y_\ell}$$

$$= -D^{2n-2}(p)\theta_1^{3-2n}(2\pi/3|p)\prod_{\substack{i=1\\i\neq k}}^{n}\theta_1(3(x_i-y_\ell)|p^3)\prod_{\substack{i=1\\i\neq \ell}}^{n}\theta_1(3(y_\ell-y_i)|p^3)$$

$$\times F_{n-1}^{\mu+\overline{1}}(x_1,\ldots,\widehat{x}_k,\ldots,x_n;y_1,\ldots,\widehat{y}_\ell,\ldots,y_n|\alpha,p)$$

Functional equations

$$S^{\mu}_{n,k}(x_1, \ldots, x_k, \ldots, x_n; \{y\}) = \sum_{r=0}^{2} F_n^{\mu+\bar{r}}(x_1, \ldots, x_k + 2\pi r/3, \ldots, x_n; \{y\}).$$

Counting zeros

- Recursion relations give $2n$ zeros at the points $x_n = y_\ell + 2\pi/3$ and $x_n = y_\ell$, $\ell = 1, \ldots, n$.
- By construction there are $2n - 2$ zeros at the points $x_n = x_\ell$, $\ell = 2, \ldots, n-1$ and at the points $x_n = y_\ell + \pi/3$, $\ell = 1, \ldots, n$.
- The relation $S^{\mu}_{n,1}(x_1, \ldots, x_n + \pi; \{y\}) = (-1)^n S^{\mu}_{n,1}(x_1, \ldots, x_n + \pi; \{y\})$ doubles zeros.
- Thus, $S^{\mu}_{n,1}(\{x\}; \{y\})$ has $8n - 4$ zeros in the interval $0 \leq x_n < 2\pi$.

Functional equations

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Conclusion

$$\sum_{r=0}^{2} F_n^{\mu+\bar{r}}(x_1, \ldots, x_k + 2\pi r/3, \ldots, x_n; \{y\}) = 0,$$

where

$$F_n^{\mu}(x_1, \ldots, x_n; y_1, \ldots, y_n) = \frac{1}{\theta_4(\alpha + 2\pi(m+n)/3)}$$
$$\times \prod_{\substack{i,j=1 \\ i<j}}^{n} \theta_1(x_i - x_j) \prod_{i,j=1}^{n} \theta_1(x_i - y_j - \pi/3) \prod_{\substack{i,j=1 \\ i<j}}^{n} \theta_1(y_i - y_j)$$
$$\times Z_n^{\mu}(x_1, \ldots, x_n; y_1, \ldots, y_n),$$