The qKZ equation, partial sum rules and punctured plane partitions

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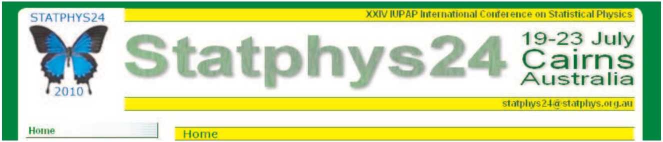
Statphys24 (2010) in Australia



http://www.statphys.org.au

Outline

Hecke Type A

Algebra with generators T_i satisfying

$$(T_i - q)(T_i + q^{-1}) = 0, \qquad i = 1, \dots, N-1,$$
$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}.$$

Projectors:

$$a_i := q - T_i, \qquad s_i := q^{-1} + T_i,$$

$$a_i s_i = s_i a_i = 0, \qquad a_i + s_i = [2], \qquad \left([x] = \frac{q^x - q^{-x}}{q - q^{-1}} \right)$$

$$a_i^2 = [2]\, a_i,$$
$$a_i a_{i+1} a_i - a_i = a_{i+1} a_i a_{i+1} - a_{i+1}.$$

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Yang-Baxter equation

Every representation of Hecke leads to an 2D integrable lattice model constructed from the R-operator

$$R_i(u) = \frac{q^u - [u]T_i}{[u+1]} = \frac{[1-u]+[u]a_i}{[u+1]} = 1 - \frac{[u]}{[u+1]}s_i,$$

which satisfies the Yang-Baxter equation

$$R_i(u)R_{i+1}(u+v)R_i(v) = R_{i+1}(v)R_i(u+v)R_{i+1}(u).$$

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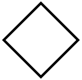
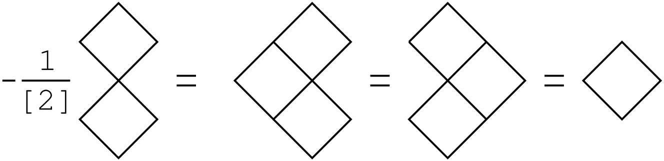
Temperley-Lieb Type A

TL quotient of Hecke Type A (generators $e_i := -a_i$)

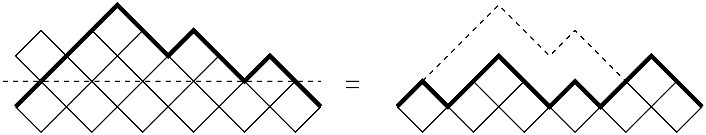
$$e_i^2 = -[2]e_i,$$
$$e_i e_{i\pm1} e_i = e_i$$

For TL, e_i can be represented pictorially by a box:

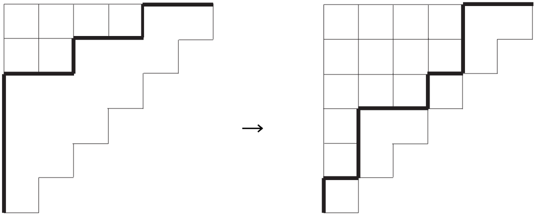
$e_i =$ 



This gives rise to a TL-action on Dyck paths:



Or on partitions:



q-Knizhnik-Zamolodchikov equation

Let $\psi_\alpha(x_1,\ldots,x_N) \in \mathbb{C}[\![q^{\pm x_1},\ldots,q^{\pm x_N}]\!]$. Consider

$$|\Psi(x_1,\ldots,x_N)\rangle = \sum_\alpha \psi_\alpha(x_1,\ldots,x_N)|\alpha\rangle.$$

Here α runs over paths of length N.

The q-Knizhnik-Zamolodchikov equations for type A are

$$R_i(x_i - x_{i+1})|\Psi\rangle = \pi_i|\Psi\rangle, \qquad |\Psi\rangle = \pi_0|\Psi\rangle, \qquad |\Psi\rangle = \pi_N|\Psi\rangle.$$

$$\pi_i\psi_\alpha(\ldots,x_i,x_{i+1},\ldots) = \psi_\alpha(\ldots,x_{i+1},x_i,\ldots),$$
$$\pi_0\psi_\alpha(x_1,\ldots) = \psi_\alpha(-x_1,\ldots),$$
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$$\begin{aligned} \pi_i\psi_\alpha(\dots,x_i,x_{i+1},\dots) &= \psi_\alpha(\dots,x_{i+1},x_i,\dots), \\ \pi_0\psi_\alpha(x_1,\dots) &= \psi_\alpha(-x_1,\dots), \\ \pi_N\psi_\alpha(\dots,x_N) &= \psi_\alpha(\dots,-\lambda-x_N). \end{aligned}$$

$|\Psi\rangle$ intertwines the path and functional representations.

The Yang-Baxter equation is a consistency condition for qKZ.

The qKZ equations for Type A can be rewritten as

$$-\sum_{\alpha}\psi_{\alpha}(x_1,\ldots,x_N)(e_i|\alpha\rangle) = \sum_{\alpha}(\hat{a}_i\psi_{\alpha})(x_1,\ldots,x_N)|\alpha\rangle.$$

where

$$\hat{a}_i = (\pi_i + 1)\frac{[x_i - x_{i+1} - 1]}{[x_i - x_{i+1}]},$$

are Hecke projectors in a functional representation.

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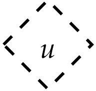
Baxterisation

Introduce the Baxterised operator

$$h_i(u) = \frac{[u+1]}{[u]} - \hat{a}_i = \hat{s}_i - \frac{[u-1]}{[u]}.$$

$h_i(u)$ satisfies the Yang-Baxter equation.
Pictorial representation:

$h_i(u) =$ 

Lemma: α has a slope at $i \Rightarrow \hat{a}_i \psi_\alpha = h_i(-1)\psi_\alpha = 0$.

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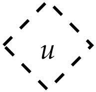
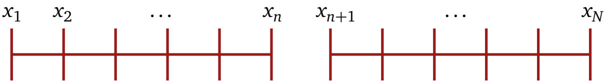


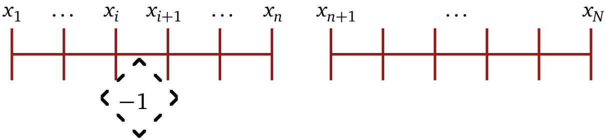
Lemma: α has a slope at $i \Rightarrow \hat{a}_i\psi_\alpha = h_i(-1)\psi_\alpha = 0$.

Reduction to scalar equation

Let Ω be the maximal Dyck path. Denote $\psi_\Omega \in \mathbb{C}[\![q^{\pm x_1}, \ldots, q^{\pm x_N}]\!]$ by

$\psi_\Omega =$ 

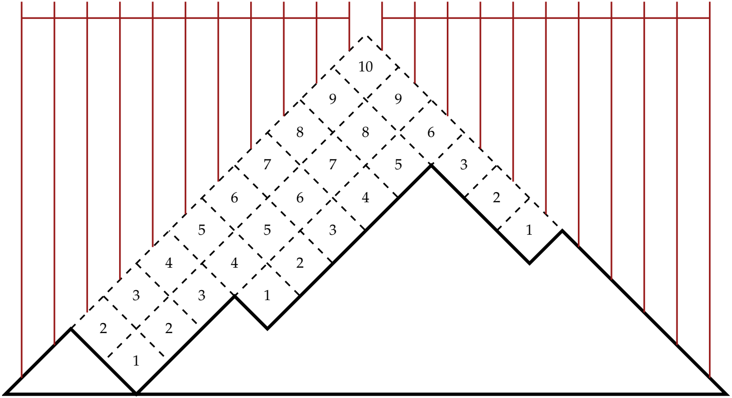
$$h_i(-1)\psi_\Omega(x_1,\ldots,x_N) = 0 \quad \Leftrightarrow$$



$= 0,$

Reduction to scalar equation

Theorem: [Kirillov and Lascoux] Solution ψ_α for each path α can be written as



Kazhdan-Lusztig elements (canonical basis).

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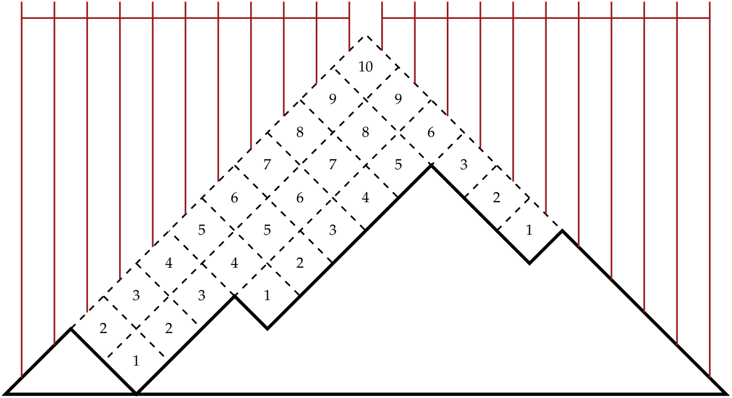

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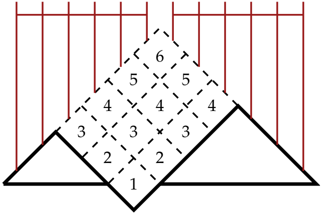
Truncation conditions on ψ_Ω:



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These conditions may fix some parameters, such as the level parameter λ in

$$\psi(\ldots, x_N) = \psi(\ldots, -\lambda - x_N).$$

For polynomial solutions, λ takes on discrete values.

Polynomial solutions for ψ_Ω can be obtained from restricted Macdonald polynomials.

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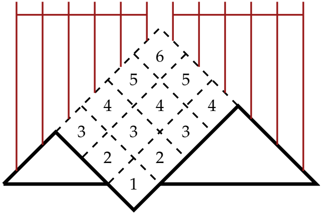

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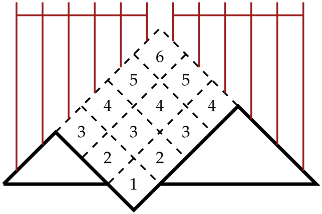

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Hecke Type B

Hecke Type A plus

$$(T_0+q^{\omega})(T_0+q^{-\omega})=0,$$
$$T_0T_1T_0T_1=T_1T_0T_1T_0.$$

Projectors:

$$a_0:=-\frac{q^{-\omega}+T_0}{q^{\omega+1}-q^{-\omega-1}},\qquad s_0:=\frac{q^{\omega}+T_0}{q^{\omega+1}-q^{-\omega-1}},$$

$$a_0s_0=s_0a_0=0,\qquad a_0+s_0=\frac{[\omega]}{[\omega+1]}.$$

$$a_0^2=\frac{[\omega]}{[\omega+1]}a_0,$$
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$$a_0 a_1 a_0 a_1 - a_0 a_1 = a_1 a_0 a_1 a_0 - a_1 a_0.$$

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Reflection Equation

Integrable boundaries are constructed from the K-operator:

$$K_0(u) = \frac{k(u,\delta) - [2u][\omega+1]a_0}{k(-u,\delta)} = 1 + \frac{[2u][\omega+1]}{k(-u,\delta)}s_0,$$

where $k(u,\delta) = [\frac{\omega+\delta}{2}+u][\frac{\omega-\delta}{2}+u]$, and δ is an additional arbitrary parameter.

The boundary Baxterised element $K_0(u)$ satisfies the reflection equation

$$K_0(v)R_1(u+v)K_0(u)R_1(u-v) = R_1(u-v)K_0(u)R_1(u+v)K_0(v).$$

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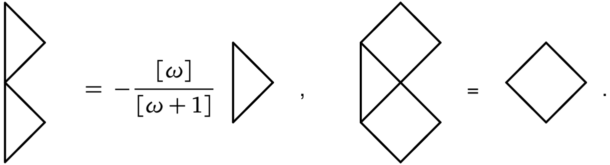
Temperley-Lieb Type B

TL quotient of Hecke Type B (generators $e_i := -a_i$)

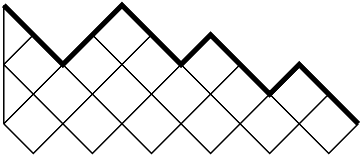
$$e_0^2 = -\frac{[\omega]}{[\omega+1]} e_0,$$

$$e_1 e_0 e_1 = e_1$$

e_0 can be represented by a half-box:



This leads to an action on Ballot paths:



The boundary equation for Type B is modified to

$$K_0(-x_1)|\Psi\rangle = \pi_0|\Psi\rangle$$

which can be rewritten as

$$-\sum_\alpha \psi_\alpha(x_1,\ldots,x_N)(e_0|\alpha\rangle) = \sum_\alpha (\hat{a}_0\psi_\alpha)(x_1,\ldots,x_N)|\alpha\rangle.$$

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$$\hat{a}_0 = -(\pi_0+1)\frac{k(-x_1,\delta_1)}{[2x_1][\omega_1+1]},$$

is the boundary Hecke projector in the functional representation.

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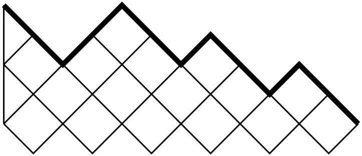

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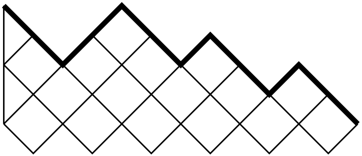

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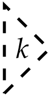
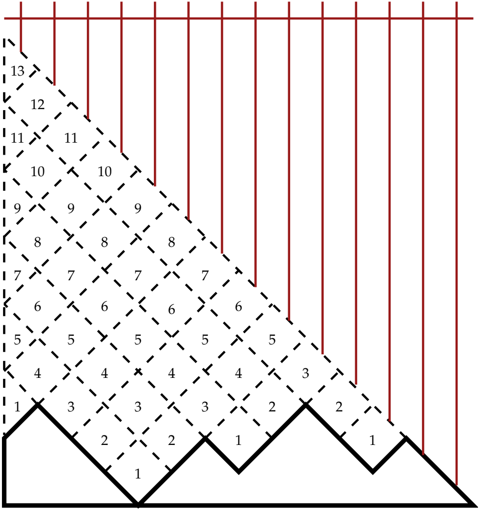
Introduce the Baxterised boundary operator

$$h_0(k) = \hat{s}_0 - \frac{[\lfloor k/2 \rfloor]\,[\omega + \lfloor (k+1)/2 \rfloor]}{[k][\omega+1]}.$$

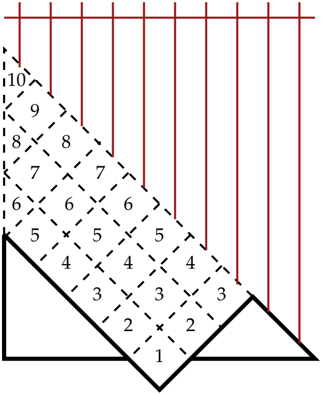
which has the pictorial representation

$h_0(k) =$ 

Solution for Type B:



Truncation conditions:



Polynomial solution of ψ_Ω can be obtained from restricted Macdonald-Koornwinder polynomials

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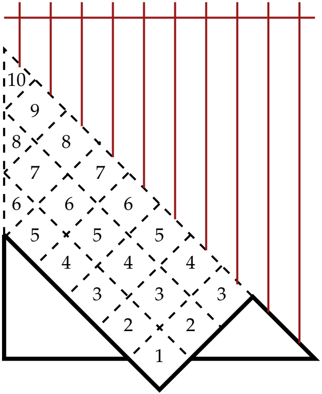

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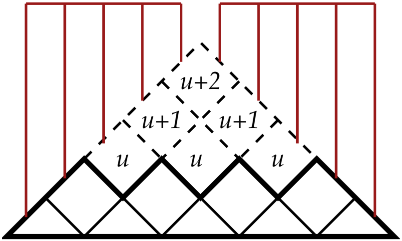
Weighted sums

Consider shifted arguments:



Conjecture:

$$S(L,p,u) = \sum_{\alpha \geq \Omega_p} \left(\frac{[u-1]}{[u]} \right)^{c_{\alpha,p}} \psi_\alpha(x_1,\ldots,x_L).$$

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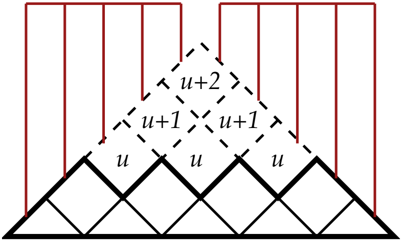

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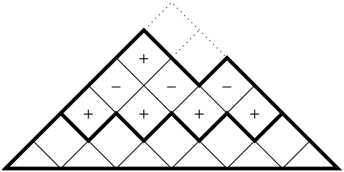


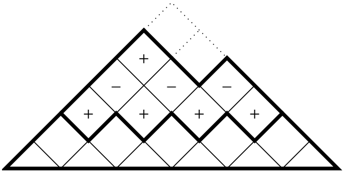
Definition of the $c_{\alpha,p}$ as the signed sum of boxes between two paths

Special values of u:

$$S_+(L,p) = S(L,p,-1) = \sum_{\alpha \geq \Omega_p} \tau^{c_{\alpha,p}} \psi_\alpha,$$

$$S_-(L,p) = S(L,p,2) = \sum_{\alpha \geq \Omega_p} \tau^{-c_{\alpha,p}} \psi_\alpha.$$

$\tau = -[2].$



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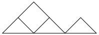
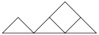
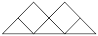
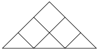
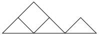
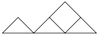
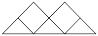
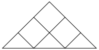
Explicit solution in the limit $x_i \to 0$ for $L = 6$:

α	ψ_α	$\tau^{\pm c_{\alpha,2}}$	$\tau^{\pm c_{\alpha,1}}$
	$1 + 5\tau^2 + 4\tau^4 + \tau^6$	1	
	$\tau(2 + 2\tau^2 + \tau^4)$	$\tau^{\pm 1}$	
	$\tau(1 + 3\tau^2 + \tau^4)$	$\tau^{\pm 1}$	
	$2\tau^2(1 + \tau^2)$	$\tau^{\pm 2}$	1
	τ^3	$\tau^{\pm 1}$	$\tau^{\pm 1}$

$$S_-(6,1) = \tau^2(3 + 2\tau^2) \qquad S_+(6,1) = \tau^2(2 + 3\tau^2)$$
$$S_-(6,2) = 6 + 13\tau^2 + 6\tau^4 + \tau^6 \qquad S_+(6,2) = 1 + 8\tau^2 + 12\tau^4 + 5\tau^6$$

Polynomials in $\tau = -[2]$ with positive coefficients

Explicit solution in the limit $x_i \to 0$ for $L = 6$:

α	ψ_α	$\tau^{\pm c_{\alpha,2}}$	$\tau^{\pm c_{\alpha,1}}$
	$1+5\tau^2+4\tau^4+\tau^6$	1	
	$\tau(2+2\tau^2+\tau^4)$	$\tau^{\pm 1}$	
	$\tau(1+3\tau^2+\tau^4)$	$\tau^{\pm 1}$	
	$2\tau^2(1+\tau^2)$	$\tau^{\pm 2}$	1
	τ^3	$\tau^{\pm 1}$	$\tau^{\pm 1}$

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Theorem

$$S_+(L,p) = \tau^{\nu_{L,p}} T(L,p,\lfloor L/2 \rfloor - p),$$
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where

$$T(L,p,k) = \det_{1 \le i,j \le p} \left(\sum_{r=0}^{2p} \binom{i+k-1}{r-i} \binom{j+L-2p-k}{2j-r} \tau^{2(2j-r)} \right)$$

$T(L,p,k)$ satisfies the discrete Hirota equation

$$T(L,p,k)T(L-2,p-2,k+2) = T(L-1,p-2,k+2)T(L-1,p,k) + \tau^2 T(L-2,p-1,k)T(L,p-1,k+2)$$

$$T(L,p,k) = \sum_{1 \le r_1 < r_2 < \ldots < r_p} \det_{1 \le i,j \le p} \left(\binom{i+k-1}{r_j - i} \right) \det_{1 \le i,j \le p} \left(\binom{i+L-2p-k}{2i-r_j} \tau^{2(2i-r_j)} \right)$$

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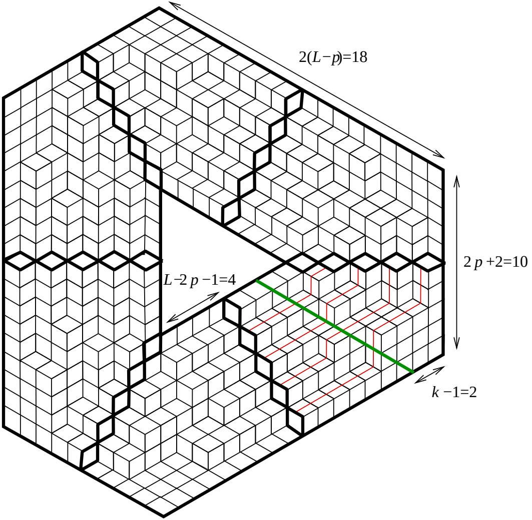
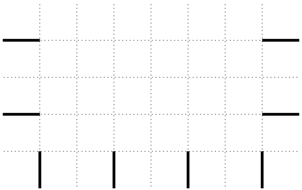
$T(L, p, k)$ enumerate weighted Punctured Cyclically Symmetric Transpose Complement Plane Partitions [Di Francesco and Zinn-Justin for $p = \lfloor (L-1)/2 \rfloor$]



FPL diagrams

Fully Packed Loop configurations on rectangular grid:



FPL diagrams on this grid are in bijection with vertically symmetric alternating-sign matrices.

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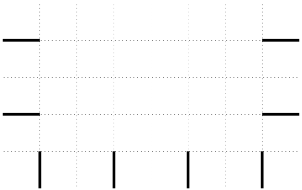

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Define $\Delta(x) = \prod_{i<j}(x_i - x_j)$, $\Delta^*(x) = \prod_{i<j}(1 - x_i x_j)$, and

$$Z_{\mathrm{UASM}}(x,y) \propto \frac{1}{\Delta(x)\Delta(y)\Delta^*(x)\Delta^*(y)} \times \det_{1\le i,j\le n}\left(\frac{1}{(q^2x_i - y_j)(q^2y_j - x_i)(q^2 - x_iy_j)(1 - q^2x_iy_j)}\right)$$

Theorem[Kuperberg]

$$A_{\mathrm{VSASM}}(2n+1;\tau^2) = Z_{UASM}(1,1), \qquad \tau = -q - q^{-1}$$

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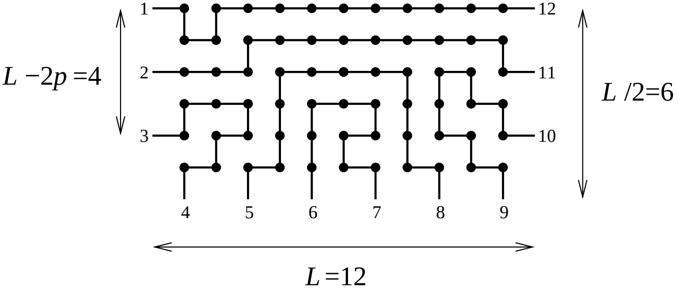
Restricted FPL diagrams



An FPL diagram for $L=12$, with $L/2-p=2$ loop lines connecting loop terminals 1,2 with $L, L-1$.

We call such FPL diagrams p-restricted.

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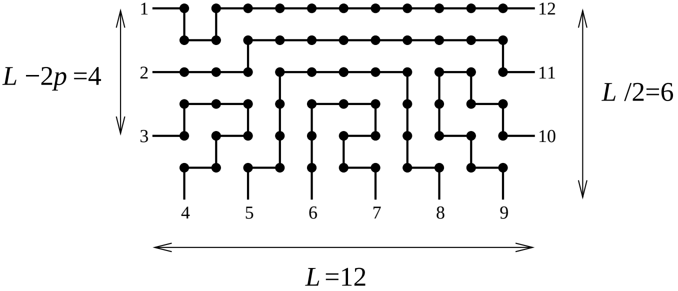

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of p-restricted FPL diagrams = $S_{\pm}(L,p)\Big|_{\tau=1}$

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Conclusion

- Factorised expressions for qKZ solutions
- Kazhdan-Lusztig form
- Partial sums from generalised shifts
- Homogeneous limit: enumeration of PCSTCPPs and p-restricted VSFPL diagrams

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