Gaudin functions of arbitrary level

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The theory of symmetric functions mostly relies on the Cauchy formula, that is

$$\det\left(\frac{1}{x-y}\right) = \frac{\Delta(x)\Delta(y)}{\prod(x-y)} \quad 1$$

"1" contains all the information about symmetric functions, except what concerns the plethysm.

Next: Gaudin-Izergin-Korepin determinant:

$$\det\left(\frac{1}{(x-y)(x-ty)}\right) = \frac{\Delta(x)\Delta(y)}{\prod(x-y)(x-ty)}\ F(\mathbf{x},\mathbf{y})$$

Useful to enumerate ASM , by specialization of t to a root of unity.

NextNext: Gaudin function of level r.

$$\det\left(\frac{1}{(x-y)\cdots(x-t^r y)}\right) = \frac{\Delta(x)\Delta(y)}{\prod(x-y)\cdots(x-t^r y)}\; F^r(\mathbf{x},\mathbf{y})$$

with $F^r(\mathbf{x},\mathbf{y})$ a function symmetrical in $\mathbf{x}$, and symmetrical in $\mathbf{y}$.

The classical case was level $r = 1$.

How to compute ?

Problem: difficult to experiment, the expansion of the determinant becomes to big.

```
matrix([seq([ seq(
    1/(x.i-y.j)/(x.i-t*y.j)/(x.i-t^2*y.j),
                                  j=1..3)],i=1..3)]);
factor(det(%));
        Error, (in minor) object too large
```

Fortunately, Newton's divided differences allow to compute with rational functions, and in particular, to increase the number of poles.

$$\frac{1}{x-z_1}\partial_1 \cdots \partial_{r-1} = \frac{1}{(x-z_1)\cdots(x-z_r)}$$

where ∂_i is the divided difference

$$f \to (f - f^{s_i})(z_i - z_{i+1})^{-1}$$

and s_i = transposition of z_i, z_{i+1}.

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	1	z_{i+1}
Young	1	z_i
Newton	0	-1
Demazure	1	0
Macdonald	t	z_i

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$$

Divided difference like operators

We can pass from Cauchy to Gaudin by introducing n sets of $r+1$ variables extending the y_j variables :

$$\mathbf{y}^j := \{y_j^0 = y_j, y_j^1, \ldots, y_j^r\}\ ,\ j = 1, \ldots, n\,,$$

and using the products of divided differences

$$\prod_{j=1}^{n} \partial_0^j \cdots \partial_{r-1}^j\,,$$

where ∂_i^j is relative to the pair y_j^i, y_j^{i+1}.

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$$\frac{(1-z_2)}{(1-z_1)(1-z_2)}\,\partial_1 =$$

$$\frac{1}{(1-z_1)(1-z_2)}$$

Divided differences increase the number of poles.

Let $z_1,\ldots,z_N$ be $N=2n$ complex indeterminates (spectral parameters). We consider the following multiple contour integrals:

$$\Psi_{a_1,\ldots,a_n} = \prod_{1\le i<j\le N} (q\,z_i - q^{-1}z_j)\times \\ \times \oint\cdots\oint \prod_{\ell=1}^{n} \frac{\mathrm{d}w_\ell}{2\pi i} \frac{\prod_{1\le \ell<m\le n}(w_m-w_\ell)(q\,w_\ell-q^{-1}w_m)}{\prod_{\ell=1}^{n}\prod_{1\le i\le a_\ell}(w_\ell-z_i)\prod_{a_\ell<i\le N}(q\,w_\ell-q^{-1}z_i)} \tag{3.7}$$

where $(a_\ell)_{\ell=1,\ldots,n}$ is any *non-decreasing* sequence of integers in $\{1,\ldots,N-1\}$. The contours catch the poles at $w_i=z_j$ but not those at $w_i=q^{-2}z_j$. These integrals are closely related to formulae for solutions of level 1 qKZ equation in the *spin* basis, as given in e.g. [17]. In appendix B, a detailed discussion of the connection between the two types of integrals is given.

We want to show the following: $\Psi_{a_1,\ldots,a_n}$ *is a homogeneous polynomial in the variables* $z_1,\ldots,z_N$ *of degree* $n(n-1)$*. Furthermore it satisfies the wheel condition: for all ordered triplets* i,j,k*,*

$$\Psi_{a_1,\ldots,a_n}(\ldots,z_i=z,\ldots,z_j=q^2z,\ldots,z_k=q^4z,\ldots)=0 \qquad 1\le i<j<k\le N \tag{3.8}$$

To prove this, we first write explicitly the residue formula for Eq. (3.7):

$$\Psi_{a_1,\ldots,a_n} = \prod_{1\le i<j\le n}(q\,z_i-q^{-1}z_j)\times \\ \times \sum_{\substack{\{k_1,\ldots,k_n\}\\ k_\ell\ne k_m,1\le k_\ell\le a_\ell}} \frac{\prod_{1\le\ell<m\le n}(z_{k_m}-z_{k_\ell})(q\,z_{k_\ell}-q^{-1}z_{k_m})}{\prod_{\ell=1}^{n}\prod_{1\le i\le a_\ell, i\ne k_\ell}(z_{k_\ell}-z_i)\prod_{a_\ell<i\le N}(q\,z_{k_\ell}-q^{-1}z_i)} \\ = \sum_{\substack{K=\{k_1,\ldots,k_n\}\\ k_\ell\ne k_m,1\le k_\ell\le a_\ell}} (-1)^{s(k_\ell)} \frac{\prod\limits_{1\le \ell<m\le n}(q\,z_{k_\ell}-q^{-1}z_{k_m}) \prod\limits_{\substack{1\le i<j\le N\\ i\notin K \text{ or } i=k_\ell, j\le a_\ell}}(q\,z_i-q^{-1}z_j)}{\prod\limits_{\ell=1}^{n}\prod\limits_{\substack{1\le i\le a_\ell\\ i\notin K \text{ or } i>k_\ell}}(z_{k_\ell}-z_i)} \tag{3.9}$$

where $(-1)^{s(k_\ell)}$ is the sign of the permutation that places the k_ℓ in increasing order.

Let us now compute the residue at $z_i\to z_j$. Note that at least one of the two integers i, j must belong to K for the residue of the summand to be non-zero. Two cases arise: a) terms where both i and j are in K, say $j=k_\ell$ and $i=k_m$ with $k_\ell<k_m\le a_\ell$ (and as always $k_m\le a_m$). Then one can switch k_ℓ and k_m: now $k_\ell=i$, $k_m=j$, $k_m<k_\ell\le a_m$

Here is the effect of using ∂_0^1 and ∂_0^2:

$$\begin{vmatrix} \frac{1}{x_1 - y_1^0} & \frac{1}{x_2 - y_1^0} \\ \frac{1}{x_1 - y_2^0} & \frac{1}{x_2 - y_2^0} \end{vmatrix} \longrightarrow \begin{vmatrix} \frac{1}{(x_1 - y_1^0)(x_1 - y_1^1))} & \frac{1}{(x_2 - y_1^0)(x_2 - y_1^1)} \\ \frac{1}{x_1 - y_2^0} & \frac{1}{x_2 - y_2^0} \end{vmatrix}$$

$$\longrightarrow \begin{vmatrix} \frac{1}{(x_1 - y_1^0)(x_1 - y_1^1))} & \frac{1}{(x_2 - y_1^0)(x_2 - y_1^1)} \\ \frac{1}{(x_1 - y_2^0)(x_1 - y_2^1)} & \frac{1}{(x_2 - y_2^0)(x_2 - y_2^1)} \end{vmatrix}$$

To get an identity, we need to compute the image of Cauchy in a second manner. The product of Cauchy by the denominator of the Gaudin function is

$$\Delta(\mathbf{x}) \left| (y_i^0)^j \, R(\mathbf{x}, \mathbf{y}^i - y_i^0) \right|_{i=1,\ldots,n,\; j=0,\ldots,n-1}$$

where $R(A,B)$ is the resultant $\prod_{a\in A}\prod_{b\in B}(a-b)$ of two sets of indeterminates A, B, and where $\mathbf{y}^i - y_i^0$ stands for $\{y_j^1, \ldots, y_j^r\}$.

We have now to compute the image of each entry under divided differences, having passed from rational functions to polynomials.

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$$R(A,B) = \prod (a-b)$$

$$= S_{\beta^\alpha}(A-B)$$

The resultant is a Schur function of a difference of alphabets

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$$\frac{\prod_b (1 - zb)}{\prod_a (1 - za)} = \sum z^n S_n(A - B)$$

Complete functions of A-B

$$0 \; 0 \; 1 \; 1 \; 2 \; 2 \; 3 \; 3$$

It is a simple exercise to arrive to a determinant of Schur functions :

$$\det \left(S_{j\Box}(\mathbf{y}^i - \mathbf{x} \right) \ ,$$

with $j\Box = j, \underbrace{n-1, \ldots, n-1}_{r} = j, (n-1)^r$.

Going back to the original variables y_i, one gets a determinant of Schur functions to express the Gaudin function :

$$F^r(\mathbf{x}, \mathbf{y})\, \Delta(\mathbf{y}) = \det \left(S_{j\Box}(y_i + \cdots + t^r y_i - \mathbf{x} \right) \ .$$

For example, for $r = 1$, $n = 3$,

$$F^1(\mathbf{x}, \mathbf{y})\Delta(\mathbf{y}) =$$

$$\begin{vmatrix} S_{022}(y_1+ty_1 - \mathbf{x}) & S_{122}(y_1+ty_1 - \mathbf{x}) & S_{222}(y_1+ty_1 - \mathbf{x}) \\ S_{022}(y_2+ty_2 - \mathbf{x}) & S_{122}(y_2+ty_2 - \mathbf{x}) & S_{222}(y_2+ty_2 - \mathbf{x}) \\ S_{022}(y_3+ty_3 - \mathbf{x}) & S_{122}(y_3+ty_3 - \mathbf{x}) & S_{222}(y_3+ty_3 - \mathbf{x}) \end{vmatrix}$$

Notice that the functions $S_{j\square}$ are subresultants. Indeed $R(A,B)$ is equal to $S_{\beta^\alpha}(A-B)$, with $\alpha = card(A)$, $\beta = card(B)$.

Here, we have sets of cardinality $r{+}1, n$, and $\square = (n{-}1)^r$. However, when A, B have a letter c in common, then $S_{(\beta-1)^{\alpha-1}}(A-B)$ becomes the resultant of $A{-}c$, $B{-}c$. This function vanishes if A, B have more than one letter in common.

Therefore, when specializing $\mathbf{x}$ into a subset of

$$\{y_1, \ldots, y_n, \ldots, t^r y_1, \ldots, t^r y_n\},$$

then each entry of the Gaudin determinant either vanishes or becomes a resultant.

This gives enough specializations (which are either 0 or products of linear factors $y_j - t^k y_i$) to characterize the Gaudin function, as stated by the next theorem.

Theorem. $F_n^r(\mathbf{x},\mathbf{y})$ *is the only linear combination of Schur functions in* $\mathbf{x}$ *(with coefficients in* $\mathbf{y}$*) indexed by partitions contained in* $((n-1)r)^n$*, which has the same specializations*

$$\mathbf{x} \subset \{y_1,\ldots,y_n,\ldots,t^r y_1,\ldots,t^r y_n\}$$

than the function

$$G_n^r(\mathbf{x},\mathbf{y}) := \frac{\Delta(\mathbf{x})}{\Delta(\mathbf{y})} \prod_i S_\square(y_i + \cdots + t^r y_i - \mathbf{x})\,.$$

For a cubic root of unity, the Izergin-Korepin determinant specializes, up to evident factors, into a Schur function in the union of $\mathbf{x}$ and $\mathbf{y}$, which can be used in the enumeration of alternating sign matrices.

For an odd level r, $F_n^r(\mathbf{x},\mathbf{y})$ possesses similar properties when $t = \exp(2\pi\sqrt{-1}/(r+2))$. The determinant is equal to

$$\det\left(\frac{x_i - y_j}{x_i^{r+2} - y_j^{r+2}}\right) = \frac{R(\mathbf{x},\mathbf{y})\Delta(\mathbf{x})\Delta(\mathbf{y})}{R(\mathbf{x}^{r+2},\mathbf{y}^{r+2})}$$

$$S_{0,0,r,r,\ldots,(n-1)r,(n-1)r}(\mathbf{x}+\mathbf{y})$$

morphism

$$f \longrightarrow f\square_\omega := \sum_{w\in\mathfrak{S}_n} \left(f \frac{\prod_{i<j}(tx_i - x_j)}{x_i - x_j} \right)^w \in \mathfrak{Sym}(\mathbf{x})\,.$$

$\square_\omega$ is a symmetrizer which sends dominant monomials onto Hall-Littlewood polynomials, up to normalization. It can be factorized in the Hecke algebra. For example, for n=3,

$$\square_\omega = \square_1 \left(\square_2 - \frac{t}{1+t} \right) \square_1$$

The usual generators of the Hecke algebra are

$$T_i := \square_i - 1\,.$$

We also need an affine operation θ, the incrementation of indices :

$$x_i\,\theta = x_{i+1}\,, \quad \text{periodicity } x_{i+n} = x_i t^{-1}\,.$$

We have now all the ingredients to cook-up the Gaudin-Izergin-Korepin function.

Theorem. *Let f be a function of 1 variable. Then*

$$f(x_1)\,R(\mathbf{x}-x_1\,,\,\mathbf{y})(1-t\theta)\cdots(1-t^{n-1}\theta)\Box_\omega$$
$$=\Big(f(x_1)x_2\cdots x_n\partial_1\cdots\partial_{n-1}\Big)\,F_n^1(\mathbf{x},\mathbf{y})\,[n]!\,.$$

In particular, when $f=1$,

$$R(\mathbf{x}-x_1\,,\,\mathbf{y})(1-t\theta)\cdots(1-t^{n-1}\theta)\Box_\omega = F_n^1(\mathbf{x},\mathbf{y})\,[n]!$$

The proof consists in testing the specializations of both sides of the equation in the points

$$\mathbf{y}\subset\{x_1,\ldots,x_n,x_2t^{-1},\ldots x_nt^{-1}\}\,.$$

Explicitely, for $n = 4$, one starts with

$$f(x_1)\,R(x_2{+}x_3{+}x_4,\mathbf{y})(1-t\theta)(1-t^2\theta)(1-t^3\theta)$$

$$\begin{aligned}= f(x_1)\,R(x_2{+}x_3{+}x_4,\mathbf{y}) - t\begin{bmatrix}3\\1\end{bmatrix} f(x_2)\,R(x_3{+}x_4{+}x_1/t,\mathbf{y})\\ + t^3\begin{bmatrix}3\\2\end{bmatrix} f(x_3)\,R(x_4{+}x_1/t{+}x_2/t,\mathbf{y})\\ - t^6 f(x_4)\,R(x_1/t{+}x_2/t{+}x_3/t,\mathbf{y})\end{aligned}$$

The sum under the symmetric group can be written

$$\sum_w \left(f(x_1) R(x_2{+}x_3{+}x_4, \mathbf{y}) \frac{\Delta_t(1234)}{\Delta(1234)} \right)^w$$

$$- t \begin{bmatrix} 3 \\ 1 \end{bmatrix} \sum_w \left(f(x_1) R(x_3{+}x_4{+}x_2 t^{-1}, \mathbf{y}) \frac{\Delta_t(2134)}{\Delta(2134)} \right)^w$$

$$+ t^3 \begin{bmatrix} 3 \\ 2 \end{bmatrix} \sum_w \left(f(x_1) R(x_4{+}x_2 t^{-1}{+}x_3 t^{-1}, \mathbf{y}) \frac{\Delta_t(3214)}{\Delta(3214)} \right)^w$$

$$- t^6 \sum_w \left(f(x_1) R(x_2 t^{-1}{+}x_3 t^{-1}{+}x_4 t^{-1}, \mathbf{y}) \frac{\Delta_t(4231)}{\Delta(4231)} \right)^w$$

using the Vandermonde, and the t-Vandermonde $\Delta_t = \prod_{i<j}(tx_i - x_j)$.

One checks the specializations of the coefficient of $f(x_1)$ in both sides. They coincide, knowing (Lagrange interpolation !) that $f(x_1)x_2x_3x_4\partial_1\partial_2\partial_3$ is equal to

$$\frac{f(x_1)x_2x_3x_4}{R(x_1, x_2{+}x_3{+}x_4)} + \frac{f(x_2)x_1x_3x_4}{R(x_2, x_1{+}x_3{+}x_4)} + \frac{f(x_3)x_1x_2x_4}{R(x_3, x_1{+}x_2{+}x_4)} + \frac{f(x_4)x_1x_2x_3}{R(x_4, x_1{+}x_2{+}x_3)} .$$

Remark due to Pasquier. $F_n^1(\mathbf{x},\mathbf{y})$ is a factor of the (non-symmetric) Macdonald polynomial of index $[2n-2,\ldots,2,0,2n-2,\ldots,2,0]$, when $q=t^6$.

Indeed, the affine Hecke algebra is not far. To introduce it in the pedestrian way that I favoured with M.P. Schützenberger, I shall use $\square_i := \square_{s_i}$ which is the morphism :

$$f \longrightarrow f\,\square_i = f(tx_i - x_{i+1})\,\partial_i\,.$$

More generally, the *Euler-Poincaré characteristic* is the

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$$f(x_1 \mid \underbrace{x_2 \cdots x_{n+1}}_{\text{symmetric}})$$

interpolation points : A, $n+1$

$$\sum_{a \in A} \frac{f(a \mid A \setminus a)}{R(a \mid A \setminus a)}$$

$$\frac{1}{1-q} = \{1, q, q^2, q^3, \dots\}$$

Lagrange Interpolation

Taking $f = 1$, we shall get a statement about Macdonald polynomials already obtained by Warnaar.

Let us use λ-rings. Given three alphabets A, B, C, the function $\sigma_1(AB - C)$ is by definition equal to $\prod_c(1-c)\prod_{a,b}(1-ab)^{-1}$, and therefore, one infers that

$$\sigma_1\left(\mathbf{xy}\frac{1-t}{1-q}\right) = \prod_{x,y}\prod_{i\geq 0}\frac{1-tq^ixy}{1-q^ixy}$$

This is the generating function of the symmetric Macdonald polynomials. Let τ_q be the following incrementation of indices:

$$x_i \, \tau_q = x_{i+1} \,, \quad \text{periodicity } x_{i+n} = q x_i \,.$$

Inspired by the preceding computation, we now want

$$\sigma_1 \left(\mathbf{xy} \frac{1-t}{1-q} \right) (1 - t\tau_q) \cdots (1 - t^n \tau_q) \, \Box_\omega \,.$$

Since

$$\mathbf{x}\frac{1-t}{1-q}\tau_q = \mathbf{x}\frac{1-t}{1-q} + x_1(t-1)\,,$$

q disappears from the computation ! Indeed

$$\sigma_1\left(\mathbf{xy}\frac{1-t}{1-q}\right)(1-t\tau_q)\cdots(1-t^n\tau_q)\,\square_\omega$$

$$= \sigma_1\left(\mathbf{xy}(1-t)\right)(1-t\tau_0)\cdots(1-t^n\tau_0)\square_\omega\,\sigma_1\left(\mathbf{xy}q\frac{1-t}{1-q}\right)$$

Thus the starting function is now

$$\sigma_1\,(\mathbf{xy}(1{-}t)) = \prod (1 - txy)(1 - xy)^{-1}$$

that one rewrites, using the variables $y_i^\vee = y_i^{-1}$, as

$$R(tx_1 + \cdots + tx_n, \mathbf{y}^\vee)/R(\mathbf{x}, \mathbf{y}^\vee)\,.$$

We are back to the functions used in the preceding theorem, which gives, as a corollary, that

$$\sigma_1\,(\mathbf{xy}(1{-}t))(1 - t\tau_0)\cdots(1 - t^n\tau_0)\square_\omega = \sigma_1(\mathbf{xy})\,\widetilde{F}_n^1(\mathbf{x}, \mathbf{y})\,[n]!\,,$$

where $\widetilde{F}_n^1(\mathbf{x}, \mathbf{y})$ is the Gaudin function

$$(x_1 \cdots x_n)^{n-1}\,F_n^1(\mathbf{x}^\vee, \mathbf{y})\,.$$

Notice that $\sigma_1\left(\mathbf{xy}(1-t)\right)$ is the generating function of Hall-Littlewood polynomials. Reintroducing the factor in q, we get the action on the generating function of Macdonald polynomials :

$$\sigma_1\left(\mathbf{xy}\frac{1-t}{1-q}\right)(1-t\tau_q)\cdots(1-t^n\tau_q)\,\square_\omega$$
$$= \sigma_1\left(\mathbf{xy}\frac{1-t}{1-q}\right)\sigma_1(t\mathbf{xy})\widetilde{F}_n^1(\mathbf{x},\mathbf{y})\,.$$

Warnaar writes differently the LHS. Indeed, there exists commuting operators $\xi_1, \ldots, \xi_n$ which act diagonally on the basis of non-symmetric Macdonald polynomials. The eigenvalues are $q^{\lambda_1} t^{n-1}, \ldots, q^{\lambda_n} t^0$ for the polynomial M_λ indexed by $\lambda : \lambda_1 \geq \cdots \geq \lambda_n \geq 0$. The symmetric functions in $\xi_1, \ldots, \xi_n$ act therefore diagonally on the symmetric Macdonald polynomials. Knowing that $\square_\omega \, \xi_i \, \square_\omega = \square_\omega \, t^{n-i} \tau_q \, \square_\omega$, one has

$$P_\lambda(\mathbf{x}; q, t) \prod_{i=1}^{n} (1 - q^{\lambda_i} t^{n-i+1})$$
$$= P_\lambda(\mathbf{x}; q, t) (1 - t\tau_q) \cdots (1 - t^n \tau_q) \, \square_\omega \, .$$

In final, the preceding theorem is Warnaar's theorem :

$$\sum_{\lambda} b_\lambda P_\lambda(\mathbf{x}; q, t) P_\lambda(\mathbf{y}; q, t) \prod_{i=1}^{n} (1 - q^{\lambda_i} t^{n-i+1})$$
$$= \sigma_1\left(\mathbf{xy}\frac{1-t}{1-q}\right) \sigma_1(t\mathbf{xy}) \widetilde{F}_n^1(\mathbf{x}, \mathbf{y}),$$

up to notations, signs and missing factorials.

This shows another link between the Izergin-Korepin determinant and Macdonald polynomials.