Multivariate Generalization of Hankel Determinants Involving Catalan Numbers and Middle Binomial Coefficients

Soichi OKADA

1 Hankel determinants

For a given sequence $\{a_m\}_{m\geq 0}$, the Hankel determinants $H_n^k(\{a_m\})$ is defined by

$$H_n^k(\{a_m\}) = \det\left(a_{i+j+k}\right)_{0\leq i,j\leq n-1}.$$

In this note, we give a generalization of the Hankel determinants for the Catalan numbers

$$c_m = \frac{1}{m+1}\binom{2m}{m} = \frac{1}{2m+1}\binom{2m+1}{m} = \frac{(2m)!}{m!(m+1)!},$$

and the middle binomial coefficients

$$b_m = \binom{2m+1}{m} \quad \text{and} \quad d_m = \binom{2m}{m}.$$

The first few temrs are

m	0	1	2	3	4	5
c_m	1	1	2	5	14	42
b_m	1	3	10	35	126	462
d_m	1	2	6	20	70	252

The Hankel determinants of the sequences $\{c_m\}$ and $\{b_m\}$ are given in [3]. (See also [1] and [4]).

Theorem 1.1.

$$H_n^k(\{c_m\}) = \begin{cases} 1 & \text{if } k=0, \\ \displaystyle\prod_{1\leq i\leq j\leq k-1} \frac{i+j+2n}{i+j} & \text{if } k\geq 1, \end{cases} \tag{1}$$

$$H_n^k(\{b_m\}) = \prod_{1\leq i\leq j\leq k} \frac{i+j-1+2n}{i+j-1}, \tag{2}$$

$$H_n^k(\{d_m\}) = 2^n \prod_{0\leq i<j\leq k-1} \frac{i+j+2n}{i+j}. \tag{3}$$

In fact, the last identity (3) was not treated in [3], but it follows easily from (2) by noting

$$\binom{2m}{m} = \binom{2m-1}{m-1} + \binom{2m-1}{m} = 2\binom{2m-1}{m-1}.$$

2 Main theorem

In this note, we give a multi-variate generalization of these determinant evaluation from the view-point of representation theory.

For a non-negative integer k, we denote by $e_k(x_1, \cdots, x_N)$ and $h_k(x_1, \cdots, x_N)$ the k-th elementary and complete symmetric polynomials in $x_1, \cdots, x_N$ repsectively. If $k < 0$, then we put $e_k = h_k = 0$. Also we put

$$e_k^{\circ} = e_k - e_{k-2}, \quad h_k^{\circ} = h_k - h_{k-2}, \quad e_k^{\triangle} = e_k - e_{k-1}, \quad h_k^{\triangle} = h_k - h_{k-1}.$$

The generating functions of these symmetric polynomials are given by

$$\sum_{k \geq 0} e_k(x_1, \cdots, x_N) t^k = \prod_{i=1}^{N} (1 + x_i t), \quad \sum_{k \geq 0} h_k(x_1, \cdots, x_N) t^k = \prod_{i=1}^{N} \frac{1}{1 - x_i t},$$

$$\sum_{k \geq 0} e_k^{\circ}(x_1, \cdots, x_N) t^k = (1 - t^2) \prod_{i=1}^{N} (1 + x_i t), \quad \sum_{k \geq 0} h_k^{\circ}(x_1, \cdots, x_N) t^k = (1 - t^2) \prod_{i=1}^{N} \frac{1}{1 - x_i t},$$

$$\sum_{k \geq 0} e_k^{\triangle}(x_1, \cdots, x_N) t^k = (1 - t) \prod_{i=1}^{N} (1 + x_i t), \quad \sum_{k \geq 0} h_k^{\triangle}(x_1, \cdots, x_N) t^k = (1 - t) \prod_{i=1}^{N} \frac{1}{1 - x_i t}.$$

In the following, we use the plethystic notation. We write $X = x_1 + \cdots + x_N$ and $e_k(X) = e_k(x_1, \cdots, x_N)$, $h_k(X) = h_k(x_1, \cdots, x_N)$, e.t.c..

For a partition $\lambda = (\lambda_1, \cdots, \lambda_k)$ of length $\leq k$, we define

$$s_{\langle \lambda \rangle, \mathbf{Sp}_{2k}}(x_1, \cdots, x_k) = \frac{\det \left(x_i^{\lambda_j + n - j + 1} - x_i^{-\lambda_j - n + j - 1} \right)_{1 \leq i,j \leq n}}{\det \left(x_i^{n - j + 1} - x_i^{-n + j - 1} \right)_{1 \leq i,j \leq n}}.$$

If $\lambda = (\lambda_1, \cdots, \lambda_k)$ is a partition of length $\leq k$, or a half-partition of length k (i.e. $\lambda_i \in \mathbb{Z} + 1/2$ and $\lambda_1 \geq \cdots \geq \lambda_k$), we define

$$s_{[\lambda], \mathbf{O}_{2k+1}}(x_1, \cdots, x_k) = \frac{\det \left(x_i^{\lambda_j + n - j + 1/2} - x_i^{-\lambda_j - n + j - 1/2} \right)_{1 \leq i,j \leq n}}{\det \left(x_i^{n - j + 1/2} - x_i^{-n + j - 1/2} \right)_{1 \leq i,j \leq n}},$$

$$s_{[\lambda], \mathbf{O}_{2k}}(x_1, \cdots, x_k) = \frac{\det \left(x_i^{\lambda_j + n - j} + x_i^{-\lambda_j - n + j} \right)_{1 \leq i,j \leq n}}{\det \left(x_i^{n - j} + x_i^{-n + j} \right)_{1 \leq i,j \leq n}}.$$

Then $s_{\langle \lambda \rangle, \mathbf{Sp}_{2k}}$ is the character (evaluated at the diagonal matrix) of the irreducible representation of the symplectic group $\mathbf{Sp}_{2k}$ with highest weight λ. Also $s_{[\lambda], \mathbf{O}_K}$ is the character of the irreducible representation of the orthogonal group $\mathbf{O}_K$ or its double cover $\mathbf{Pin}_K$ with "highest weight" λ.

These characters can be written in terms of elementary symmetric polynomials. (See [2].) If λ is a partition $(1^p) = (\overbrace{1, \cdots, 1}^{p}, 0, \cdots, 0)$ or a half-partition $(1^p) + 1/2 = (\overbrace{3/2, \cdots, 3/2}^{p}, 1/2, \cdots, 1/2)$, then we have

$$s_{[(1^p)], \mathbf{O}_{2k+1}}(z_1, \cdots, z_k) = e_p(z_1, \cdots, z_k, z_1^{-1}, \cdots, z_k^{-1}, 1),$$

$$s_{[(1^p)+1/2],\mathbf{O}_{2k+1}}(z_1,\cdots,z_k)=\prod_{i=1}^k(z_i^{1/2}+z_i^{-1/2})\cdot e_p^{\circ}(z_1,\cdots,z_k,z_1^{-1},\cdots,z_k^{-1}),$$
$$s_{[(1^p)],\mathbf{Sp}_{2k}}(z_1,\cdots,z_k)=e_p^{\circ}(z_1,\cdots,z_k,z_1^{-1},\cdots,z_k^{-1}),$$
$$s_{[(1^p)],\mathbf{O}_{2k}}(z_1,\cdots,z_k)=e_p(z_1,\cdots,z_k,z_1^{-1},\cdots,z_k^{-1}),$$
$$s_{[(1^p)+1/2],\mathbf{O}_{2k}}(z_1,\cdots,z_k)=\prod_{i=1}^k(z_i^{1/2}+z_i^{-1/2})\cdot e_p^{\triangle}(z_1,\cdots,z_k,z_1^{-1},\cdots,z_k^{-1}).$$

In general, for a partition λ of length $\leq k$, we have the following Jacobi–Trudi formulae :

$$s_{[\lambda],\mathbf{O}_{2k+1}}(z_1,\cdots,z_k)=\frac{1}{2}\det\left(e_{{}^t\lambda_i-i+j}(Z_k+1)+e_{{}^t\lambda_i-i-j}(Z_k+1)\right)_{0\leq i,j\leq\lambda_1-1},$$
$$s_{[\lambda+1/2],\mathbf{O}_{2k+1}}(z_1,\cdots,z_k)=\frac{1}{2}\prod_{i=1}^k(z_i^{1/2}+z_i^{-1/2})\cdot\det\left(e^{\circ}_{{}^t\lambda_i-i+j}(Z_k)+e^{\circ}_{{}^t\lambda_i-i-j}(Z_k)\right)_{0\leq i,j\leq\lambda_1-1},$$
$$s_{\langle\lambda\rangle,\mathbf{Sp}_{2k}}(z_1,\cdots,z_k)=\frac{1}{2}\det\left(e^{\circ}_{{}^t\lambda_i-i+j}(Z_k)+e^{\circ}_{{}^t\lambda_i-i-j}(Z_k)\right)_{0\leq i,j\leq\lambda_1-1},$$
$$s_{[\lambda],\mathbf{O}_{2k}}(z_1,\cdots,z_k)=\frac{1}{2}\det\left(e_{{}^t\lambda_i-i+j}(Z_k)+e_{{}^t\lambda_i-i-j}(Z_k)\right)_{0\leq i,j\leq\lambda_1-1},$$
$$s_{[\lambda+1/2],\mathbf{O}_{2k}}(z_1,\cdots,z_k)=\frac{1}{2}\prod_{i=1}^k(z_i^{1/2}+z_i^{-1/2})\cdot\det\left(e^{\triangle}_{{}^t\lambda_i-i+j}(Z_k)+e^{\triangle}_{{}^t\lambda_i-i-j}(Z_k)\right)_{0\leq i,j\leq\lambda_1-1},$$

where $Z_k=z_1+z_1^{-1}+\cdots+z_k+z_k^{-1}$. We note that

$$s_{[\lambda+1/2],\mathbf{O}_{2k+1}}(z_1,\cdots,z_k)=\prod_{i=1}^k(z_i^{1/2}+z_i^{-1/2})\cdot s_{\langle\lambda\rangle,\mathbf{Sp}_{2k}}(z_1,\cdots,z_k).$$

We define

$$s'_{[\lambda],\mathbf{O}_{2k}}(z_1,\cdots,z_k)=\frac{1}{2}\det\left(e^{\triangle}_{{}^t\lambda_i-i+j}(Z_k)+e^{\triangle}_{{}^t\lambda_i-i-j}(Z_k)\right)_{0\leq i,j\leq\lambda_1-1}.$$

Now we can state our main theorem.

Theorem 2.1. Let $z_1,\cdots,z_k,x_1,\cdots,x_{n-1},y_1,\cdots,y_{n-1}$ be indeterminates. Then we have

$$\det\left(s_{[(1^{k+i+j})],\mathbf{O}_{2(k+i+j)+1}}(z_1,\cdots,z_k,x_1,\cdots,x_i,y_1,\cdots,y_j)\right)_{0\leq i,j\leq n-1}$$
$$=s_{[(n^k)],\mathbf{O}_{2k+1}}(z_1,\cdots,z_k), \tag{4}$$
$$\det\left(s_{\langle(1^{k+i+j})\rangle,\mathbf{Sp}_{2(k+i+j)}}(z_1,\cdots,z_k,x_1,\cdots,x_i,y_1,\cdots,y_j)\right)_{0\leq i,j\leq n-1}$$
$$=s_{\langle(n^k)\rangle,\mathbf{Sp}_{2k}}(z_1,\cdots,z_k), \tag{5}$$
$$\det\left(s_{[(1^{k+i+j})],\mathbf{O}_{2(k+i+j)}}(z_1,\cdots,z_k,x_1,\cdots,x_i,y_1,\cdots,y_j)\right)_{0\leq i,j\leq n-1}$$
$$=2^{n-1}s_{[(n^k)],\mathbf{O}_{2k}}(z_1,\cdots,z_k), \tag{6}$$
$$\det\left(s'_{[(1^{k+i+j})],\mathbf{O}_{2(k+i+j)}}(z_1,\cdots,z_k,x_1,\cdots,x_i,y_1,\cdots,y_j)\right)_{0\leq i,j\leq n-1}$$
$$=s'_{[(n^k)],\mathbf{O}_{2k}}(z_1,\cdots,z_k). \tag{7}$$

By substituting $z_1=\cdots=z_k=x_1=\cdots=x_{n-1}=y_1=\cdots=y_{n-1}=1$, we obtain the determinant evaluations in Theorem 1.1. In fact, we have the following specializations.

Lemma 2.2.

$$s_{[(1^k)],\mathbf{O}_{2k+1}}(1,\cdots,1) = b_k,$$
$$s_{[(1^k)],\mathbf{Sp}_{2k}}(1,\cdots,1) = c_{k+1},$$
$$s_{[(1^k)],\mathbf{O}_{2k}}(1,\cdots,1) = d_k,$$
$$s'_{[(1^k)],\mathbf{O}_{2k}}(1,\cdots,1) = c_k.$$

Proof. Since we have

$$e_p(\overbrace{1,\cdots,1}^{n}) = \binom{n}{p},$$

we see that

$$\begin{aligned}
s_{[(1^k)],\mathbf{O}_{2k+1}}(1,\cdots,1) &= \binom{2k+1}{k} = b_k,\\
s_{[(1^k)],\mathbf{Sp}_{2k}}(1,\cdots,1) &= \binom{2k}{k} - \binom{2k}{k-2} = \frac{(2k)!}{k!(k+2)!}\cdot((k+2)(k+1)-k(k-1))\\
&= \frac{(2k)!}{k!(k+2)!}\cdot 2(2k+1) = \frac{(2k+2)!}{(k+1)!(k+2)!} = c_{k+1},\\
s_{[(1^k)],\mathbf{O}_{2k}}(1,\cdots,1) &= \binom{2k}{k} = d_k,\\
s'_{[(1^k)],\mathbf{O}_{2k}}(1,\cdots,1) &= \binom{2k}{k} - \binom{2k}{k-1} = \frac{(2k)!}{k!(k+1)!}\cdot(k+1-k) = \frac{(2k)!}{k!(k+1)!} = c_k.
\end{aligned}$$

□

Lemma 2.3. (1) For a non-negative interger or a non-negative half-integer n, we have

$$s_{[(n^k)],\mathbf{O}_{2k+1}}(1,\cdots,1) = \prod_{1\le i\le j\le k} \frac{2n+i+j-1}{i+j-1},$$

(2) For a non-negative integer n, we have

$$s_{\langle(n^k)\rangle,\mathbf{Sp}_{2k}}(1,\cdots,1) = \prod_{1\le i\le j\le k} \frac{2n+i+j}{i+j},$$

(3) For a non-negative integer or a negative half-integer n, we have

$$s_{[(n^k)],\mathbf{O}_{2k}}(1,\cdots,1) = 2\prod_{0\le i<j\le k-1} \frac{2n+i+i}{i+j}.$$

For a non-negative integer n, we have

$$s'_{[(n^k)],\mathbf{O}_{2k}}(1,\cdots,1) = \prod_{1\le i\le j\le k-1} \frac{2n+i+j}{i+j}.$$

Proof. (1) By applying Weyl's dimension formula to the positive root system

$$\{\varepsilon_i : 1\le i\le k\} \cup \{\varepsilon_i \pm \varepsilon_j : 1\le i<j\le k\}$$

with the Weyl vector $\rho = (k-1/2, k-3/2, \cdots, 3/2, 1/2)$ and the highest weight $\lambda = (n, n, \cdots, n)$, we have

$$s_{[(n^k)],\mathbf{O}_{2k+1}}(1,\cdots,1) = \prod_{p=1}^{k} \frac{n+k-p+1/2}{k-p+1/2} \prod_{1\le p<q\le k} \frac{(n+k-p+1/2)-(n+k-q+1/2)}{(k-p+1/2)-(k-q+1/2)}$$

$$\times \prod_{1\le p<q\le k} \frac{(n+k-p+1/2)+(n+k-q+1/2)}{(k-p+1/2)+(k-q+1/2)}$$

$$= \prod_{p=1}^{k} \frac{2n+2(k-p+1)-1}{2(k-p+1)-1} \prod_{1\le p<q\le k} \frac{2n+(k-p+1)+(k-q+1)-1}{(k-p+1)+(k-q+1)-1}$$

$$= \prod_{i=1}^{k} \frac{2n+2i-1}{2i-1} \prod_{1\le i<j\le k} \frac{2n+j+i-1}{j+i-1}$$

$$= \prod_{1\le i\le j\le k} \frac{2n+i+j-1}{i+j-1},$$

(2) By applying Weyl's dimension formula to the positive root system

$$\{2\varepsilon_i : 1 \le i \le k\} \cup \{\varepsilon_i \pm \varepsilon_j : 1 \le i < j \le k\}$$

with the Weyl vector $\rho = (k, k-1, \cdots, 2, 1)$ and the highest weight $\lambda = (n, n, \cdots, n)$, we have

$$s_{\langle (n^k)\rangle, \mathbf{Sp}_{2k}}(1, \cdots, 1) = \prod_{p=1}^{k} \frac{n+k-p+1}{k-p+1} \prod_{1\le p<q\le k} \frac{(n+k-p+1)-(n+k-q+1)}{(k-p+1)-(k-q+1)}$$

$$\times \prod_{1\le p<q\le k} \frac{(n+k-p+1)+(n+k-q+1)}{(k-p+1)+(k-q+1)}$$

$$= \prod_{p=1}^{k} \frac{2n+2(k-p+1)}{2(k-p+1)} \prod_{1\le p<q\le k} \frac{2n+(k-p+1)+(k-q+1)}{(k-p+1)+(k-q+1)}$$

$$= \prod_{i=1}^{k} \frac{2n+2i}{2i} \prod_{1\le i<j\le k} \frac{2n+j+i}{j+i}$$

$$= \prod_{1\le i\le j\le k} \frac{2n+i+j}{i+j}.$$

(3) We note that the irreducible representation of $\mathbf{O}_{2n}$ with "highest weight" $(n, n, \cdots, n)$ is decomposed into the direct sum of two irreducible representations of $\mathbf{SO}_{2n}$ of the same dimension with highest weight $(n, n, \cdots, n, n)$ and $(n, n, \cdots, n, -n)$. By applying Weyl's dimension formula to the positive root system

$$\{\varepsilon_i \pm \varepsilon_j : 1 \le i < j \le k\}$$

with the Weyl vector $\rho = (k-1, k-2, \cdots, 1, 0)$ and the highest weight $\lambda = (n, n, \cdots, n)$, we have

$$s_{[(n^k)], \mathbf{O}_{2k}}(1, \cdots, 1) = 2 \prod_{1\le p<q\le k} \frac{(n+k-p)-(n+k-q)}{(k-p)-(k-q)} \prod_{1\le p<q\le k} \frac{(n+k-p)+(n+k-q)}{(k-p)+(k-q)}$$

$$= 2 \prod_{1\le p<q\le k} \frac{2n+(k-p)+(k-q)}{(k-p)+(k-q)}$$

$$= 2 \prod_{0\le i<j\le k-1} \frac{2n+j+i}{j+i}.$$

By repracing n by $n+1/2$, we have

$$s_{[(n^k)+1/2], \mathbf{O}_{2k}}(1, \cdots, 1) = 2 \prod_{0\le i<j\le k-1} \frac{2n+1+i+j}{i+j}.$$

By replacing $i+1$ by i, we have

$$\prod_{0\le i<j\le k-1}(2n+1+j+i)=\prod_{1\le i\le j\le k-1}(2n+i+j).$$

Also we see that

$$\begin{aligned}\prod_{0\le i<j\le k-1}(i+j)&=\prod_{j=1}^{k-1}(k-1)\cdot\prod_{1\le i<j\le k-1}(i+j)=2^{-(k-1)}\prod_{j=1}^{k-1}(2k-2)\cdot\prod_{1\le i<j\le k-1}(i+j)\\&=2^{-(k-1)}\prod_{1\le i\le j\le k-1}(i+j).\end{aligned}$$

Hence we have

$$s_{[(n^k)+1/2],\mathbf{O}_{2k}}(1,\cdots,1)=2^k\prod_{1\le i\le j\le k-1}\frac{2n+i+j}{i+j},$$

that is,

$$s'_{[(n^k)],\mathbf{O}_{2k}}(1,\cdots,1)=\prod_{1\le i\le j\le k-1}\frac{2n+i+j}{i+j}.$$

□

3 Proof

In this section, we provide a proof of Theorem 2.1 by applying row and column transformations to the matrices on the left hand side. We begin with the following lemma.

Lemma 3.1. (1) For a set of variables A and B, we have

$$e_r(A+B)=\sum_{s=0}^{r}e_{r-s}(A)e_s(B),$$
$$e^{\circ}_r(A+B)=\sum_{s=0}^{r}e^{\circ}_{r-s}(A)e_s(B),$$
$$e^{\triangle}_r(A+B)=\sum_{s=0}^{r}e^{\triangle}_{r-s}(A)e_s(B).$$

(2) If $A_p=a_1+a_1^{-1}+\cdots+a_p+a_p^{-1}$, then we have

$$e_{p+r}(A_p)=e_{p-r}(A_p),$$
$$e_{p+r}(A_p+1)=e_{p-r+1}(A_p+1),$$
$$e^{\circ}_{p+r}(A_p)=-e^{\circ}_{p-r+2}(A_p),$$
$$e^{\triangle}_{p+r}(A_p)=-e^{\triangle}_{p-r+1}(A_p).$$

Proof. (1) By taking the coefficient of t^r in

$$\sum_{r\ge0}e_r(A+B)t^r=\prod_{a\in A}(1+at)\prod_{b\in B}(1+bt)=\left(\sum_{r\ge0}e_r(A)t^r\right)\left(\sum_{r\ge0}e_r(B)t^r\right),$$

we have

$$e_r(A+B)=\sum_{s=0}^{r} e_{r-s}(A)e_s(B).$$

By using this, we see that

$$\begin{aligned}
e_r^{\circ}(A+B) &= e_r(A+B)-e_{r-2}(A+B)=\sum_{s=0}^{r} e_{r-s}(A)e_s(B)-\sum_{s=0}^{r-2} e_{r-2-s}(A)e_s(B)\\
&=\sum_{s=0}^{r} e_{r-s}^{\circ}(A)e_s(B),\\
e_r^{\triangle}(A+B) &= e_r(A+B)-e_{r-1}(A+B)=\sum_{s=0}^{r} e_{r-s}(A)e_s(B)-\sum_{s=0}^{r-1} e_{r-1-s}(A)e_s(B)\\
&=\sum_{s=0}^{r} e_{r-s}^{\triangle}(A)e_s(B).
\end{aligned}$$

□

Lemma 3.2. Let $x_1,\cdots,x_n$ be indeterminants and put $X_r=x_1+x_1^{-1}+\cdots+x_r+x_r^{-1}$.

(1) The two matrices

$$H^{\circ}=\Big((-1)^{i-j}h_{i-j}^{\circ}(X_{j+1}+1)\Big)_{0\le i,j\le n-1} \quad \text{and} \quad E=\Big(e_{i-j}(X_i+1)\Big)_{0\le i,j\le n-1}$$

are inverse to each other.

(2) The two matrices

$$H=\Big((-1)^{i-j}h_{i-j}(X_{j+1})\Big)_{0\le i,j\le n-1} \quad \text{and} \quad E^{\circ}=\Big(e_{i-j}^{\circ}(X_i)\Big)_{0\le i,j\le n-1}$$

are inverse to each other.

(3) The two matrices

$$H^{\circ}=\Big((-1)^{i-j}h_{i-j}^{\circ}(X_{j+1})\Big)_{0\le i,j\le n-1} \quad \text{and} \quad E=\Big(e_{i-j}(X_i)\Big)_{0\le i,j\le n-1}$$

are inverse to each other.

(4) The two matrices

$$H^{\triangle}=\Big((-1)^{i-j}h_{i-j}^{\triangle}(X_{j+1})\Big)_{0\le i,j\le n-1} \quad \text{and} \quad E^{\triangle}=\Big(e_{i-j}^{\triangle}(X_i)\Big)_{0\le i,j\le n-1}$$

are inverse to each other.

Proof. (1) We show that $EH^{\circ}=I$. Since E and H° are lower-triangular matrices with diagonal entries 1, it is enough to show

$$\sum_{k=j}^{i} e_{i-k}(X_i+1)\cdot(-1)^{k-j}h_{k-j}^{\circ}(X_{j+1}+1)=0 \quad \text{if } i>j.$$

The left hand side is the coefficient of t^{i-j} in the power series

$$\left(\sum_{p\ge 0} e_p(X_i+1)t^p\right)\cdot\left(\sum_{q\ge 0} h_q^{\circ}(X_{j+1}+1)(-t)^q\right)$$

$$= (1+t)\prod_{k=1}^{i}(1+x_kt)(1+x_k^{-1}t)\cdot(1-t^2)\frac{1}{1+t}\prod_{k=1}^{j+1}\frac{1}{(1+x_kt)(1+x_k^{-1}t)}$$

$$= (1-t^2)\prod_{k=j+2}^{i}(1+x_kt)(1+x_k^{-1}t).$$

Hence the coefficient of t^{i-j} for $i>j$ is given by

$$e_{i-j}(x_{j+2},x_{j+2}^{-1},\cdots,x_i,x_i^{-1})-e_{i-j-2}(x_{j+2},x_{j+2}^{-1},\cdots,x_i,x_i^{-1}),$$

which is equal to 0 by Lemma 3.1 (2).

(2) We show that $E^\circ H=I$. Since E° and H are lower-triangular matrices with diagonal entries 1, it is enough to show

$$\sum_{k=j}^{i}e^\circ_{i-k}(X_i)\cdot(-1)^{k-j}h_{k-j}(X_{j+1})=0 \quad \text{if } i>j.$$

The left hand side is the coefficient of t^{i-j} in the power series

$$\left(\sum_{p\geq 0}e^\circ_p(X_i)t^p\right)\cdot\left(\sum_{q\geq 0}h_q(X_{j+1})(-t)^q\right).$$

$$= (1-t^2)\prod_{k=1}^{i}(1+x_kt)(1+x_k^{-1}t)\cdot\prod_{k=1}^{j+1}\frac{1}{(1+x_kt)(1+x_k^{-1}t)}$$

$$= (1-t^2)\prod_{k=j+2}^{i}(1+x_kt)(1+x_k^{-1}t).$$

Hence the coefficient of t^{i-j} for $i>j$ is given by

$$e_{i-j}(x_{j+2},x_{j+2}^{-1},\cdots,x_i,x_i^{-1})-e_{i-j-2}(x_{j+2},x_{j+2}^{-1},\cdots,x_i,x_i^{-1}),$$

which is equal to 0 by Lemm 3.1 (2).

(3) We show that $EH^\circ=I$. Since E and H° are lower-triangular matrices with diagonal entries 1, it is enough to show

$$\sum_{k=j}^{i}e_{i-k}(X_i)\cdot(-1)^{k-j}h^\circ_{k-j}(X_{j+1})=0 \quad \text{if } i>j.$$

The left hand side is the coefficient of t^{i-j} in the power series

$$\left(\sum_{p\geq 0}e_p(X_i)t^p\right)\cdot\left(\sum_{q\geq 0}h^\circ_q(X_{j+1})(-t)^q\right)$$

$$= \prod_{k=1}^{i}(1+x_kt)(1+x_k^{-1}t)\cdot(1-t^2)\prod_{k=1}^{j+1}\frac{1}{(1+x_kt)(1+x_k^{-1}t)}$$

$$= (1-t^2)\prod_{k=j+2}^{i}(1+x_kt)(1+x_k^{-1}t).$$

Hence the coefficient of t^{i-j} for $i > j$ is given by

$$e_{i-j}(x_{j+2}, x_{j+2}^{-1}, \cdots, x_i, x_i^{-1}) - e_{i-j-2}(x_{j+2}, x_{j+2}^{-1}, \cdots, x_i, x_i^{-1}),$$

which is equal to 0 by Lemma 3.1 (2).

(4) We show that $E^{\triangle} H^{\triangle} = I$. Since $E^{\triangle}$ and $H^{\triangle}$ are lower-triangular matrices with diagonal entries 1, it is enough to show

$$\sum_{k=j}^{i} e_{i-k}^{\triangle}(X_i) \cdot (-1)^{k-j} h_{k-j}^{\triangle}(X_{j+1}) = 0 \quad \text{if } i > j.$$

The left hand side is the coefficient of t^{i-j} in the power series

$$\begin{aligned}
&\left(\sum_{p\geq 0} e_p^{\triangle}(X_i) t^p\right) \cdot \left(\sum_{q\geq 0} h_q^{\triangle}(X_{j+1})(-t)^q\right). \\
&= (1-t)\prod_{k=1}^{i}(1+x_k t)(1+x_k^{-1}t) \cdot (1+t)\prod_{k=1}^{j+1}\frac{1}{(1+x_k t)(1+x_k^{-1}t)} \\
&= (1-t^2)\prod_{k=j+2}^{i}(1+x_k t)(1+x_k^{-1}t).
\end{aligned}$$

Hence the coefficient of t^{i-j} for $i > j$ is given by

$$e_{i-j}(x_{j+2}, x_{j+2}^{-1}, \cdots, x_i, x_i^{-1}) - e_{i-j-2}(x_{j+2}, x_{j+2}^{-1}, \cdots, x_i, x_i^{-1}),$$

which is equal to 0 by Lemma 3.1 (2). □

A key to the proof is the following Lemma.

Lemma 3.3. Let $w_1, \cdots, w_l$ and $x_1, x_2, \cdots$ be indeterminates and let $W_l = w_1 + w_1^{-1} + \cdots + w_l + w_l^{-1}$ and $X_k = x_1 + x_1^{-1} + \cdots + x_k + x_k^{-1}$. (Here we understand that $X_0 = 0$.)

(1) We have

$$\Big((-1)^{i-j} h_{i-j}^{\circ}(X_{j+1}+1)\Big)_{i,j=0}^{n-1} \cdot \Big(e_{l+i}(W_l + X_i + 1)\Big)_{i=0}^{n-1} = \Big(e_{l-i}(W_l+1)\Big)_{i=0}^{n-1},$$

equivalenty

$$\sum_{j=0}^{i}(-1)^{i-j} h_{i-j}^{\circ}(X_{j+1}+1) e_{l+j}(W_l + X_j + 1) = e_{l-i}(W_l + 1).$$

(2) We have

$$\Big((-1)^{i-j} h_{i-j}(X_{j+1})\Big)_{i,j=0}^{n-1} \cdot \Big(e_{l+i}^{\circ}(W_l + X_i)\Big)_{i=0}^{n-1} = \Big(e_{l-i}^{\circ}(W_l)\Big)_{i=0}^{n-1},$$

equivalently

$$\sum_{j=0}^{i}(-1)^{i-j} h_{i-j}(X_{j+1}) e_{l+j}^{\circ}(W_l + X_j) = e_{l-i}^{\circ}(W_l).$$

(3) We have

$$\left((-1)^{i-j}h^{\circ}_{i-j}(X_{j+1})\right)_{i,j=0}^{n-1}\cdot\left(e_{l+i}(W_l+X_i)\right)_{i=0}^{n-1}=\left(2^{\chi(i\neq 0)}e_{l-i}(W_l)\right)_{i=0}^{n-1},$$

equivalently

$$\sum_{j=0}^{i}(-1)^{i-j}h^{\circ}_{i-j}(X_{j+1})e_{l+j}(W_l+X_j)=\begin{cases}e_l(W_l) & \text{if } i=0,\\ 2e_{l-i}(W_l) & \text{if } i>0.\end{cases}$$

(4) We have

$$\left((-1)^{i-j}h^{\triangle}_{i-j}(X_{j+1})\right)_{i,j=0}^{n-1}\cdot\left(e^{\triangle}_{l+i}(W_l+X_i)\right)_{i=0}^{n-1}=\left(e^{\triangle}_{l-i}(W_l)\right)_{i=0}^{n-1},$$

equivalently,

$$\sum_{j=0}^{i}(-1)^{i-j}h^{\triangle}_{i-j}(X_{j+1})e^{\triangle}_{l+j}(W_l+X_j)=e^{\triangle}_{l-i}(W_l).$$

Proof. (1) It follows from Lemma 3.1 that

$$e_{l+j}(W_l+X_j+1)=\sum_{s=0}^{2j}e_{l+j-s}(W_l+1)e_s(X_j),$$
$$e_{l+r}(W_l+1)=e_{l-r+1}(W_l)\quad\text{and}\quad e_{j+r}(X_j)=e_{j-r}(X_j).$$

Hence we have

$$\begin{aligned}
e_{l+j}(W_l+X_j+1)&=\sum_{s=0}^{j-1}e_{l+j-s}(W_l+1)e_s(X_j)+\sum_{s=j}^{2j}e_{l+j-s}(W_l+1)e_s(X_j)\\
&=\sum_{s=0}^{j-1}e_{l-j+s+1}(W_l+1)e_s(X_j)+\sum_{s=j}^{2j}e_{l+j-s}(W_l+1)e_{2j-s}(X_j)\\
&=\sum_{s=0}^{j-1}e_{l-j+s+1}(W_l+1)e_s(X_j)+\sum_{t=0}^{j}e_{l+j-(2j-t)}(W_l+1)e_t(X_j)\quad(t=2j-s)\\
&=\sum_{s=0}^{j-1}e_{l-j+s+1}(W_l+1)e_s(X_j)+\sum_{t=0}^{j}e_{l-j+t}(W_l+1)e_t(X_j)\\
&=\sum_{t=0}^{j}e_{l-j+t}(W_l+1)\left(e_{t-1}(X_j)+e_t(X_j)\right)\\
&=\sum_{t=0}^{j}e_{l-j+t}(W_l+1)e_t(X_j+1).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\sum_{j=0}^{i}(-1)^{i-j}h^{\circ}_{i-j}(X_{j+1}+1)e_{l+j}(W_l+X_j+1)\\
&\quad=\sum_{j=0}^{i}(-1)^{i-j}h^{\circ}_{i-j}(X_{j+1}+1)\sum_{t=0}^{j}e_{l-j+t}(W_l+1)e_t(X_j+1)
\end{aligned}$$

$$= \sum_{k=0}^{i} e_{l-k}(W_l+1) \sum_{j=k}^{i} (-1)^{i-j} h^{\circ}_{i-j}(X_{j+1}+1) e_{j-k}(X_j+1)$$
$$= \sum_{k=0}^{i} e_{l-k}(W_l+1)\delta_{ik}$$
$$= e_{l-i}(W_l+1),$$

by using Lemma 3.2 (1).

(2) It follows from Lemma 3.1 that

$$e^{\circ}_{l+j}(W_l+X_j) = \sum_{s=0}^{2j} e^{\circ}_{l+j-s}(W_l) e_s(X_j),$$
$$e^{\circ}_{l+r}(W_l) = -e^{\circ}_{l-r+2}(W_l), \quad e^{\circ}_{l+1}(W_l) = 0. \quad \text{and} \quad e_{j+r}(X_j) = e_{j-r}(X_j).$$

Hence we have

$$\begin{aligned}
e^{\circ}_{l+j}(W_l+X_j) &= \sum_{s=0}^{j-2} e^{\circ}_{l+j-s}(W_l)e_s(X_j) + e^{\circ}_{l+1}(W_l)e_{j-1}(X_j) + \sum_{s=j}^{2j} e^{\circ}_{l+j-s}(W_l)e_s(X_j) \\
&= \sum_{s=0}^{j-2} \left(-e_{l-j+s+2}(W_l)\right) e_s(X_j) + 0 \cdot e_{j-1}(X_j) + \sum_{s=j}^{2j} e^{\circ}_{l+j-s}(W_l)e_{2j-s}(X_j) \\
&= \sum_{s=0}^{j-2} \left(-e_{l-j+s+2}(W_l)\right) e_s(X_j) + \sum_{t=0}^{j} e^{\circ}_{l+j-(2j-t)}(W_l)e_t(X_j) \quad (t = 2j-s) \\
&= \sum_{s=0}^{j-2} \left(-e_{l-j+s+2}(W_l)\right) e_s(X_j) + \sum_{t=0}^{j} e^{\circ}_{l-j+t}(W_l)e_t(X_j) \\
&= \sum_{t=0}^{j} e^{\circ}_{l-j+t}(W_l) \left(-e_{t-2}(X_j) + e_t(X_j)\right) \\
&= \sum_{t=0}^{j} e^{\circ}_{l-j+t}(W_l)e^{\circ}_t(X_j).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\sum_{j=0}^{i} (-1)^{i-j} h_{i-j}(X_{j+1}) e^{\circ}_{l+j}(W_l+X_j) \\
&\quad = \sum_{j=0}^{i} (-1)^{i-j} h_{i-j}(X_{j+1}) \sum_{t=0}^{j} e^{\circ}_{l-j+t}(W_l)e^{\circ}_t(X_j) \\
&\quad = \sum_{k=0}^{i} e^{\circ}_{l-k}(W_l) \sum_{j=k}^{i} (-1)^{i-j} h_{i-j}(X_{j+1}) e^{\circ}_{j-k}(X_j) \\
&\quad = \sum_{k=0}^{i} e^{\circ}_{l-k}(W_l)\delta_{ik} \\
&\quad = e^{\circ}_{l-i}(W_l),
\end{aligned}$$

by using Lemma 3.2 (2).

(3) It follows from Lemma 3.1 that

$$e_{l+j}(W_l + X_j) = \sum_{s=0}^{2j} e_{l+j-s}(W_l)e_s(X_j),$$

$$e_{l+r}(W_l) = e_{l-r}(W_l) \quad \text{and} \quad e_{j+r}(X_j) = e_{j-r}(X_j).$$

Hence we have

$$\begin{aligned}
e_{l+j}(W_l + X_j) &= \sum_{s=0}^{j-1} e_{l+j-s}(W_l)e_s(X_j) + e_l(W_l)e_j(X_j) + \sum_{s=j+1}^{2j} e_{l+j-s}(W_l)e_s(X_j) \\
&= \sum_{s=0}^{j-1} e_{l-j+s}(W_l)e_s(X_j) + e_l(W_l)e_j(X_j) + \sum_{s=j+1}^{2j} e_{l+j-s}(W_l)e_{2j-s}(X_j) \\
&= \sum_{s=0}^{j-1} e_{l-j+s}(W_l)e_s(X_j) + e_l(W_l)e_j(X_j) + \sum_{t=0}^{j-1} e_{l+j-(2j-t)}(W_l)e_t(X_j) \quad (t = 2j - s) \\
&= \sum_{s=0}^{j-1} e_{l-j+s}(W_l)e_s(X_j) + e_l(W_l)e_j(X_j) + \sum_{t=0}^{j-1} e_{l-j+t}(W_l)e_t(X_j) \\
&= 2\sum_{t=0}^{j-1} e_{l-j+t}(W_l)e_t(X_j) + e_l(W_l)e_j(X_j) \\
&= \sum_{t=0}^{j} 2^{\chi(i>0)} e_{l-j+t}(W_l)e_t(X_j).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\sum_{j=0}^{i}(-1)^{i-j}h^{\circ}_{i-j}(X_{j+1})e_{l+j}(W_l + X_j) \\
&\quad = \sum_{j=0}^{i}(-1)^{i-j}h^{\circ}_{i-j}(X_{j+1})\sum_{t=0}^{j} 2^{\chi(i>0)} e_{l-j+t}(W_l)e_t(X_j). \\
&\quad = 2^{\chi(i>0)}\sum_{k=0}^{i} e_{l-k}(W_l)\sum_{j=k}^{i}(-1)^{i-j}h^{\circ}_{i-j}(X_{j+1})e_{j-k}(X_j) \\
&\quad = 2^{\chi(i>0)}\sum_{k=0}^{i} e_{l-k}(W_l)\delta_{ik} \\
&\quad = 2^{\chi(i>0)} e_{l-i}(W_l),
\end{aligned}$$

by using Lemma 3.2 (3).

(4) It follows from Lemma 3.1 that

$$e^{\triangle}_{l+j}(W_l + X_j) = \sum_{s=0}^{2j} e^{\triangle}_{l+j-s}(W_l)e_s(X_j),$$

$$e^{\triangle}_{l+r}(W_l) = -e^{\triangle}_{l-r+1}(W_l), \quad \text{and} \quad e_{j+r}(X_j) = e_{j-r}(X_j).$$

Hence we have

$$e^{\triangle}_{l+j}(W_l + X_j) = \sum_{s=0}^{j-1} e^{\triangle}_{l+j-s}(W_l)e_s(X_j) + \sum_{s=j}^{2j} e^{\triangle}_{l+j-s}(W_l)e_s(X_j)$$

$$
\begin{aligned}
&= \sum_{s=0}^{j-1} \left(-e^{\triangle}_{l-j+s+1}(W_l)\right) e_s(X_j) + \sum_{s=j}^{2j} e^{\triangle}_{l+j-s}(W_l) e_{2j-s}(X_j) \\
&= \sum_{s=0}^{j-1} \left(-e^{\triangle}_{l-j+s+1}(W_l)\right) e_s(X_j) + \sum_{t=0}^{j} e^{\triangle}_{l+j-(2j-t)}(W_l) e_t(X_j) \quad (t = 2j - s) \\
&= \sum_{s=0}^{j-1} \left(-e^{\triangle}_{l-j+s+1}(W_l)\right) e_s(X_j) + \sum_{t=0}^{j} e^{\triangle}_{l-j+t}(W_l) e_t(X_j) \\
&= \sum_{t=0}^{j} e^{\triangle}_{l-j+t}(W_l) \left(-e_{t-1}(X_j) + e_t(X_j)\right) \\
&= \sum_{t=0}^{j} e^{\triangle}_{l-j+t}(W_l) e^{\triangle}_t(X_j).
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
&\sum_{j=0}^{i} (-1)^{i-j} h^{\triangle}_{i-j}(X_{j+1}) e^{\triangle}_{l+j}(W_l + X_j) \\
&\quad = \sum_{j=0}^{i} (-1)^{i-j} h^{\triangle}_{i-j}(X_{j+1}) \sum_{t=0}^{j} e^{\triangle}_{l-j+t}(W_l) e^{\triangle}_t(X_j). \\
&\quad = \sum_{k=0}^{i} e^{\triangle}_{l-k}(W_l) \sum_{j=k}^{i} (-1)^{i-j} h^{\triangle}_{i-j}(X_{j+1}) e^{\triangle}_{j-k}(X_j) \\
&\quad = \sum_{k=0}^{i} e^{\triangle}_{l-k}(W_l) \delta_{ik} \\
&\quad = e^{\triangle}_{l-i}(W_l),
\end{aligned}
$$

by using Lemma 3.2 (4). □

Lemma 3.4. Let $w_1, \cdots, w_l$ and $y_1, y_2, \cdots$ be indeterminates and put $Y_j = y_1 + y_1^{-1} + \cdots + y_j + y_j^{-1}$.

(1) We have

$$
\left(e_{r+j}(W + Y_j)\right)_{j=0}^{n-1} \cdot \left((-1)^{j-i} h^{\circ}_{j-i}(Y_{i+1})\right)_{i,j=0}^{n-1} = \left(2^{-\chi(j=0)}(e_{r+j}(W) + e_{r-j}(W))\right)_{j=0}^{n-1},
$$

equivalently,

$$
\sum_{i=0}^{j} e_{r+i}(W + Y_i) \cdot (-1)^{j-i} h^{\circ}_{j-i}(Y_{i+1}) = 2^{-\chi(j=0)}(e_{r+j}(W) + e_{r-j}(W)).
$$

(2) We have

$$
\left(e^{\circ}_{r+j}(W + Y_j)\right)_{j=0}^{n-1} \cdot \left((-1)^{j-i} h^{\circ}_{j-i}(Y_{i+1})\right)_{i,j=0}^{n-1} = \left(2^{-\chi(j=0)}(e^{\circ}_{r+j}(W) + e^{\circ}_{r-j}(W))\right)_{j=0}^{n-1},
$$

equivalently,

$$\sum_{i=0}^{j} e^{\circ}_{r+i}(W+Y_i)\cdot(-1)^{j-i}h^{\circ}_{j-i}(Y_{i+1}) = 2^{-\chi(j=0)}(e^{\circ}_{r+j}(W)+e^{\circ}_{r-j}(W)).$$

(3) We have

$$\left(e^{\triangle}_{r+j}(W+Y_j)\right)_{j=0}^{n-1}\cdot\left((-1)^{j-i}h^{\circ}_{j-i}(Y_{i+1})\right)_{i,j=0}^{n-1} = \left(2^{-\chi(j=0)}(e^{\triangle}_{r+j}(W)+e^{\triangle}_{r-j}(W))\right)_{j=0}^{n-1},$$

equivalently,

$$\sum_{i=0}^{j} e^{\triangle}_{r+i}(W+Y_i)\cdot(-1)^{j-i}h^{\circ}_{j-i}(Y_{i+1}) = 2^{-\chi(j=0)}(e^{\triangle}_{r+j}(W)+e^{\triangle}_{r-j}(W)).$$

Proof. For $*=\emptyset$, $\circ$, or $\triangle$, we have, by using Lemma 3.1,

$$\begin{aligned}
e^*_{r+i}(W+Y_i) &= \sum_{s=0}^{i-1} e^*_{r+i-s}(W)e_s(Y_i) + e^*_r(W)e_i(Y_i) + \sum_{s=i+1}^{2i} e^*_{r+i-s}(W)e_s(Y_i)\\
&= \sum_{s=0}^{i-1} e^*_{r+i-s}(W)e_s(Y_i) + e^*_r(W)e_i(Y_i) + \sum_{s=i+1}^{2i} e^*_{r+i-s}(W)e_{2i-s}(Y_i)\\
&= \sum_{s=0}^{i-1} e^*_{r+i-s}(W)e_s(Y_i) + e^*_r(W)e_i(Y_i) + \sum_{t=0}^{i-1} e^*_{r+i-(2i-t)}(W)e_t(Y_i)\\
&= \sum_{s=0}^{i-1} e^*_{r+i-s}(W)e_s(Y_i) + e^*_r(W)e_i(Y_i) + \sum_{t=0}^{i-1} e^*_{r-i+t}(W)e_t(Y_i)\\
&= \sum_{s=0}^{i} \left(e^*_{r+i-s}(W)+e^*_{r-i+s}(W)\right)2^{-\chi(s=i)}e_s(Y_i)\\
&= \sum_{t=0}^{i} \left(e^*_{r+t}(W)+e^*_{r-t}(W)\right)2^{-\chi(t=0)}e_{i-t}(Y_i) \quad (t=i-s)
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\sum_{i=0}^{j} e_{r+i}(W+Y_i)\cdot(-1)^{j-i}h^{\circ}_{j-i}(Y_{i+1})\\
&= \sum_{i=0}^{j}\sum_{t=0}^{i} \left(e^*_{r+t}(W)+e^*_{r-t}(W)\right)2^{-\chi(t=0)}e_{i-t}(Y_i)\cdot(-1)^{j-i}h^{\circ}_{j-i}(Y_{i+1})\\
&= \sum_{t=0}^{j} 2^{-\chi(t=0)}\left(e^*_{r+t}(W)+e^*_{r-t}(W)\right)\sum_{i=t}^{j}(-1)^{j-i}e_{t-i}(Y_i)h_{j-i}(Y_{i+1})\\
&= \sum_{t=0}^{j} 2^{-\chi(t=0)}\left(e^*_{r+t}(W)+e^*_{r-t}(W)\right)\delta_{jt}\\
&= 2^{-\chi(j=0)}\left(e^*_{r+j}(W)+e^*_{r-j}(W)\right).
\end{aligned}$$

□

Proof of Theorem 2.1. It follows from Lemma 3.3 that

$$\Big((-1)^{i-j}h^{\circ}_{i-j}(X_{j+1}+1)\Big)_{i,j=0}^{n-1}\cdot\Big(e_{k+i+j}(Z_k+X_i+Y_j+1)\Big)_{i=0}^{n-1}=\Big(e_{k-i+j}(Z_k+Y_j+1)\Big)_{i.j=0}^{n-1},$$

$$\Big((-1)^{i-j}h_{i-j}(X_{j+1})\Big)_{i,j=0}^{n-1}\cdot\Big(e^{\circ}_{k+i+j}(Z_k+X_i+Y_j)\Big)_{i,j=0}^{n-1}=\Big(e^{\circ}_{k-i+j}(Z_k+Y_j)\Big)_{i,j=0}^{n-1},$$

$$\Big((-1)^{i-j}h^{\circ}_{i-j}(X_{j+1})\Big)_{i,j=0}^{n-1}\cdot\Big(e_{k+i+j}(Z_k+X_i+Y_j)\Big)_{i,j=0}^{n-1}=\Big(2^{\chi(i\neq 0)}e_{k-i+j}(Z_k+Y_j)\Big)_{i,j=0}^{n-1},$$

$$\Big((-1)^{i-j}h^{\triangle}_{i-j}(X_{j+1})\Big)_{i,j=0}^{n-1}\cdot\Big(e^{\triangle}_{k+i+j}(Z_k+X_i+Y_j)\Big)_{i,j=0}^{n-1}=\Big(e^{\triangle}_{k-i+j}(Z_k+Y_j)\Big)_{i=0}^{n-1}.$$

And it follows from Lemma 3.4 that

$$\begin{aligned}&\Big(e_{k-i+j}(Z_k+Y_j+1)\Big)_{i,j=0}^{n-1}\cdot\Big((-1)^{j-i}h^{\circ}_{j-i}(Y_{i+1})\Big)_{i,j=0}^{n-1}\\&\quad=\Big(2^{-\chi(j=0)}(e_{k-i+j}(Z_k+1)+e_{k-i-j}(Z_k+1))\Big)_{i,j=0}^{n-1}\\&\Big(e^{\circ}_{k-i+j}(Z_k+Y_j)\Big)_{i,j=0}^{n-1}\cdot\Big((-1)^{j-i}h^{\circ}_{j-i}(Y_{i+1})\Big)_{i,j=0}^{n-1}\\&\quad=\Big(2^{-\chi(j=0)}(e^{\circ}_{k-i+j}(Z_k)+e^{\circ}_{k-i-j}(Z_k))\Big)_{i,j=0}^{n-1}\\&\Big(2^{\chi(i>0)}e_{k-i+j}(Z_k+Y_j)\Big)_{i,j=0}^{n-1}\cdot\Big((-1)^{j-i}h^{\circ}_{j-i}(Y_{i+1})\Big)_{i,j=0}^{n-1}\\&\quad=\Big(2^{\chi(i>0)}2^{-\chi(j=0)}(e_{k-i+j}(Z_k)+e_{k-i-j}(Z_k))\Big)_{i,j=0}^{n-1}\\&\Big(e^{\triangle}_{k-i+j}(Z_k+Y_j)\Big)_{i,j=0}^{n-1}\cdot\Big((-1)^{j-i}h^{\circ}_{j-i}(Y_{i+1})\Big)_{i,j=0}^{n-1}\\&\quad=\Big(2^{-\chi(j=0)}(e^{\triangle}_{k-i+j}(Z_k)+e^{\triangle}_{k-i-j}(Z_k))\Big)_{i,j=0}^{n-1}.\end{aligned}$$

Since the matrices

$$\Big((-1)^{i-j}h^{*}_{i-j}(X_{j+1}(+1))\Big)_{0\le i,j\le n-1}\quad\text{and}\quad\Big((-1)^{j-i}h^{\circ}_{j-i}(Y_{i+1})\Big)_{0\le i,j\le n-1}$$

are triangular matrices with diagonal entries 1 and have determinants 1, we see that

$$\begin{aligned}\det\Big(e_{k+i+j}(Z_k+X_i+Y_j+1)\Big)_{i=0}^{n-1}&=\det\Big(2^{-\chi(j=0)}(e_{k-i+j}(Z_k+1)+e_{k-i-j}(Z_k+1))\Big)_{i,j=0}^{n-1}\\&=\frac{1}{2}\det\Big(e_{k-i+j}(Z_k+1)+e_{k-i-j}(Z_k+1)\Big)_{i,j=0}^{n-1},\\\det\Big(e^{\circ}_{k+i+j}(Z_k+X_i+Y_j)\Big)_{i,j=0}^{n-1}&=\det\Big(2^{-\chi(j=0)}(e^{\circ}_{k-i+j}(Z_k)+e^{\circ}_{k-i-j}(Z_k))\Big)_{i,j=0}^{n-1}\\&=\frac{1}{2}\det\Big(e^{\circ}_{k-i+j}(Z_k)+e^{\circ}_{k-i-j}(Z_k)\Big)_{i,j=0}^{n-1}\\\det\Big(e_{k+i+j}(Z_k+X_i+Y_j)\Big)_{i,j=0}^{n-1}&=\det\Big(2^{\chi(i>0)}2^{-\chi(j=0)}(e_{k-i+j}(Z_k)+e_{k-i-j}(Z_k))\Big)_{i,j=0}^{n-1}\\&=2^{n-1}\cdot\frac{1}{2}\cdot\det\Big(e_{k-i+j}(Z_k)+e_{k-i-j}(Z_k)\Big)_{i,j=0}^{n-1}\\\det\Big(e^{\triangle}_{k+i+j}(Z_k+X_i+Y_j)\Big)_{i,j=0}^{n-1}&=\det\Big(2^{-\chi(j=0)}(e^{\triangle}_{k-i+j}(Z_k)+e^{\triangle}_{k-i-j}(Z_k))\Big)_{i,j=0}^{n-1}\\&=\frac{1}{2}\det\Big(e^{\triangle}_{k-i+j}(Z_k)+e^{\triangle}_{k-i-j}(Z_k)\Big)_{i,j=0}^{n-1}.\end{aligned}$$

Comparing with Jacobi–Trudi formula, we complete the proof. □

4 Concluding remarks

By using the Desnanot–Jacobi formula, we obtain the following relations among the classical group characters corresponding rectangular-shaped Young diagrams. Let B be a square matrix and we denote by B_k^i (resp. $B_{k,l}^{i,j}$) the submatrix of B obtained by removing row i and column k (resp rows i, j and columns k, l). Then the Desnanot–Jacobi formula says that

$$\det B \cdot \det B_{k,l}^{i,j} = \det B_k^i \cdot \det B_l^j - \det B_l^i \cdot \det B_k^j.$$

Applying this formula to the matrices in Theorem 2.1, we have

Theorem 4.1. (1) For a non-negative integer or a half integer $n \geq 2$, we have

$$\begin{aligned} & s_{[n^k],\mathbf{O}_{2k+1}}(z_1,\cdots,z_k)\cdot s_{[(n-2)^{k+2}],\mathbf{O}_{2k+5}}(z_1,\cdots,z_k,x,y) \\ &\quad = s_{[(n-1)^k],\mathbf{O}_{2k+1}}(z_1,\cdots,z_k)\cdot s_{[(n-1)^{k+2}],\mathbf{O}_{2k+5}}(z_1,\cdots,z_k,x,y) \\ &\qquad - s_{[(n-1)^{k+1}],\mathbf{O}_{2k+3}}(z_1,\cdots,z_k,x)\cdot s_{[(n-1)^{k+1}],\mathbf{O}_{2k+3}}(z_1,\cdots,z_k,y). \end{aligned}$$

(2) For a non-negative integer $n \geq 2$, we have

$$\begin{aligned} & s_{\langle n^k\rangle],\mathbf{Sp}_{2k}}(z_1,\cdots,z_k)\cdot s_{\langle (n-2)^{k+2}\rangle,\mathbf{O}_{2k+4}}(z_1,\cdots,z_k,x,y) \\ &\quad = s_{\langle (n-1)^k\rangle,\mathbf{Sp}_{2k}}(z_1,\cdots,z_k)\cdot s_{\langle (n-1)^{k+2}\rangle,\mathbf{Sp}_{2k+4}}(z_1,\cdots,z_k,x,y) \\ &\qquad - s_{\langle (n-1)^{k+1}\rangle,\mathbf{Sp}_{2k+2}}(z_1,\cdots,z_k,x)\cdot s_{\langle (n-1)^{k+1}\rangle,\mathbf{Sp}_{2k+2}}(z_1,\cdots,z_k,y), \end{aligned}$$

(3) For a non-negative integer or a half integer $n \geq 2$, we have

$$\begin{aligned} & s_{[n^k],\mathbf{O}_{2k}}(z_1,\cdots,z_k)\cdot s_{[(n-2)^{k+2}],\mathbf{O}_{2k+4}}(z_1,\cdots,z_k,x,y) \\ &\quad = s_{[(n-1)^k],\mathbf{O}_{2k}}(z_1,\cdots,z_k)\cdot s_{[(n-1)^{k+2}],\mathbf{O}_{2k+4}}(z_1,\cdots,z_k,x,y) \\ &\qquad - s_{[(n-1)^{k+1}],\mathbf{O}_{2k+2}}(z_1,\cdots,z_k,x)\cdot s_{[(n-1)^{k+1}],\mathbf{O}_{2k+2}}(z_1,\cdots,z_k,y), \end{aligned}$$

Proof. (1) If n is an integer, the identity follows from (4). If n is a half-integer, the identity follows from (5), by multiplying the both sides by $\prod_{i=1}^k (z_i^{1/2}+z_i^{-1/2})^2(x^{1/2}+x^{-1/2})(y^{1/2}+y^{-1/2})$.

(2) The identity follows from (5).

(3) If n is an integer, the identity follows from (4). From (7), we have

$$\begin{aligned} & s'_{[n^k],\mathbf{O}_{2k}}(z_1,\cdots,z_k)\cdot s'_{[(n-2)^{k+2}],\mathbf{O}_{2k+4}}(z_1,\cdots,z_k,x,y) \\ &\quad = s'_{[(n-1)^k],\mathbf{O}_{2k}}(z_1,\cdots,z_k)\cdot s'_{[(n-1)^{k+2}],\mathbf{O}_{2k+4}}(z_1,\cdots,z_k,x,y) \\ &\qquad - s'_{[(n-1)^{k+1}],\mathbf{O}_{2k+2}}(z_1,\cdots,z_k,x)\cdot s'_{[(n-1)^{k+1}],\mathbf{O}_{2k+2}}(z_1,\cdots,z_k,y). \end{aligned}$$

By multiplying the both sides by $\prod_{i=1}^k (z_i^{1/2}+z_i^{-1/2})^2(x^{1/2}+x^{-1/2})(y^{1/2}+y^{-1/2})$, we obtain the case where n is a half-integer. □

The following problems would be interesting and important.

Problem 4.2. Find nice specializations of the identities in Thoemre 2.1, which are q-analogues of Theorem 1.1.

Problem 4.3. Give a Gessel–Viennot type proof to Theorem 2.1.

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