Exact valence bond entanglement entropy in the XXZ and related spin chains

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22 April, 2008

Collaboration with H. Saleur

Summary

1. Entanglement entropy in spin chains
   - von Neumann entanglement entropy
   - Valence bond entanglement entropy
   - XXZ chain and the Temperley-Lieb algebra
2. Computation of S_{VB} and its moments
   - Relation to TL ground state
   - Mapping to a boundary Coulomb gas problem
3. Combinatorial features
   - Bond percolation
   - Antiferromagnetic bond percolation
   - Epilogue

Résumé

VON NEUMANN ENTANGLEMENT ENTROPY

- $|\Phi\rangle$ pure state of a bipartite system $A \cup B$
- $\rho = |\Phi\rangle\langle\Phi|$ density matrix
- $\rho_A = \mathrm{Tr}_B \rho$ reduced density matrix for A
- $S_{\mathrm{vN}}(A) = -\mathrm{Tr}_A\, \rho_A \ln \rho_A$

SOME SIMPLE PROPERTIES

- $S_{\mathrm{vN}}(A) = S_{\mathrm{vN}}(B)$
- $S_{\mathrm{vN}}(A) = 0$ if A and B are non interacting
- Otherwise, not a very “intuitive” concept !

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APPLICATION TO $d = 1$ SPIN CHAINS (Cardy-Calabrese)

- N spins ; A segment of length $L \ll N$
- $S_{\mathrm{vN}}(A) \sim \frac{c}{3} \ln L$ (critical) or $S_{\mathrm{vN}}(A) \sim \frac{c}{6} \ln \xi$ (non critical)
- Useful numerically for determining criticality

EXAMPLE : THE XXZ SPIN CHAIN

- Hamiltonian $H = -\frac{1}{2} \sum_i \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z \right)$
- If $\Delta = -\cos(\pi e_0)$, then $c = 1 - \frac{6e_0^2}{1-e_0}$

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Valence bond entanglement entropy (F. Alet et al.)

- More "intuitive" (geometrical) measure of entanglement
- $S_{\rm VB} = \ln 2 \times \langle N_{\rm c} \rangle$
- $N_{\rm c}$ = number of valence bonds (spin-0 projections)
- Numerical observation for XXX chain $(e_0 = 0)$: $S_{\rm VB} \approx S_{\rm vN}$

Main results

- $S_{\rm VB}$ admits a purely geometrical interpretation
- $S_{\rm VB} \neq S_{\rm vN}$
- Exact computation of $S_{\rm VB}$ for all $e_0 \in [0,1]$
- Exact results in finite size for $e_0 = \frac{1}{3}$
- Extensions to spin chains other than XXZ

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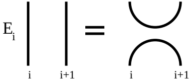
Reformulation of the XXZ chain as a loop model

- Hamiltonian $H = -\sum_i E_i$, with E_i = TL generator
- One strand per spin ; E_i acts on a pair of strands



- Algebraic relations : Set $n = 2\cos(\pi e_0)$. Then

$$\begin{aligned} (E_i)^2 &= nE_i \\ E_iE_j &= E_jE_i \text{ for } |i-j| \geq 2 \\ E_iE_{i\pm1}E_i &= E_i \end{aligned}$$

- Each closed loop has a weight n
- Lines are domain walls in the $Q = n^2$ state Potts model

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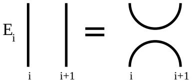

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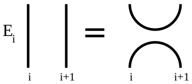

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JONES-WENZL PROJECTORS IN THE TL ALGEBRA

- Define projector P_N on N strands such that

$$E_i P_N = P_N E_i = 0 \text{ for all } i = 1, 2, \ldots, N-1$$

- Interpretation : P_N projects onto (maximal) spin $N/2$
- Easy recursive proof that

$$\begin{aligned} P_1 &= 1 \\ P_{N+1} &= P_N - \frac{U_{N-1}(n/2)}{U_N(n/2)} P_N E_N P_N \end{aligned}$$

- In particular, for $N = 2$ we have $P_2 = 1 - \frac{1}{n} E_1$
- By orthogonality, E_1 is projector on spin 0

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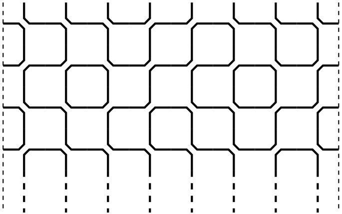
Identification of the valence bonds

- Typical time evolution of XXZ chain in TL basis



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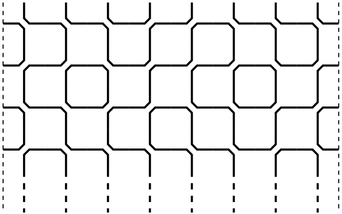

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Relation between S_{VB} and TL ground state

- Suppose $n = 2\cos(\pi e_0) \geq 0$
- Then ground state $|\Omega\rangle \in$ sector with 0 through-lines
- Basis states $|w_i\rangle$ are arc configurations
- Define non-symmetric matrix h_{ij} by

$$H|w_i\rangle = \sum_j h_{ij}|w_j\rangle$$

- Eigenvalues and (right) eigenvectors of h are those of H

$$|\Omega\rangle = \sum_i \lambda_i |w_i\rangle, \qquad \lambda_i > 0$$

- $N_{\mathrm{c}}(w_i)$ is number of valence bonds connecting a fixed interval to the outside in basis state $|w_i\rangle$

$$\langle N_{\mathrm{c}}\rangle(\Omega) = \frac{\sum_i \lambda_i N_{\mathrm{c}}(w_i)}{\sum_i \lambda_i}$$

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Mapping to a boundary Coulomb gas problem

- $|\Omega\rangle$ from path integral of Euclidian theory on half cylinder
- $|\Omega\rangle$ common to ∞ family of commuting transfer matrices
  - Instead of H, take isotropic Potts model transfer matrix
- Redistribute $n = 2\cos(\pi e_0)$ over loop orientations
  - On square lattice, give weight $\exp\left(\pm\frac{i\pi e_0}{4}\right)$ to left/right turns
- Boundary half loops get weight $n_{\mathrm{b}} = 2\cos\left(\frac{\pi e_0}{2}\right) = \sqrt{n+2}$
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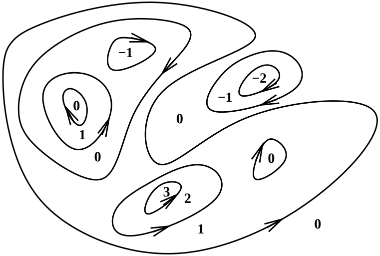
From oriented loops to heights

- Use oriented loops as level lines of a height field Φ
  - Convention : elementary height step is $\pm\pi$



- In continuum limit, height model becomes free bosonic field

$$S = \frac{g}{4\pi} \int \mathrm{d}\mathbf{x} \, (\partial \Phi(\mathbf{x}))^2 \text{ with } g = 1 - e_0$$

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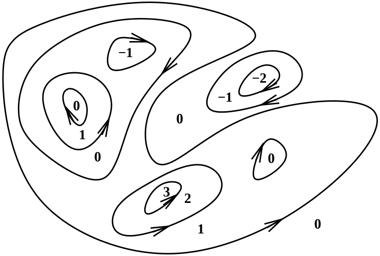

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Boundary conditions

- Height Φ has Neumann boundary conditions
- Height correlator

$$\langle \Phi(\mathbf{x}) \Phi(\mathbf{x}') \rangle = -\frac{1}{g} \ln |\mathbf{x} - \mathbf{x}'|^2$$

Change weights of valence bonds

- Single out interval of length L (“the subsystem”)
- Insert vertex operators

$$V_{\pm} = \exp[i(\pm e_1 + e_0/2)\Phi]$$

at interval end points

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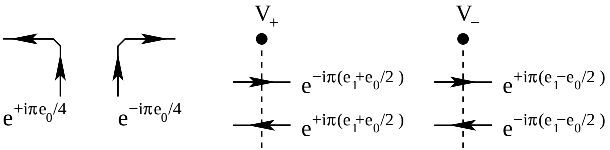

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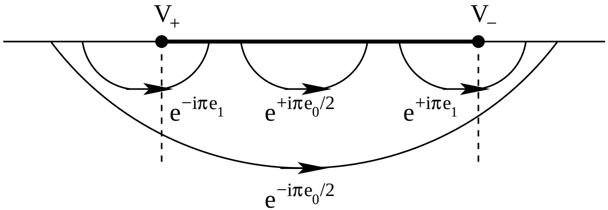

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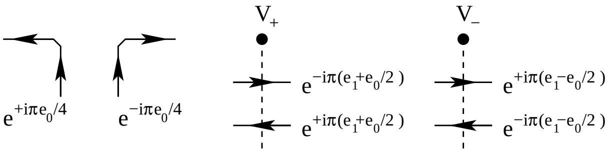

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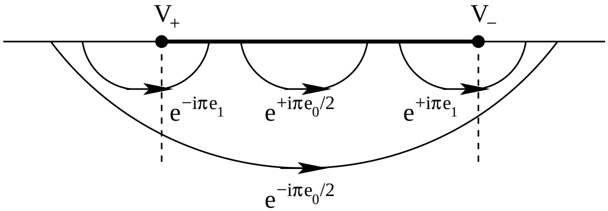

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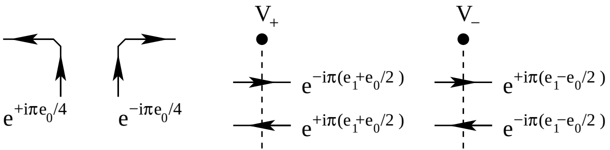

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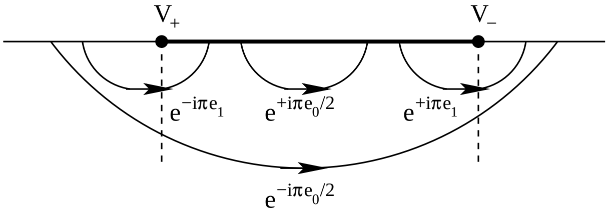

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Result for $S_{\rm VB}$

- Let the configuration $\mathcal{C}$ have N_0 bulk loops, $N_{\rm c}$ valence bonds, and $N_{\rm b}$ other boundary half loops

$$\langle V_+(0) V_-(L) \rangle = \frac{\sum_{\mathcal{C}} n^{N_0} n_{\rm b}^{N_{\rm b}} w^{N_{\rm c}}}{\sum_{\mathcal{C}} n^{N_0} n_{\rm b}^{N_{\rm b}+N_{\rm c}}} \propto L^{-2h}$$

- The *boundary* conformal weight of $V_\pm$ is [Kostov, Ponsot, Serban 2004]

$$h = \frac{4e_1^2 - e_0^2}{4g}$$

- Taking derivatives wrt w and setting $w = n_{\rm b}$ in the end :

$$\langle N_{\rm c} \rangle = \frac{e_0}{\pi(1-e_0)} \frac{2\cos(\pi e_0/2)}{\sin(\pi e_0/2)} \ln L$$

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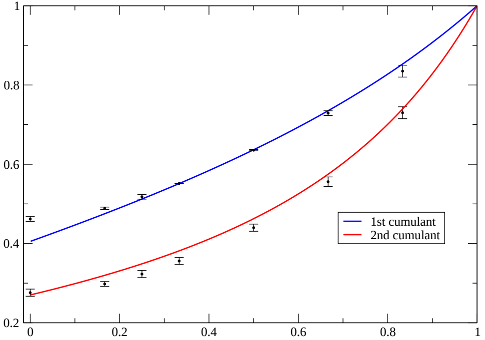
NUMERICAL VERIFICATION

- Slopes c_k of cumulants $\mathcal{C}_k = (c_k/\pi^k) \ln \tilde{L}$
  - As functions of e_0 ; for cylinder circumferences $L \leq 32$
  - L replaced by "conformal distance" $\tilde{L} = \frac{N}{\pi} \sin \frac{L\pi}{N}$



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Bond percolation ($e_0 = \frac{1}{3}$)

- Square lattice loop model, axial transfer direction
- $|\Omega\rangle$ is combinatorial [Mitra-Nienhuis-deGier-Batchelor. . .]
- $\mathrm{Cat}_{N/2}$ basis states written in parenthesis notation
- If $w_{(((\ldots)))} = 1$ then $w_{()()()\ldots} = A_{N/2-1}$ and $\sum w = A_{N/2}$

Valence bond problem in finite size

- Probability $p_{L,s}(N)$ of s excess (among first L symbols
- Clearly $p_{L,s}(N) = 0$ if $s > L$ or $L + s > N$.
  Also $p_{L,s}(N) = p_{N-L,s}(N)$
- Henceforth take both L and s to be even

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COMBINATORIAL RESULTS (conjectures)

- Define

$$w_k = N^2 - k^2$$

$$W_{L,s} = \frac{\prod_{j=-(s/2-1)}^{s/2-1} (w_{L+2j})^{\left\lfloor \frac{s+2-2|j|}{4} \right\rfloor}}{2^{L(L+1)/2} \prod_{j=0}^{L/2-1} (w_{2j+1})^{L/2-j}}$$

- Then $p_{L,s} = W_{L,s} P_{L,s}(N^2)$, where $P_{L,s}$ are polynomials of $\ell \equiv N^2$ with integer coefficients, of degree $\frac{(L-s)(L+s+2)}{8}$
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$$
\begin{aligned}
P_{2,2}(\ell) &= 5 \\
P_{4,2}(\ell) &= 15(-864 + 124\,\ell + 47\,\ell^2) \\
P_{4,4}(\ell) &= 66 \\
P_{6,2}(\ell) &= 3(-16795238400 + 4571106304\,\ell - 244070080\,\ell^2 - 369 \\
&\quad -4127020\,\ell^4 + 482449\,\ell^5) \\
P_{6,4}(\ell) &= 15(4858560 - 1245584\,\ell - 19492\,\ell^2 + 16551\,\ell^3) \\
P_{6,6}(\ell) &= 2431 \\
P_{8,2}(\ell) &= 4(1502554792919040000000 - 33639181895938867200 \\
&\quad +3478729799725350912\,\ell^2 - 221554086142103552\,\ell^3 \\
&\quad -4053084606375936\,\ell^4 + 329853079885056\,\ell^5 - 45675 \\
&\quad +1060141642094\,\ell^7 - 665466145512\,\ell^8 + 1164357162 \\
P_{8,4}(\ell) &= -31849825016217600 0 + 9742295253830860 8\,\ell - 815 \\
&\quad +21681560951641 6\,\ell^3 + 29351838165760\,\ell^4 + 104660
\end{aligned}
$$

- $P_{L,L} = \mathrm{TSPP}(L)$ (totally symmetric plane partitions)

$$P_{L,L}(\ell) = \prod_{j=1}^{L} \frac{(2j+L-1)!}{(j+L-1)!} \frac{(2j-2)!}{(3j-2)!}$$

- Matches earlier conjecture [Nienhuis-Mitra]
- Analogous to “emptiness formation probability” in 6V model

- For $\langle N_c \rangle \simeq k \log(L) + S_0$ this gives

$(L, L+2)$	k	S_0
$(2,4)$	0.55510	0.86523
$(4,6)$	0.55742	0.86201
$(6,8)$	0.55624	0.86414

- Using structural properties : $k = 0.5517 \pm 0.0003$
- Our exact asymptotic result : $k = 0.551329$

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Résumé

Antiferromagnetic bond percolation

- Negative percolation threshold $p_{\mathrm{c}} = \frac{1-\sqrt{3}}{2} = -0.366025\cdots$
- Combinatorial features of $|\Omega\rangle$ not previously studied
- If $w_{(((\ldots)))} = 1$, then $\sum w = H_{N/2}$ (half-turn symmetric ASM)
- Much more is true…

- In general, must distinguish parities of $N/2$ and $L/2$

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Emptiness formation for $N/2$ odd and $L/2$ even

$$
\begin{aligned}
W_{L,L}^{\rm odd} &= N^{2\lfloor\frac{L+2}{4}\rfloor} \frac{\prod_{j=-\lfloor\frac{L-2}{4}\rfloor}^{\lfloor\frac{L-2}{4}\rfloor} w_{2+4\lfloor\frac{L}{4}\rfloor+4j}^{\lfloor\frac{L+2}{4}\rfloor-|j|}}{4^{\lfloor\frac{L+2}{4}\rfloor(2\lfloor\frac{L+2}{4}\rfloor-1)} \prod_{j=0}^{\lfloor\frac{L-2}{4}\rfloor} w_{4j}^{2(\lfloor\frac{L+2}{4}\rfloor-j)}} \\
p_{L,s} &= W_{L,s}^{\rm odd} P_{L,s}^{\rm odd}(N^2) \\
P_{L,L}^{\rm odd}(\ell) &= (V_{L/4})^2
\end{aligned}
$$

- Here

$$V_n = (-3)^{n^2} \prod_{i=1}^{2n+1} \prod_{j=1}^{n} \frac{3(2j-i)+1}{2j-i+2n+1}$$

is the number of $(2n+1)\times(2n+1)$ vertically symmetric alternating sign matrices (VSASM), alias the number of $2n \times 2n$ off-diagonally symmetric alternating sign matrices (OSASM)

Résumé

1. Entanglement entropy in spin chains
   - von Neumann entanglement entropy
   - Valence bond entanglement entropy
   - XXZ chain and the Temperley-Lieb algebra
2. Computation of S_{VB} and its moments
   - Relation to TL ground state
   - Mapping to a boundary Coulomb gas problem
3. Combinatorial features
   - Bond percolation
   - Antiferromagnetic bond percolation
   - Epilogue

Epilogue

- We also have exact asymptotic results for the following cases
  - Antiferromagnetic bond percolation
  - $n = 2$ Brauer model with $\Delta = 1 - \frac{2}{(w+1)^2}$
  - Higher-spin generalisations...
- Validated by precise numerical checks