The Asymmetric Exclusion Process : An Integrable Model for Non-Equilibrium Statistical Mechanics

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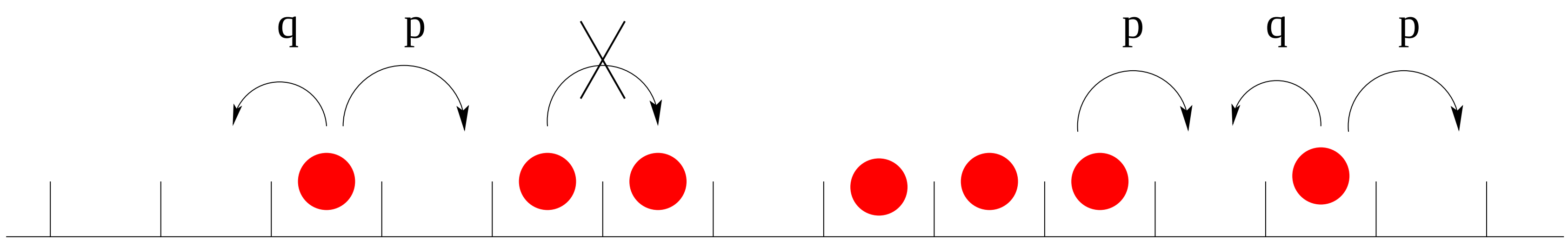
ASEP



Asymmetric Exclusion Process. A paradigm for non-equilibrium Statistical Mechanics.

EXCLUSION : Hard core-interaction ; at most 1 particle per site.

ASYMMETRIC : External driving ; breaks detailed-balance

PROCESS : Stochastic Markovian dynamics ; no Hamiltonian

ORIGINS

- Interacting Brownian Processes (Spitzer, Harris, Liggett).
- Driven diffusive systems (KLS).
- Transport of Macromolecules through thin vessels.
  Motion of RNA templates.
- Hopping conductivity in solid electrolytes.
- Directed Polymers in random media. Reptation models.

APPLICATIONS

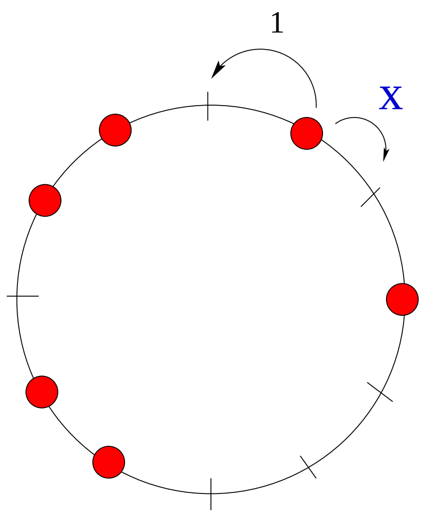
- Traffic flow.
- Sequence matching. Brownian motors.

1. Spectral Properties of the Markov Matrix (O. Golinelli)

2. Fluctuations of the current (S. Prolhac)

3. Multispecies exclusion processes and Matrix Ansatz (M. Evans and P. Ferrari)

Markov Equation for the ASEP



L SITES

N PARTICLES

$$\Omega = \binom{L}{N}$$

CONFIGURATIONS

x asymmetry parameter

$P_t(x_1, \dots, x_N)$: Prob. of config. $1 \le x_1 < \dots < x_N \le L$ at time t.

$$\frac{\mathrm{dP_t}}{\mathrm{d}t} = \sum_i \left[P_t(x_1, \dots, x_i - 1, \dots, x_N) - P_t(x_1, \dots, x_i, \dots x_N)\right] = M P_t$$

$(x = 0)$ The sum is restricted to $x_{i-1} < x_i - 1$.

ASEP : An Integrable System

MAPPING TO A NON-HERMITIAN SPIN CHAIN

$$M = \sum_{l=1}^{L} \left(\mathbf{S}_l^+ \mathbf{S}_{l+1}^- + x \mathbf{S}_l^- \mathbf{S}_{l+1}^+ + \frac{1+x}{4} \mathbf{S}_l^z \mathbf{S}_{l+1}^z - \frac{1+x}{4} \right)$$

Complex Eigenvalues $M\psi = E\psi$:

- Ground State : $E = 0$, $P = \Omega^{-1}$ (non-degenerate).
- Excited States : $\Re(E) < 0$ (Perron-Frobenius).

Excitations correspond to relaxation times

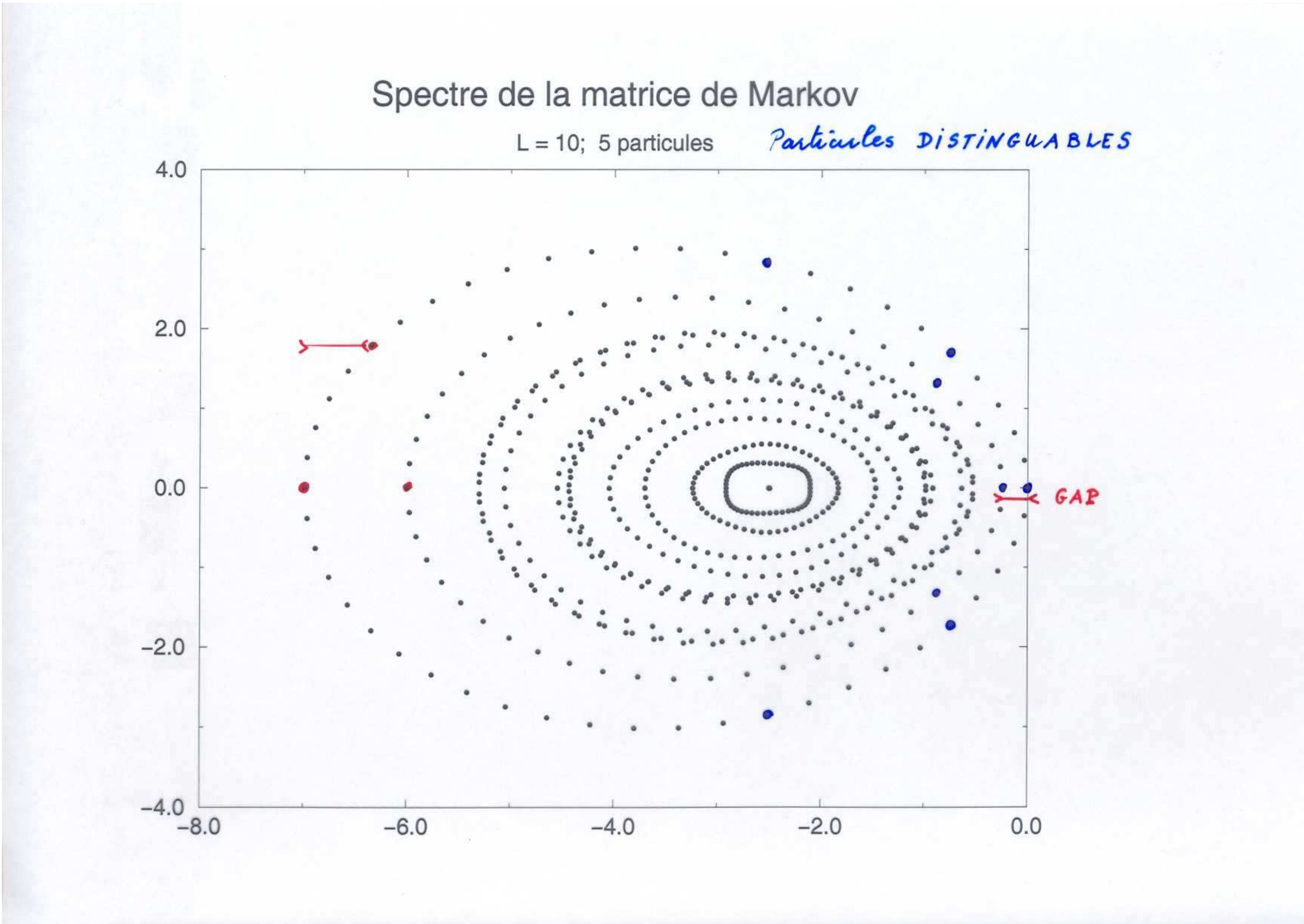
TASEP : $x = 0$

1. TASEP on a ring : Spectral Properties

- SPECTRAL GAP : Largest relaxation time T. How does it depend on the size L of the system : $T \sim L^z$?

- DEGENERACIES in the Markov Matrix : Hidden symmetries.

Example of a spectrum



Bethe Ansatz for TASEP

Eigenvectors of M as linear combinations of plane waves, with pseudo-momenta given by $z_1, \ldots z_N$:

$$\psi(x_1, \ldots, x_N) = \det\left(\frac{2^{x_j}(z_i+1)^{j-x_j}}{(z_i-1)^j}\right) \quad \text{for} \quad 1 \leq i, j \leq N$$

- ψ is an eigenfunction with eigenvalue $\mathbf{E = \frac{1}{2}(-N + \sum_j z_j)}$.
- Cancellation of the two-particle collision terms $(x_{k-1} = x_k - 1)$.
- Bethe Equations

$$\mathbf{(1 - z_i)^N (1 + z_i)^{L-N} = -2^L \prod_{j=1}^{N} \frac{z_j - 1}{z_j + 1}} \quad \text{for} \quad i = 1, \ldots N$$

Note that the r.h.s. is a constant independent of i.

Procedure for solving the Bethe Equations

- For any given value of Y, *SOLVE* $(1 - z_i)^N (1 + z_i)^{L-N} = Y$.
  The roots are located on Cassini Ovals
- *CHOOSE N roots* $z_{c(1)}, \ldots z_{c(N)}$ amongst the L available roots, with a *choice set* $c : \{c(1), \ldots, c(N)\} \subset \{1, \ldots, L\}$.
- *SOLVE* the self-consistent equation $\mathbf{A_c}(\mathbf{Y}) = \mathbf{Y}$ where

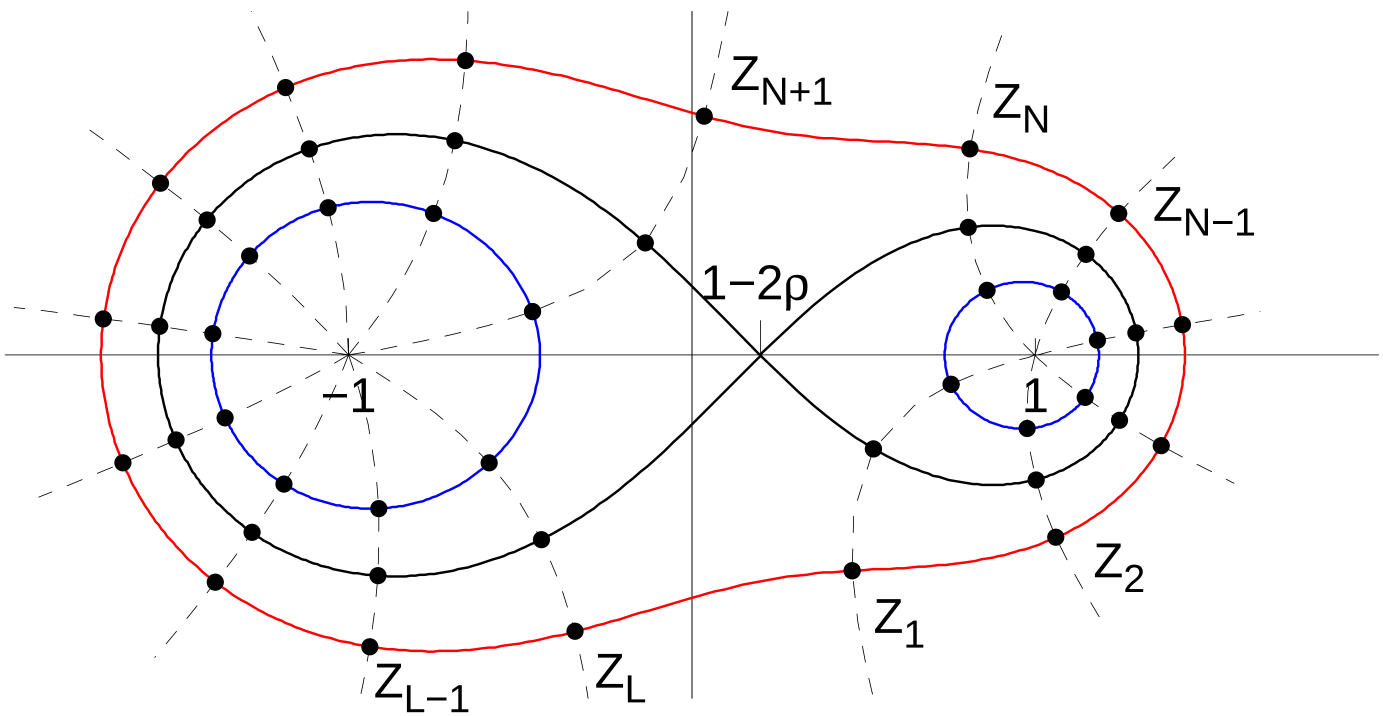
$$A_c(Y) = -2^L \prod_{j=1}^{N} \frac{z_{c(j)} - 1}{z_{c(j)} + 1} \, .$$

- *DEDUCE* from the value of Y, the $z_{c(j)}$'s and the energy corresponding to the choice set c :

$$2E_c(Y) = -N + \sum_{j=1}^{N} z_{c(j)} .$$

Labelling the roots of the Bethe Equations

The loci of the roots are remarquable curves : **Cassini Ovals**



Calculation of the GAP

An original method : **EXACT** combinatorial formulae for $A_0(Y)$ and $E_0(Y)$ for any finite values of L and N :

$$\log \frac{A_0(Y)}{Y} = \sum_{k=1}^{\infty} \binom{kL}{kN} \frac{Y^k}{k 2^{kL}}$$

$$E_0(Y) = -\sum_{k=1}^{\infty} \binom{kL-2}{kN-1} \frac{Y^k}{k 2^{kL}}$$

These expressions are analytically continued in $C - [1, \infty)$. When $L \to \infty$, $A_0(Y)$ and $E_0(Y)$ become the polylogarithm functions $Li_{3/2}$ and $Li_{5/2}$, respectively.

Calculation of the first excited state by solving transcendental equations. For a density ρ :

$$E_1 = \quad -2\sqrt{\rho(1-\rho)}\frac{6.509189337\ldots}{L^{3/2}} \pm \frac{2i\pi(2\rho-1)}{L}\,.$$

RELAXATION OSCILLATIONS

Higher excitations. Opposite side of the spectrum. Tagged particle.

SPECTRAL DEGENERACIES

NATURAL SYMMETRIES OF TASEP :

- Translation T : $MT = TM$. Momentum k
- Charge-conjugation C + Reflection R : $M(CR) = (CR)M$.

These natural symmetries do not commute $(CR)T = T^{-1}(CR)$ $\rightarrow$
The spectrum of M is composed of singlets for $(k = \pm 1)$ and doublets $(k, k^{\star})$ for $(k \neq \pm 1)$.

A NUMERICAL OBSERVATION FOR TASEP :

Unexpected degeneracies of certain orders with specific numbers of multiplets appear.

The highest degeneracy order $\sim 2^{L/6}$ (at half-filling).
Can we calculate these numbers ? Can we explain their origin ?

L	N	$m(1)$	$m(2)$	$m(6)$	$m(20)$	$m(70)$
2	1	2				
4	2	4	1			
6	3	8	6			
8	4	16	24	1		
10	5	32	80	10		
12	6	64	240	60	1	
14	7	128	672	280	14	
16	8	256	1792	1120	112	1
18	9	512	4608	4032	672	18

Spectral degeneracies in the TASEP at half filling.

$m(d)$ is the number of multiplets with degeneracy d.

ρ	L	N	$m(1)$	$m(2)$	$m(3)$	$m(4)$	$m(5)$	$m(15)$
1/3	9	3	81		1			
	12	4	459		12			
	15	5	2673		90	15		
	18	6	15849		540	270		1
	21	7	95175		2835	2835	189	21
1/4	16	4	1816			1		
	20	5	15424			20		
	24	6	133456			240	36	
1/5	25	5	53125				1	
2/5	15	6	4975	15				

Examples of spectral degeneracies in the TASEP at filling $\rho \neq 1/2$.

A symmetry of the Bethe equations

Let us call $\delta = \gcd(L, N)$.
The L Bethe roots form δ packages, each of cardinality L/δ.
The roots composing the package $\mathcal{P}_s$ have the indices $\{s, s+\delta, s+2\delta, \ldots, s+(L/\delta-1)\,\delta\}$ with $1 \le s \le \delta$.

Consider a choice set c (*i.e.*, a choice of N roots amongst the L available ones). Suppose there exist packages $\mathcal{P}_s$ and $\mathcal{P}_t$ such that

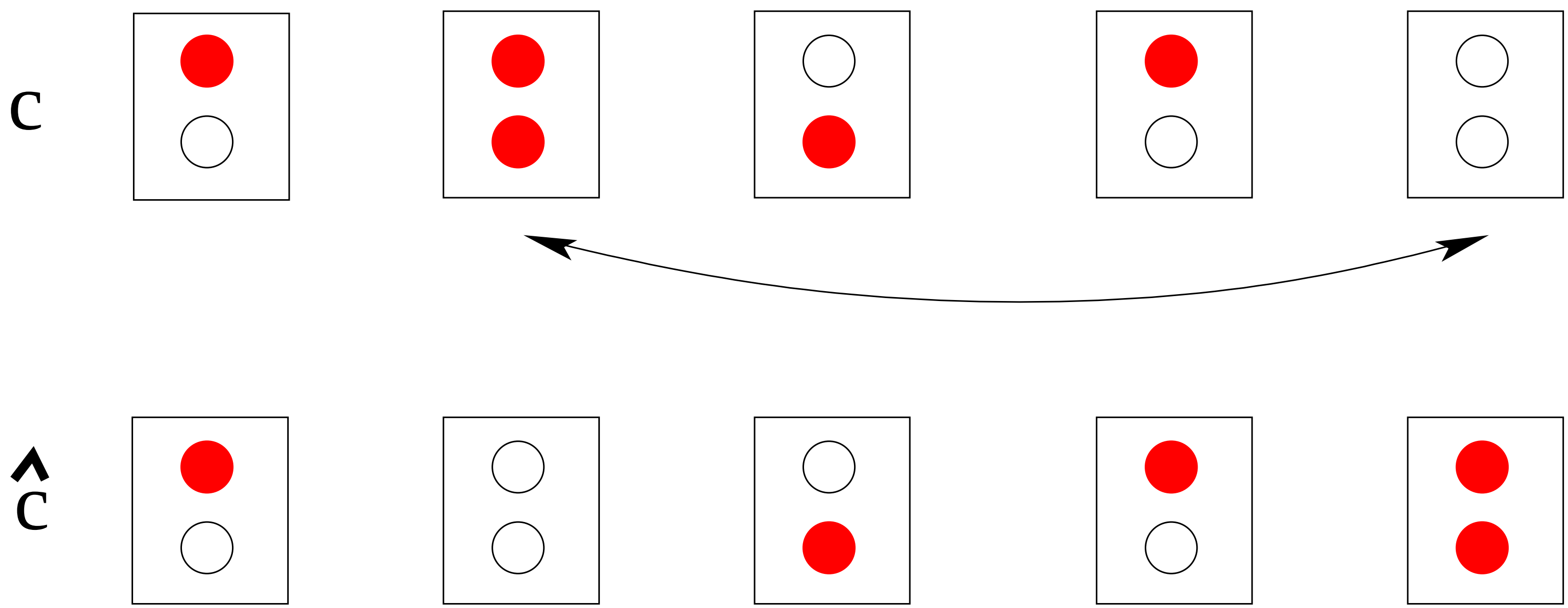
$$\mathcal{P}_s \subset c \quad \text{and} \quad \mathcal{P}_t \cap c = \emptyset\,.$$

The choice set $\hat{\mathbf{c}} = (c \backslash \mathcal{P}_s) \cup \mathcal{P}_t$ obtained from c by exchanging $\mathcal{P}_s$ and $\mathcal{P}_t$ corresponds to the same self-consistent equation and to the same eigenvalue as c.

Equivalence classes of choice sets by 'Package-swapping'.

L = 10 and N = 5 : 5 PACKAGES EACH OF 2 ROOTS

CHOOSE 5 ROOTS AMONGST THE 10 AVAILABLE



c AND $\hat{c}$ **THE SAME EIGENVALUE**

Calculation of the degeneracies

The number Ω of possible choice sets is the same as the dimension of the matrix M.

We suppose that there is a one to one correspondence :

choice sets $\leftrightarrow$ **solutions** of the Bethe Equations.

- 'package swapping' equivalence **classes** $\leftrightarrow$ **multiplets** in spectrum
- **cardinality** of a class $\leftrightarrow$ **order** of the multiplet
- # of classes of cardinality $d \leftrightarrow$ # of multiplets of order d.

Calculation of the degeneracies : a problem in combinatorics.

All the numbers in the tables can be determined.

Half-filling : $d_r = \begin{pmatrix} 2r \\ r \end{pmatrix}, \ \ m(d_r) = \begin{pmatrix} N \\ 2r \end{pmatrix} 2^{N-2r}, \ \ 0 \le r \le \frac{N}{2}$

2. Current Fluctuations and Large Deviations

- TRANSPORT PROPERTIES : modified because of interactions

$$\mathcal{D}_{TASEP} \neq \mathcal{D}_{Free}$$

- LONG RANGE CORRELATIONS : Non-Gaussian behaviour, non-vanishing higher cumulants.

Current statistics as an eigenvalue problem

Statistics of Y_t, total distance covered by all the particles between 0 and t.

Deformation of the Markov Matrix M by adding a jump-counting fugacity $\gamma : M(\gamma) = M_0 + \mathrm{e}^{\gamma} M_+ + \mathrm{e}^{-\gamma} M_-$

In the long time limit, $t \to \infty$

$$\left\langle \mathrm{e}^{\gamma Y_t} \right\rangle \simeq \mathrm{e}^{E(\gamma) t}$$

$E(\gamma)$ eigenvalue of $M(\gamma)$ with maximal real part.

Equivalently, $F(j)$, the large-deviation function of the current

$$P\left(\frac{Y_t}{t} = j\right) \sim e^{-tF(j)}$$

is the Legendre transform of $E(\gamma)$.

Bethe Ansatz for current statistics

The Bethe Equations are given by

$$z_i^L = (-1)^{N-1} \prod_{j=1}^{N} \frac{x\mathrm{e}^{-\gamma} z_i z_j - (1+x) z_i + \mathrm{e}^{\gamma}}{x\mathrm{e}^{-\gamma} z_i z_j - (1+x) z_j + \mathrm{e}^{\gamma}}$$

The eigenvalues of $M(\gamma)$ are

$$E(\gamma; z_1, z_2 \dots z_N) = \mathrm{e}^{\gamma} \sum_{i=1}^{N} \frac{1}{z_i} + x\mathrm{e}^{-\gamma} \sum_{i=1}^{N} z_i - N(1+x)\,.$$

The Bethe equations do not decouple unless $x = 0$.

TASEP CASE $x = 0$ (Derrida Lebowitz 1998)

$E(\gamma)$ is calculated by Bethe Ansatz to all orders in γ, thanks to the decoupling property of the Bethe equations.

Mean Total current :

$$J = \lim_{t\to\infty} \frac{\langle Y_t \rangle}{t} = \frac{n(L-n)}{L-1}$$

Diffusion Constant :

$$D = \lim_{t\to\infty} \frac{\langle Y_t^2 \rangle - \langle Y_t \rangle^2}{t} = \frac{Ln(L-n)}{(L-1)(2L-1)} \frac{C_{2L}^{2n}}{(C_L^n)^2}$$

Exact formula for the large deviation function.

In the general case $x \neq 0$, NO DECOUPLING.

After a change of variable, $y_i = \frac{1-\mathrm{e}^{-\gamma} z_i}{1-x\mathrm{e}^{-\gamma} z_i}$, the Bethe equations read

$$\mathrm{e}^{L\gamma} \left(\frac{1-y_i}{1-xy_i} \right)^L = -\prod_{j=1}^{N} \frac{y_i - xy_j}{xy_i - y_j} \quad \text{for} \quad i = 1 \dots N\,.$$

Let T be auxiliary variable playing a symmetric role w.r.t. all the y_i :

$$\mathrm{e}^{L\gamma} \left(\frac{1-T}{1-xT} \right)^L = -\prod_{j=1}^{N} \frac{T - xy_j}{xT - y_j} \quad \text{for} \quad i = 1 \dots N\,.$$

$$i.e.\; P(T) = \mathrm{e}^{L\gamma}(1-T)^L \prod_{j=1}^{N} (xT - y_j) + (1-xT)^L \prod_{j=1}^{N} (T - xy_j) = 0.$$

But $P(y_i) = 0$ (Bethe Eqs.). Thus, $Q(T) = \prod_{i=1}^{N} (T - y_i)$ divides $P(T)$:

$Q(T)$ DIVIDES $e^{L\gamma}(1-T)^L Q(xT) + (1-xT)^L x^N Q(T/x)$.

There exists a polynomial $R(T)$ such that

$$Q(T)R(T) = e^{L\gamma}(1-T)^L Q(xT) + x^N (1-xT)^L Q(T/x)$$

Functional Bethe Ansatz (Baxter's TQ equation).

This equation is solved perturbatively w.r.t. γ.

- Mean Current : $J = (1-x)\frac{N(L-N)}{L-1} \sim (1-x)L\rho(1-\rho)$ for $L \to \infty$
- Diffusion Constant :

$$D = (1-x)\frac{2L}{L-1}\sum_{k>0} k^2 \frac{C_L^{N+k}}{C_L^N}\frac{C_L^{N-k}}{C_L^N}\left(\frac{1+x^k}{1-x^k}\right)$$

$$D \sim 4\phi L\rho(1-\rho)\int_0^\infty du \frac{u^2}{\tanh \phi u} e^{-u^2}$$

when $L \to \infty$ and $x \to 1$ with fixed value of $\phi = \frac{(1-x)\sqrt{L\rho(1-\rho)}}{2}$.

Third cumulant

When time $t \to \infty$, $\frac{\langle Y_t^3 \rangle - 3\langle Y_t^2 \rangle \langle Y_t \rangle + 2\langle Y_t \rangle^3}{t} \to E_3$

Non-vanishing Skewness E_3 $\to$ Non Gaussian fluctuations.

When $L \to \infty$ and $x \to 1$ keeping $\phi = \frac{(1-x)\sqrt{L\rho(1-\rho)}}{2}$ fixed,

$$\frac{E_3}{\phi(\rho(1-\rho))^{3/2} L^{5/2}} \simeq -\frac{4\pi}{3\sqrt{3}} +$$

$$12 \int_0^\infty dudv \frac{(u^2+v^2)e^{-u^2-v^2} - (u^2+uv+v^2)e^{-u^2-uv-v^2}}{\tanh \phi u \tanh \phi v}$$

For $\phi \to \infty$, we recover the known TASEP limit :

$$E_3 \simeq \left(\frac{3}{2} - \frac{8}{3\sqrt{3}} \right) \pi (\rho(1-\rho))^2 L^3$$

$$
\begin{aligned}
\frac{E_3}{6L^2} &= \frac{1-x}{L-1}\sum_{i>0}\sum_{j>0}\frac{C_L^{N+i}C_L^{N-i}C_L^{N+j}C_L^{N-j}}{(C_L^N)^4}(i^2+j^2)\frac{1+x^i}{1-x^i}\,\frac{1+x^j}{1-x^j} \\
&- \frac{1-x}{L-1}\sum_{i>0}\sum_{j>0}\frac{C_L^{N+i}C_L^{N+j}C_L^{N-i-j}}{(C_L^N)^3}\,\frac{i^2+ij+j^2}{2}\,\frac{1+x^i}{1-x^i}\,\frac{1+x^j}{1-x^j} \\
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&- \frac{1-x}{L-1}\sum_{i>0}\frac{C_L^{N+i}C_L^{N-i}}{(C_L^N)^2}\,\frac{i^2}{2}\left(\frac{1+x^i}{1-x^i}\right)^2 \\
&+ (1-x)\frac{N(L-N)}{4(L-1)(2L-1)}\,\frac{C_{2L}^{2N}}{(C_L^N)^2} \\
&- (1-x)\frac{N(L-N)}{6(L-1)(3L-1)}\,\frac{C_{3L}^{3N}}{(C_L^N)^3}
\end{aligned}
$$

The symmetric case $x = 1$

Odd moments, such as the mean current vanish. For $L \to \infty$,

$$E\left(\frac{\gamma}{L}\right) \sim \frac{\rho(1-\rho)}{L}\gamma^2 + \frac{1}{L^2}\sum_{k=1}^{\infty}\left(\frac{B_{2k-2}}{k!(k-1)!}[\rho(1-\rho)]^k\gamma^{2k}\right)$$

- B_j : Bernoulli Numbers.
- Leading order (in $1/L$) : Gaussian fluctuations.
- Subleading (in $1/L^2$) : Non-Gaussian correction.
- Phase transition for a finite value of γ.

3. Multispecies Exclusion Models.

- Stationary state of generalized exclusion processes.
- Relation to the Matrix Ansatz for the stationary measure.

Definition of the N-TASEP

N classes of particles and holes with hierarchical priority rules.
During an infinitesimal time step dt, the following processes take place on each bond with probability dt :

$$I\ 0 \rightarrow 0\ I \qquad \text{for} \quad N \geq M \geq 1$$

$$I\ J \rightarrow J\ I \qquad \text{for} \quad N \geq J > I \geq 1$$

Particles can always overtake holes (= 0-th class particles).
First-class particles have highest priority etc...
There are P_I particles of class I. Total number of configurations :

$$\Omega = \frac{L!}{P_0!P_1!P_2!\dots P_N!}$$

Stationary Measure ?

Matrix Ansatz for the 2-TASEP

An algebraic description of the Stationary Measure (DEHP, 1994).

Configuration represented by a string e.g. 01220211.

Stationary weight :

$$p(01220211) = \frac{1}{Z}\mathrm{Tr}(EDAAEADD)$$

Replace 0 by E, 1 by D and 2 by A.

The operators A, D and E satisfy the *quadratic algebra*

$$\begin{aligned} DE &= D + E \\ DA &= A \\ AE &= A \end{aligned}$$

e.g. $p(01220211) \propto \mathrm{Tr}(D^2EA^3) = \mathrm{Tr}((D^2 + D + E)A^3) \propto 3\mathrm{Tr}(A^3)$

Representations of the quadratic algebra

Infinite dimensional : $D = 1 + \delta$ where δ =right-shift.

$E = 1 + \epsilon$ where ϵ =left-shift.

$A = |1\rangle\langle 1| = [\delta, \epsilon]$ projector on first coordinate.

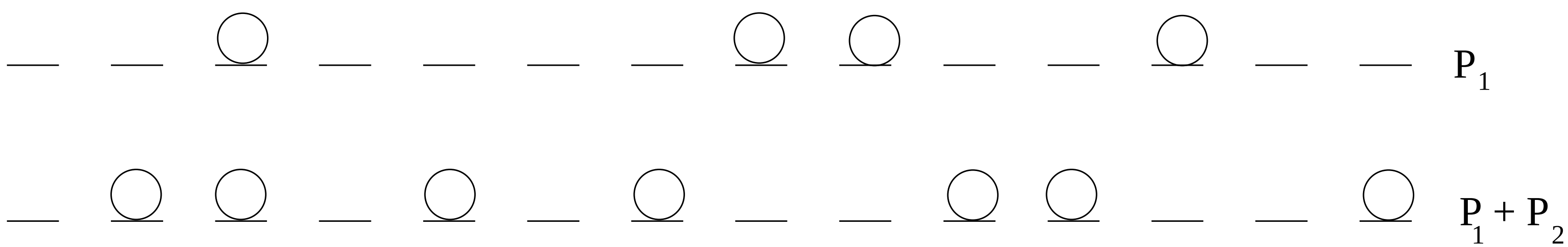
$$D = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & \\ 0 & 0 & 1 & 1 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}, \; E = D^{\dagger}, \; A = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ . & . & . & . \end{pmatrix}$$

- Matrix Ansatz : Stationary state properties (currents, correlations, fluctuations).
- Proof that the stationary measure *is not given by a Boltzmann-Gibbs measure* (E. Speer).
- Combinatorial Interpretation of these operators ?
- No Matrix Ansatz was known for N-TASEP models (for $N \geq 3$.)

Geometric Construction of the 2-TASEP stationary measure

(Omer Angel, Pablo Ferrari, James Martin)

A procedure to construct a configuration of the 2-TASEP with P_1 First Class Particles and P_2 Second Class Particles starting from two independent configurations of the 1 species TASEP.



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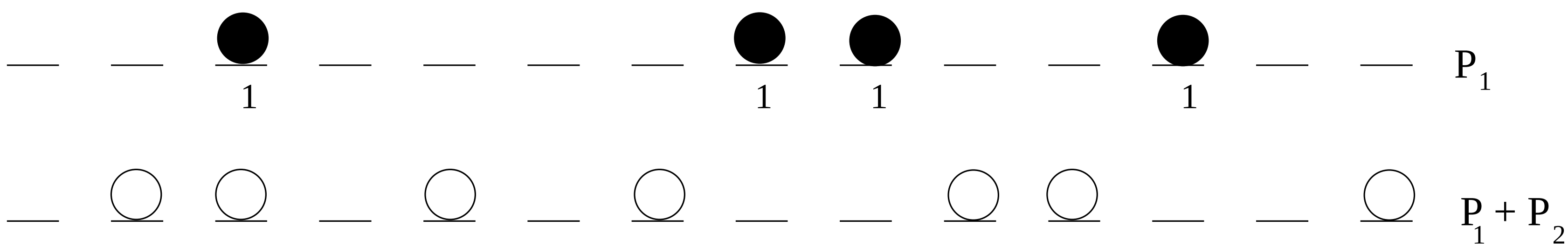

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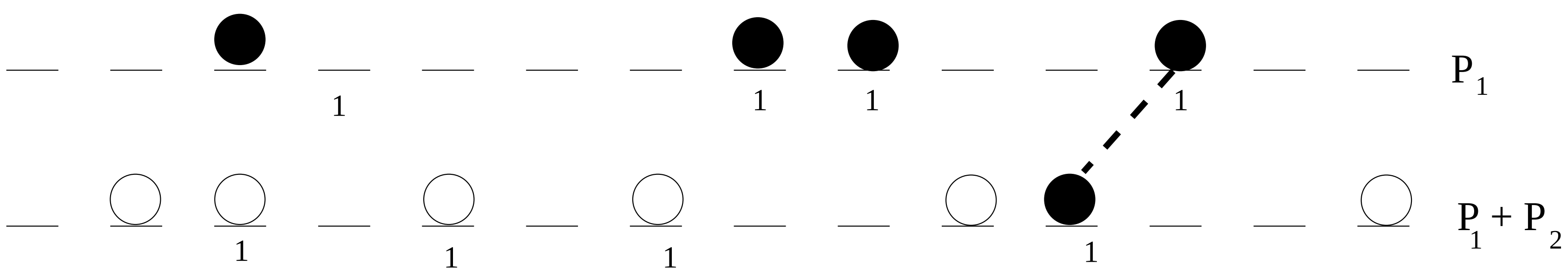

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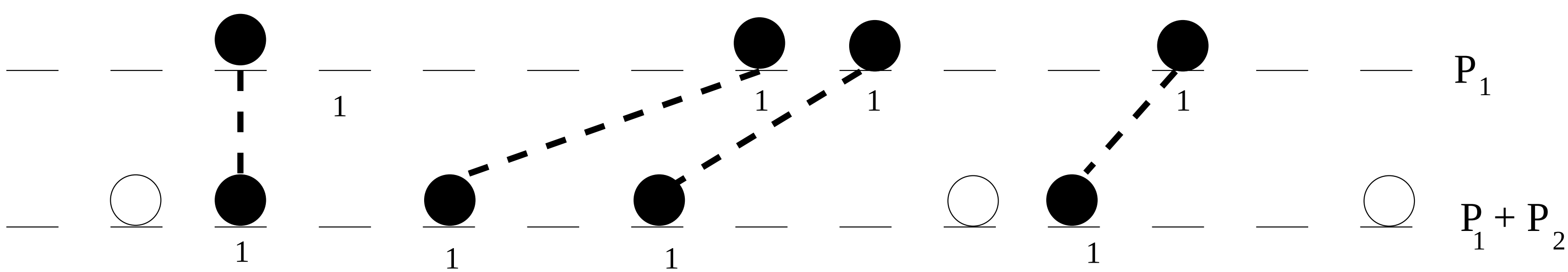

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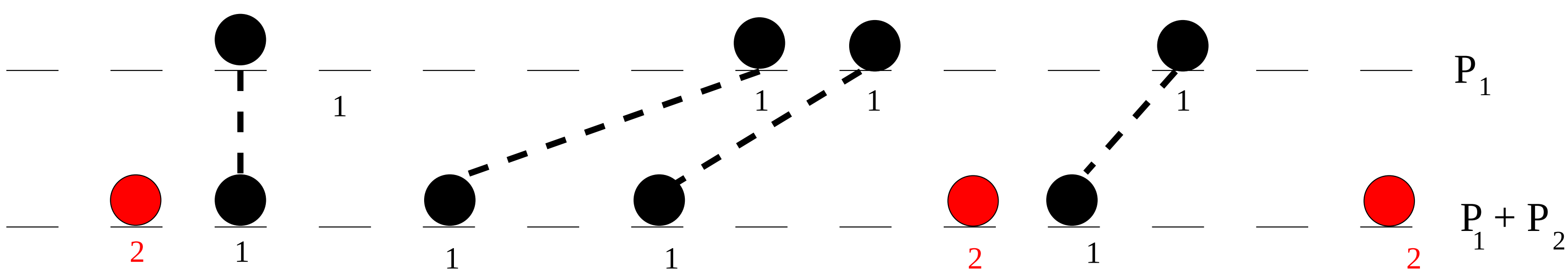
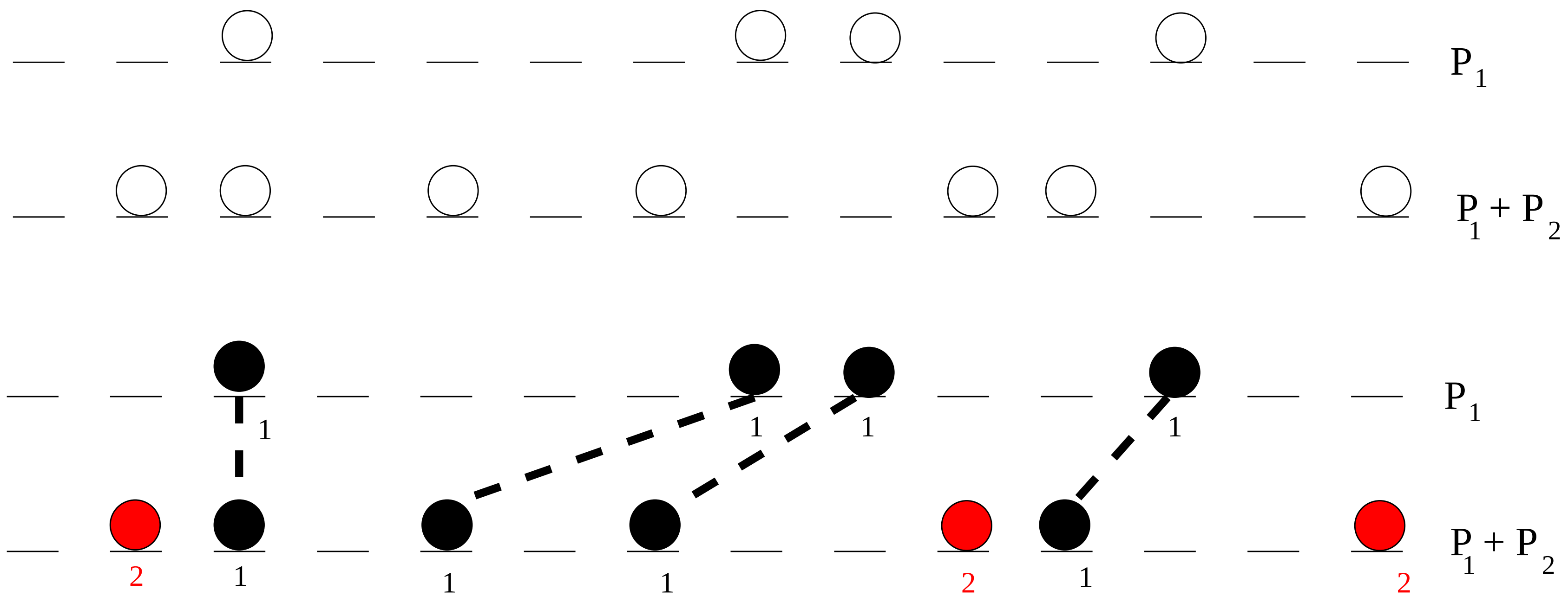


FROM 2 LINES OF TASEP TO 2−TASEP



This construction is NOT one-to one : the weight of a 2-TASEP configuration is proportional to the total number of ways you can generate it by this construction.

Fundamental Remarks :

- A 1 (on the 1st line) can not be located above a 2 (on the 2nd line).

- **Factorisation Property :** All the 1's (on the 2nd line) situated between two 2's MUST be linked to 1's (on the 1st line) that are located between the positions of the two 2's *(No Crossing Condition)*.

- **'Pushing' Procedure :** The *'ancestors'* of a string of the type 210102 are the strings obtained by pushing the 1's to the right i.e., 210102, 210012, 201102, 201012, 200112.

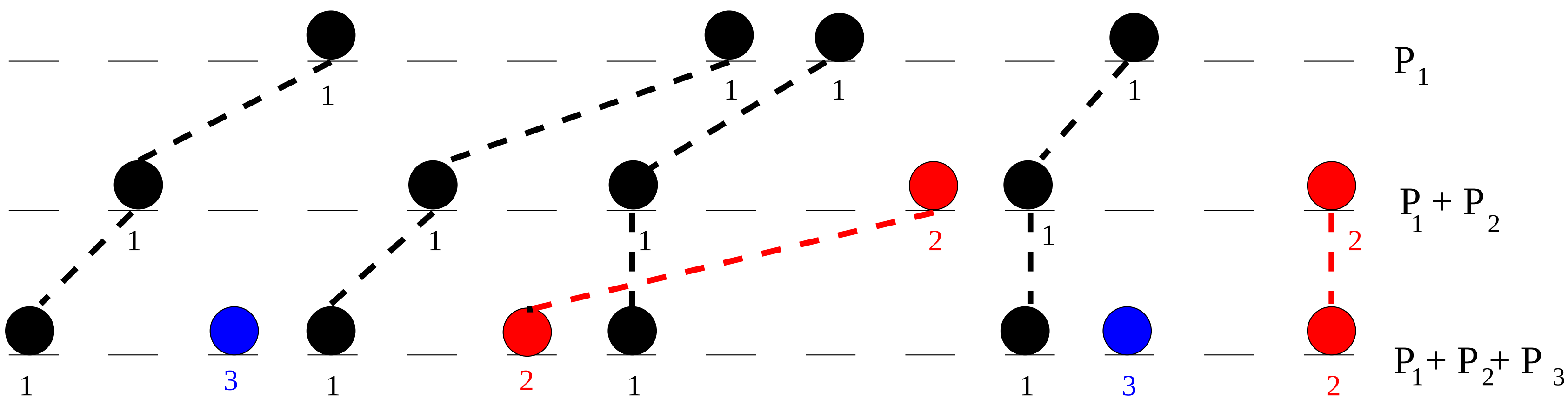
These properties uniquely characterize the stationary weights.

THE MATRIX ANSATZ PERFORMS AUTOMATICALLY THE COMBINATORICS UNDERLYING THE GEOMETRIC CONSTRUCTION OF THE WEIGHTS.

- **Factorisation Property :** A is a **PROJECTOR.**

- **Pushing Procedure :** D and E are **SHIFT OPERATORS** (right-shift and left-shift, respectively).

From 3 lines of TASEP to a 3-TASEP



The weight of a 3-TASEP configuration is proportional to the total number of ways you can generate it by this construction.

- REVERT the graphical procedure → ALGORITHM for constructing all ancestors of a given N-TASEP configuration.
- ENCODE this reverse algorithm into operators → ALGEBRA.
- CALCULATE the stationary weights → TRACES over this algebra.

NESTED MATRIX ANSATZ

Hierarchical construction based upon tensor products of the original algebra, using the D, A and E matrices and the shift operators.

For the 3-TASEP :

$$
\begin{aligned}
\hat{\mathbf{P}}_0 &= \mathbf{1} \otimes \mathbf{1} \otimes E + \mathbf{1} \otimes \epsilon \otimes A + \epsilon \otimes \mathbf{1} \otimes D\,. \\
\hat{\mathbf{P}}_1 &= \mathbf{1} \otimes \mathbf{1} \otimes D + \delta \otimes \epsilon \otimes A + \delta \otimes \mathbf{1} \otimes E \\
\hat{\mathbf{P}}_2 &= A \otimes \mathbf{1} \otimes A + A \otimes \delta \otimes E \\
\hat{\mathbf{P}}_3 &= A \otimes A \otimes E
\end{aligned}
$$

For the N-TASEP :

- EXPLICIT construction of all the matrices.
- DIRECT PROOF that the Matrix Ansatz leads to the stationary measure : independent and purely algebraic proof.
- FACTORISATION properties of the stationary measure.

EXACT SOLUTION OF THE N SPECIES ASEP :

Backward jumps allowed (rate $x \neq 0$)

$\longrightarrow$ Tensor products of a **deformation** of the initial quadratic algebra. Replace the shift-operators by deformed shift-operators :

$$\delta\epsilon = 1 \rightarrow \delta\epsilon - x\epsilon\delta = 1.$$

The stationary measure was not known in this case (NO GRAPHICAL CONSTRUCTION).

CONCLUSIONS

The asymmetric exclusion process can be studied by using a variety of techniques : Bethe Ansatz, Matrix Product Ansatz, Combinatorics, Orthogonal polynomials, Random Matrix Theory...

Relevant for mathematics (interacting particle processes, generalization of the Brownian Motion) and for Statistical Mechanics (classical N-Body problem out of equilibrium).

Can be used as a paradigm to study the behaviour of systems far from equilibrium in low dimensions : Dynamical phase transitions, Non-Gaussian fluctuations, Non-Gibbs measures, Fluctuations Theorems.