The bead model

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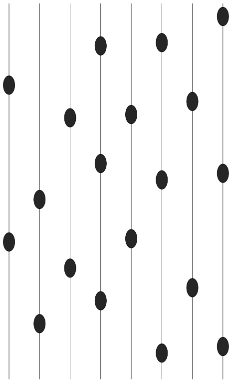
Bead configurations

collection of **threads** $\mathbb{Z} \times \mathbb{R}$
on which beads are strung

- locally finite
- interlacing

$\Omega = \{\text{set of beads configurations}\}$



Goal: describe "Uniform" measure(s) on Ω

Finite volume case

Finite number of threads of finite length $\ell_1, \ldots, \ell_p$
Finite number on beads on each thread $n_1, \ldots, n_p$
Ω bounded convex subset of $\mathbb{R}^n$, $n = \sum n_i$.
Uniform measure: normalized Lebesgue measure.

Infinite volume: notion of Gibbs measure.

- ergodic under the action of $\mathbb{Z} \times \mathbb{R}$
- conditionally uniform

Qualitative results

Possible reformulation as a model of *random surfaces with simple attractive potential* on $\mathbb{Z}^2$.

Theorem (Sheffield 2004)

There exists a two parameter family of ergodic Gibbs measures for this model

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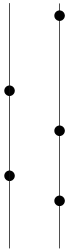

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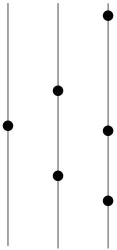

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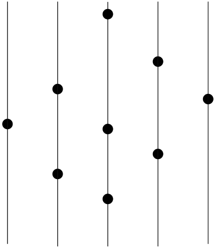
- does not give formulae to compute probability of events.
- one obvious parameter: the average density, set to 1.
- can estimate roughly probability of some rare events $\simeq \varepsilon^{n^2}$: Repulsion



The bead process is fully determinantal

Theorem

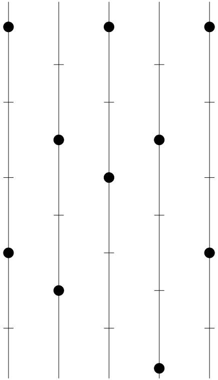
*For a fixed density, the set of Gibbs measure is $(P_\gamma)_{\gamma\in(-1,1)}$.
For all $\gamma \in (-1,1)$, (Ω, P_γ) is a determinantal point process on $\mathbb{Z}\times\mathbb{R}$ with kernel*

$$J_\gamma(x;y) = \begin{cases} \displaystyle\int_{[-1,1]} e^{iy\phi}(\gamma + i\phi\sqrt{1-\gamma^2})^x \frac{\mathrm{d}\phi}{2\pi}, \text{ if } x \geq 0 \\ \displaystyle\int_{\mathbb{R}\setminus[-1,1]} e^{iy\phi}(\gamma + i\phi\sqrt{1-\gamma^2})^x \frac{\mathrm{d}\phi}{2\pi}, \text{ if } x < 0 \end{cases}$$

In particular, for $x = 0$, $J_\gamma(0, y'-y) = \frac{\sin(y'-y)}{\pi(y'-y)}$ (bulk of the GUE).

$$\rho_\gamma^{(n)}(x_1, y_1; \ldots; x_n, y_n) = \det\big(J_\gamma(x_i - x_j; y_i - y_j)\big)$$

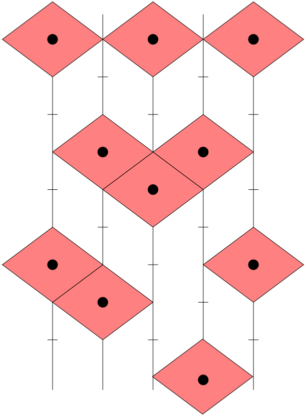
Proof: by approximation
Suppose that a bead cannot take any position on the real line but is constrained to sit at a lattice site with mesh size ϵ.



Proof: by approximation

Suppose that a bead cannot take any position on the real line but is constrained to sit at a lattice site with mesh size ϵ.

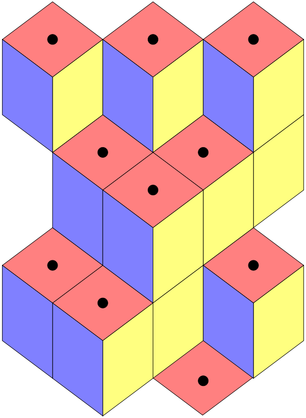
- Place a horizontal rhombus on each bead



Proof: by approximation

Suppose that a bead cannot take any position on the real line but is constrained to sit at a lattice site with mesh size ϵ.

- Place a horizontal rhombus on each bead
- Fill the rest with rhombi



Proof: by approximation

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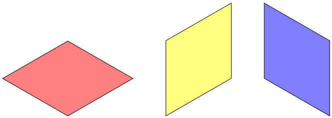
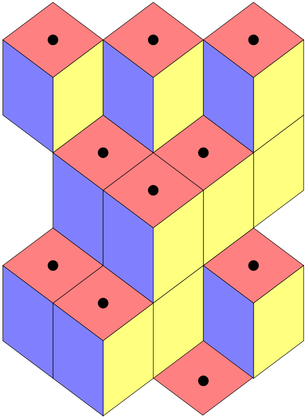
- Place a horizontal rhombus on each bead
- Fill the rest with rhombi
- Bijection with tilings of the plane with rhombi

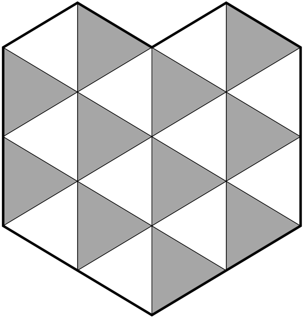




Mathematical study of these tilings started with Kasteleyn; Temperley,Fisher ($\sim$1960's).

Tilings with rhombi

Domain tiled with these rhombi : union of black and white equilateral triangles.



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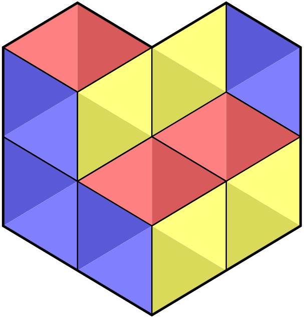

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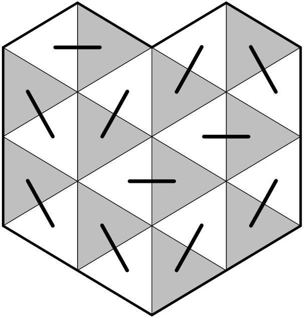

- tiling: pairing of white triangles with a black neighbouring triangle.
- Kasteleyn matrix

$$K_{w,b} = \begin{cases} 1 \text{ if } w \sim b \\ 0 \text{ otherwise} \end{cases}$$

- $Z = \sum_{\sigma} \prod_{w} K_{w,\sigma w}$

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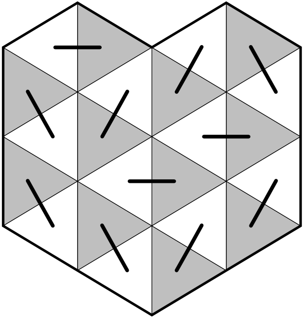

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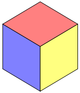

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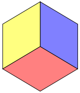

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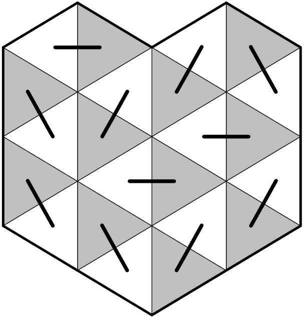

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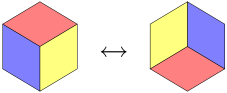
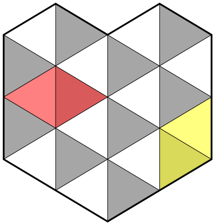


Correlations in a finite region

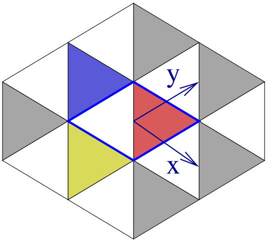


$R_j = (w_j, b_j)$.

$$P(R_1, \ldots, R_p) = \frac{\text{number of tilings with these rhombi}}{\text{total number of tilings}}$$

$$= \frac{\det K|_{\hat{R}_1, \ldots, \hat{R}_n}}{\det K} = \det K^{-1}_{b_i, w_j}$$

Tilings of the whole plane



Correlations also given by determinants of submatrices of K^{-1} [Kenyon; Kenyon, Okounkov, Sheffield]

K periodic: its inverse can be computed by Fourier transform

$$K^{-1}(b_{x',y'}, w_{x,y}) = \iint \frac{z^{-(y-y')} w^{x'-x}}{1 + w^{-1} + z} \frac{\mathrm{d}z}{2i\pi z} \frac{\mathrm{d}w}{2i\pi w}$$

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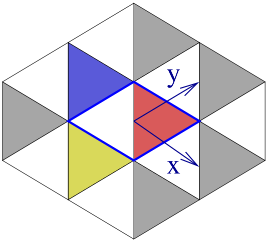

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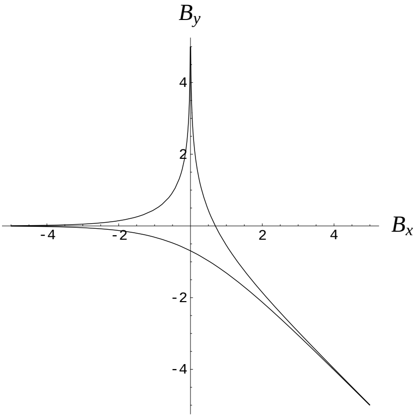
[Kenyon; Kenyon, Okounkov, Sheffield]
K periodic: its inverse(s) can be computed by Fourier transform

$$K^{-1}(b_{x',y'}, w_{x,y}) = \iint_{\substack{|z|=e^{B_x} \\ |w|=e^{B_y}}} \frac{z^{-(y-y')} w^{x'-x}}{1+w^{-1}+z} \frac{\mathrm{d}z}{2i\pi z} \frac{\mathrm{d}w}{2i\pi w}$$

B_x, B_y: two parameters. For any value, we get an ergodic Gibbs measure on tilings of the plane with rhombi.

Phase diagram



(Kenyon, Okounkov, Sheffield)
Amœba of the spectral curve

$$P(z,w) = 1 + z + w^{-1}$$

image of $\{P(z,w) = 0\}$ by

$$(z,w) \mapsto (\log|z|, \log|w|)$$

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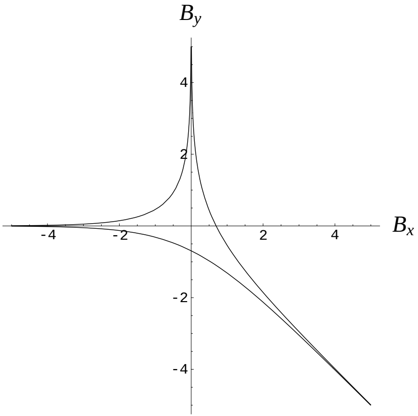

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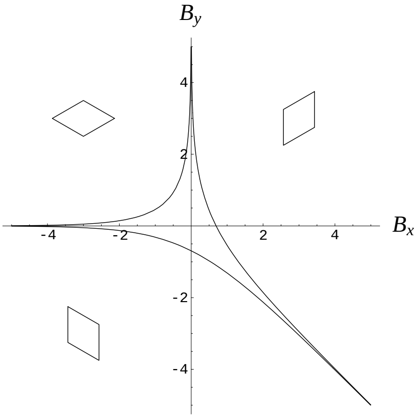

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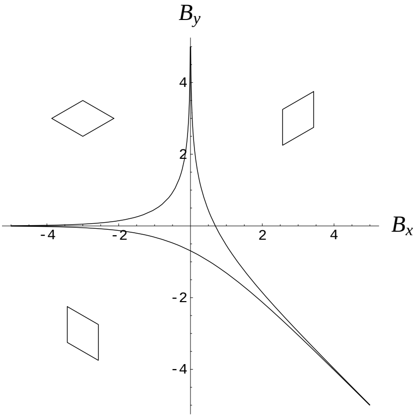

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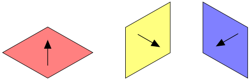
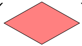
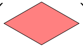
- 1 liquid phase
- 3 frozen phases
- The rhombi behave like magnetic dipoles



$B_{\text{vert}} \to -\infty \Rightarrow P\Big($  $\Big) \to 0.$

With a proper vertical scaling (mesh size $\epsilon \to 0$ simultaneously), we get a continuous limit.

parameter γ ?

$B_{\mathrm{vert}} \to -\infty \Rightarrow P\Big($  $\Big) \to 0.$

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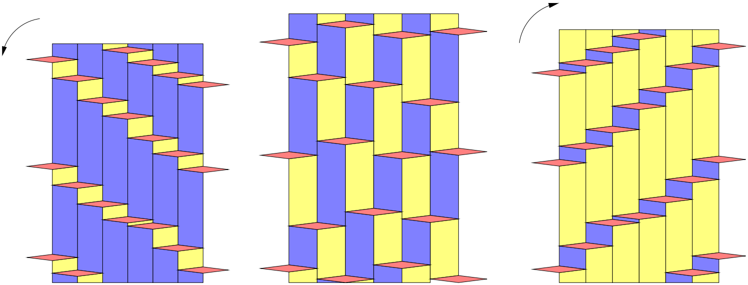
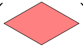


$\gamma < 0$ $\gamma = 0$ $\gamma > 0$

$$B_{\text{vert}} \to -\infty \Rightarrow P\Big($$  $$\Big) \to 0.$$

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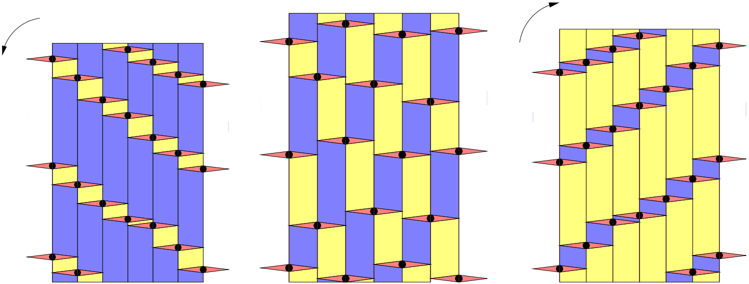

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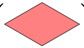
$B_{\text{vert}} \to -\infty \Rightarrow P\Bigg($  $\Bigg) \to 0.$

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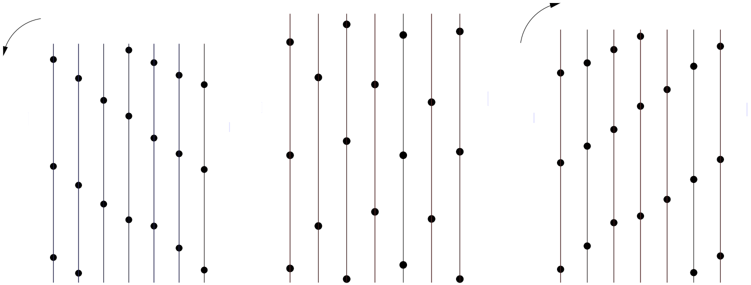

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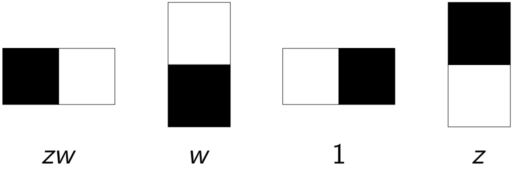
Other bipartite dimer models

[KOS] The phase diagram is given by the amoeba of a *spectral curve* (Harnack curve)

Theorem

In the tentacles, the statics of defects described in the limit by the bead process.

Example: dominos



Variants of the infinite bead process: cylinder

Tranfer matrix approach + Bethe Ansatz:
Stationary measure is the distribution of eigenvalues of random unitary matrix.
Question: discrete dynamics on the unitary group reducing to this evolution on the eigenvalues ?

Corner of the bead model: GUE minors process

Theorem (Baryshnikov,2001)

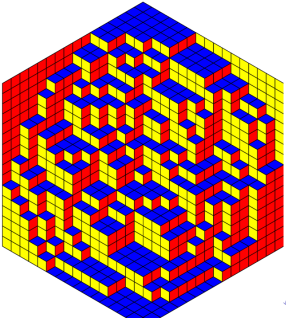
Let H_n be a matrix from the GUE ensemble. Let $\lambda_1^{(p)} \leq \cdots \leq \lambda_p^{(p)}$ be the eigenvalues of minor of size p. Given $\lambda_1^{(n)} \leq \lambda_n^{(n)}$, the distribution of $\left(\lambda_j^p\right)_{\substack{1\leq j\leq p \\ 1\leq p\leq n-1}}$ is uniform over the simplex:

$$\forall p \in \{1, \ldots, n-1\}, \forall j \in \{1, \ldots, p\} \lambda_j^{(p+1)} \leq \lambda_j^{(p)} \leq \lambda_{j+1}^{(p+1)}$$

Johansson, Nordenstam:

- this process is fully determinantal.
- statistics of horizontal rhombi in the random tiling of a large hexagon.

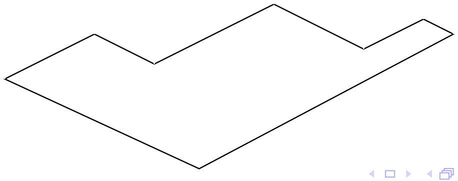
Proper scaling limit of their kernel gives J_γ



Skew plane partitions

The same phenomenon was discovered simultaneously by Okounkov and Reshetikhin in 3d skew plane partitions.

- Young diagram, interpreted as the floor of a room with infinitely high back wall
- pile cubes against the back wall.
- probability of a configuration $\propto q^{\mathrm{Vol}}$, with $0 < q < 1$



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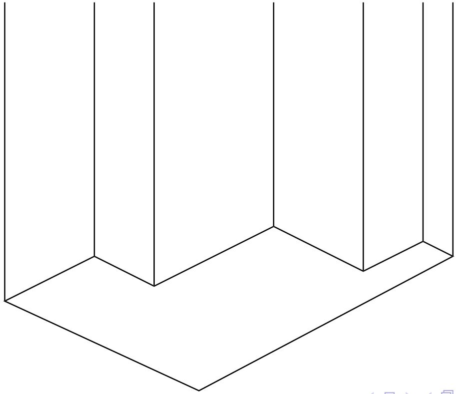

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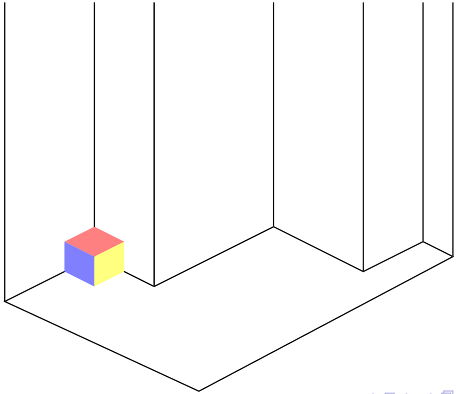

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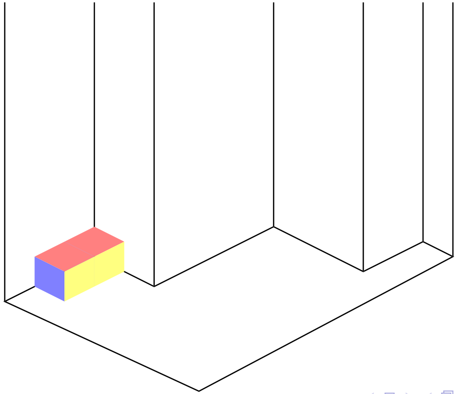

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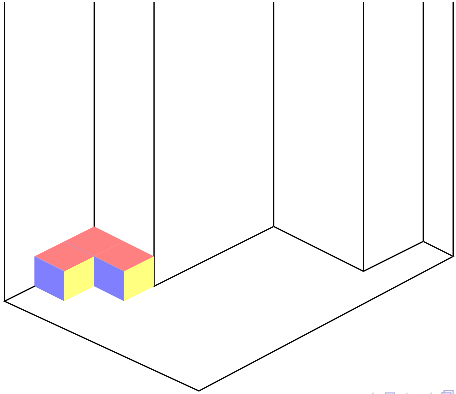

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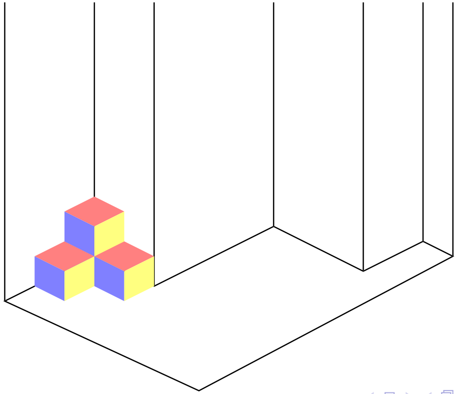

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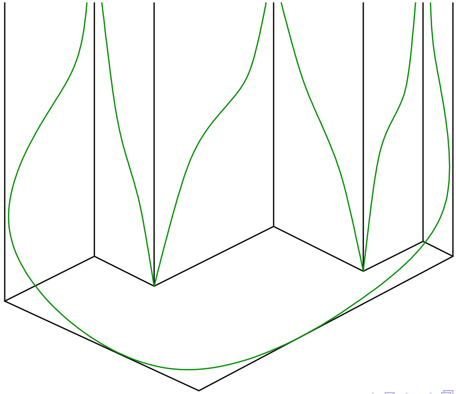
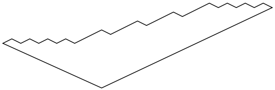


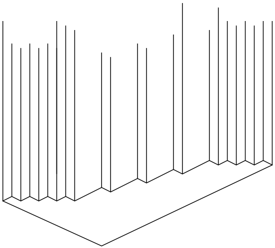
skew plane partitions with arbitrary slope

joint work with S. Mkrtchyan, N. Reshetikhin, P. Tingley
instead of slope ± 1 for the backwall
alternance of few upward/downward steps:



skew plane partitions with arbitrary slope

joint work with S. Mkrtchyan, N. Reshetikhin, P. Tingley
instead of slope ± 1 for the backwall
alternance of few upward/downward steps:



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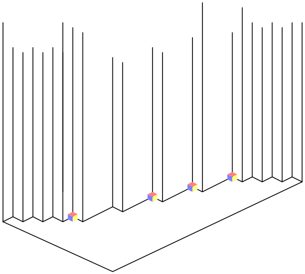

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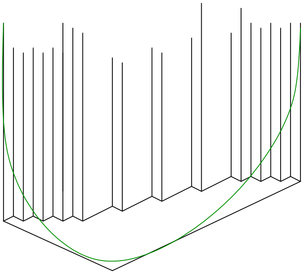


Theorem

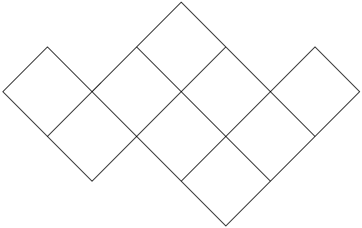
- *Limit shape: no cusps*
- *Close to the back wall, statistics of top fo piles converge to the bead process,* $\gamma = \cos(\pi \times \mathit{slope})$

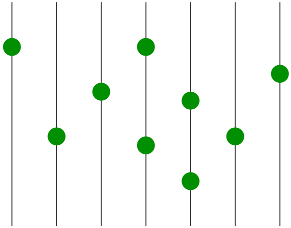
Perspectives

Interlacing point processes arise in several places

Example:

N. Elkies: number of standard skew Young tableaux with shape $\lambda = |\lambda|! \times \text{vol(bead configs.)}$.





Dimer techniques (Kasteleyn, Kenyon-Okounkov) to study limit shape for these objects when their size go to ∞.