Counting Lattice Paths With The Kernel Method

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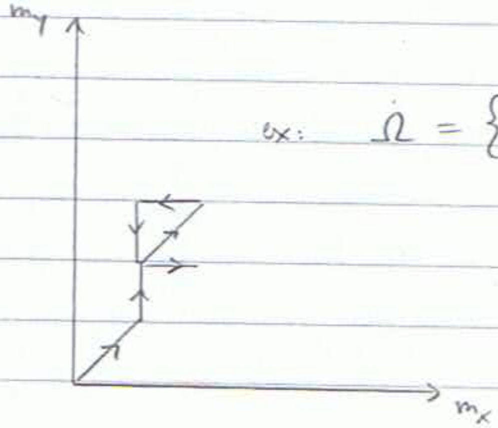
1. Lattice paths in the quarter plane $\mathbb{N}_0 \times \mathbb{N}_0$

Q: What is the # of n-step walks in $\mathbb{N}_0^2$ starting at $\mathcal{O}$ and ending at (m_x, m_y), taking steps out of a fixed set Ω

ex: $\Omega = \{ \nearrow, \downarrow, \leftarrow \}$



(i) $\Omega = \{ \uparrow, \rightarrow \}$

(ii) $\Omega = \{ \uparrow, \downarrow, \rightarrow \}$

(iii) $\Omega = \{ \uparrow, \downarrow, \rightarrow, \leftarrow \}$

(iv) $\Omega = \{ \nwarrow, \nearrow, \searrow \}$

(v) $\Omega = \{ \uparrow, \nearrow, \downarrow, \swarrow \}$

2. Generating functions and functional equations

$$G(x,y;t) = \sum_{n,m_x,m_y} c_n^{m_x,m_y} \, x^{m_x} y^{m_y} t^n$$

$G(x,y;t)$ satisfies a functional equation

(i) $$G(x,y;t) = 1 + t(x+y)\,G(x,y;t)$$

(ii) $$G(x,y;t) = 1 + t\left(x+y+\frac{1}{y}\right)G(x,y;t) - t\frac{1}{y}\,G(x,0;t)$$

(iii) $$G(x,y;t) = 1 + t\left(x+y+\frac{1}{x}+\frac{1}{y}\right)G(x,y;t)$$
$$- t\frac{1}{x}\,G(0,y;t) - t\frac{1}{y}\,G(x,0;t)$$

(iv) $$G(x,y;t) = 1 + t\left(\frac{x}{y}+\frac{y}{x}+xy\right)G(x,y;t)$$
$$- t\frac{y}{x}\,G(0,y;t) - t\frac{x}{y}\,G(x,0;t)$$

(v) $$G(x,y;t) = 1 + t\left(y+\frac{1}{y}+xy+\frac{1}{xy}\right)G(x,y;t)$$
$$-t\frac{1}{xy}\,G(0,y;t) - t\left(\frac{1}{y}+\frac{1}{xy}\right)G(x,0;t)$$
$$+t\frac{1}{xy}\,G(0,0;t)$$

3. The Kernel Method

Write functional equation as

$$\underbrace{K(x,y;t)}_{\text{Kernel}}\, G(x,y;t) = \dots\, G(x,0;t) + \dots\, G(0,y;t) + \dots\, G(0,0;t)$$

Consider pairs (x,y) for which $LHS = 0$, i.e.

$$\underline{K(x,y;t) = 0}$$

and solve $RHS = 0$ by eliminating y

(i) $\left[1 - t(x+y)\right] G(x,y;t) = 1$

$K(x,y) = 0$ not necessary, simply get

$$G(x,y;t) = \frac{1}{1-t(x+y)} \qquad \text{(rational)}$$

(ii) $\left[1 - t\left(x + y + \frac{1}{y}\right)\right] G(x,y;t) = 1 - t\frac{1}{y}\, G(x,0;t)$

$K(x,y) = 0 \rightsquigarrow y^2 - \left(\frac{1}{t} - x\right)y + 1 = 0 \qquad y_{1,2} = \frac{1}{2}\left(\frac{1}{t} - x\right) \mp \sqrt{\frac{1}{4}\left(\frac{1}{t} - x\right)^2 - 1}$

$y_1 \sim t \qquad y_2 \sim \frac{1}{t} \qquad (y_1 y_2 = 1)$

write $K(x,y) = -\frac{t}{y}(y - y_1)(y - y_2)$

$\rightsquigarrow \quad G(x,0;t) = \frac{1}{t} Y_1$

and $\quad G(x,y;t) = \dfrac{1 - t\frac{1}{y} G(x,0;t)}{K(x,y)} = \dfrac{1 - \cancel{t}\frac{1}{y}\frac{1}{\cancel{t}} Y_1}{-\frac{t}{y}(y-Y_1)(y-Y_2)}$

$$= \frac{1}{t(Y_2 - y)} \qquad \text{(algebraic)}$$

(the classical Kernel method, Knuth)

(iii) $\left[1 - t\left(x + \frac{1}{x} + y + \frac{1}{y}\right)\right] G(x,y;t) = 1 - t\frac{1}{y} G(x,0;t) - t\frac{1}{x} G(0,y;t)$

$K(x,y)$ has symmetries $x \leftrightarrow \frac{1}{x}$, $y \leftrightarrow \frac{1}{y}$, $x \leftrightarrow y$

$K(x,y) = 0 \rightsquigarrow Y_1 = f(x + \frac{1}{x}) \sim t \qquad Y_2 = \frac{1}{Y_1} \sim \frac{1}{t}$

iterate: $x \to Y_1 \to \frac{1}{x} \to \frac{1}{Y_1} = Y_2 \to x$

$K(x, Y_1) = 0$, $K(\frac{1}{x}, Y_1) = 0 \qquad$ NB: Y_2 not admissible here

RHS gives $\quad xY_1 = tx\, G(x,0;t) - tY_1\, G(0,Y_1;t)$

$\quad \frac{1}{x} Y_1 = t\frac{1}{x} G(\frac{1}{x},0;t) - tY_1\, G(0,Y_1;t)$

so that

$$tx\,G(x,0;t) - t\frac{1}{x} G(\frac{1}{x},0;t) = \left(x - \frac{1}{x}\right) Y_1$$

and $\quad G(x,0;t) = \frac{1}{tx} \times$ "Positive Part of" $\left(x - \frac{1}{x}\right) Y_1$

Extraction of the positive part:

$$Y_1 = \sum_{\ell \in \mathbb{Z}} a_\ell(t)\, x^\ell$$

From $Y_1 = t\left(1 + \frac{1}{x} Y_1\right)\left(1 + x Y_1\right)$

by Lagrange inversion

$$a_\ell(t) = \sum_{k=0}^{\infty} \frac{t^{2k+|\ell|+1}}{2k+|\ell|+1} \binom{2k+|\ell|+2}{k+|\ell|} \binom{2k+|\ell|+1}{k}$$

(Bernardi)

so that $$G(x,0;t) = \sum_{\ell=0}^{\infty} \frac{a_\ell(t) - a_{\ell+2}(t)}{t}\, x^\ell$$

and $$G(x,y;t) = \frac{1 - t\frac{1}{y} G(x,0;t) - t\frac{1}{x} G(y,0;t)}{1 - t\left(x + \frac{1}{x} + y + \frac{1}{y}\right)}$$

(d-finite)

(the obstinate kernel method, Bousquet-Mélou)

$$G(0,0;t) = \sum_{n=0}^{\infty} c_n t^n, \quad c_n = C_n C_{n+1}$$

applies also to "Kreweras-walks" $\{\nearrow, \downarrow, \leftarrow\}$,

where iteration on the kernel gives a cycle of length 6.

(iv) $$\left[1-t\left(\frac{x}{y}+\frac{y}{x}+xy\right)\right] G(x,y;t) = 1 - t\frac{x}{y}G(x,0;t) - t\frac{y}{x}G(0,y;t)$$

$$K(x,y)=0 \rightsquigarrow \frac{1}{y^2} - \frac{1}{tx}\frac{1}{y} + \frac{1}{x^2} + 1 = 0$$

$y_{1,2}$ satisfy $\frac{1}{y_1}+\frac{1}{y_2} = \frac{1}{t}\frac{1}{x}$ $\rightsquigarrow$ 3-term recurrence

iteration results in $x_n = y_1^n(x)$ given by $\frac{1}{x_n} = \alpha\lambda^n + \beta\lambda^{-n}$

$\lambda + \frac{1}{\lambda} = \frac{1}{t}$

one can check that $x_n \sim x t^n$

$t x_n G(x_n,0;t) = x_n x_{n+1} - t x_{n+1} G(x_{n+1},0;t)$ leads to

$$G(x,0;t) = \frac{1}{x^2 t}\sum_{k=0}^{\infty}(-1)^k x_k x_{k+1}$$

(not d-finite)

and

$$G(x,y;t) = \frac{1 - t\frac{x}{y}G(x,0;t) - t\frac{y}{x}G(y,0;t)}{1-t\left(\frac{x}{y}+\frac{y}{x}+xy\right)}$$

(the iterative kernel method, Rechnitzer, Janse van Rensburg, Mishna, P)

$$G(1,1;t) = \sum_{n=0}^{\infty} c_n t^n \quad : \quad c_n \sim \left(1 - 2\sum_{n=0}^{\infty}\frac{(-1)^n}{F_{2n}F_{2n+2}}\right)3^n + O(8^{n/2})$$

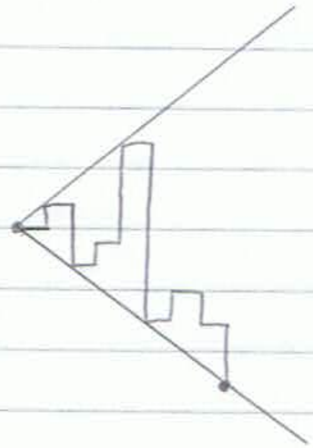
(v) open problem ("Gessel-walks")

Conjecture: # of 2n-step walks returning to origin

is given by $16^n \dfrac{(5/6)_n (1/2)_n}{(2)_n (5/3)_n}$

Zeilberger: "holy grail" of lattice path counting

4. Partially directed self-avoiding paths in the wedge $|y| \le x$



n horizontal steps

m up steps

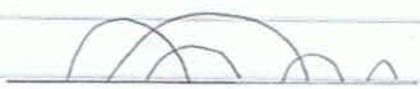
$$G(x,y) = \sum c_{n,m} x^n y^m$$

iteration kernel method gives

$$G(x,y) = C\left(\frac{x}{1-y}\right) \sum_{k=0}^{\infty} (-1)^k y^{\binom{k+1}{2}} \left(C\left(\frac{x}{1-y}\right) - 1\right)^k$$

where $C(t) = 1 + t C^2(t)$ Catalan GF (not d-finite)

bijection to



matchings of 2n points with m crossings (last workshop)