Trees versus Connected Graphs I

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Reference

Anders Björner, The homology and shellability of matroids and geometric lattices
Chapter 7, pp. 226–283
in Neil White (ed.), Matroid Applications, (Encyclopedia of Mathematics and its Applications #40), Cambridge University Press, Cambridge, 1992.

Graphs and trees

- Fix a vertex set U with n elements.
- A graph G is specified by a set of edges.
- $g_n = 2^{\binom{n}{2}}$.
- A graph is connected if there is a vertex that is connected by a path to every other vertex.
- $c_n \geq 2^{\binom{n}{2}-(n-1)}$.
- A tree T is a minimal connected graph. Each tree has $n-1$ edges.
- $t_n = n^{n-2}$.

Exchange properties of trees

- Each tree is close to each other tree.
- $x \in T_1 - T_2$ implies $\exists y \in T_2 - T_1$ such that $T = T_1 - x \cup y$ is a tree.
- This is the exchange property.
- Proof: $T_1 - x$ gives vertex set U with components V, W. Some edge $y \neq x$ in T_2 must connect V, W.

- Each tree is close to each other tree.

- $x \in T_1 - T_2$ implies $\exists y \in T_2 - T_1$ such that $T = T_2 \cup x - y$ is a tree.
- This is the dual exchange property.
- Proof: $T_2 \cup x$ has a circuit C. Some edge $y \neq x$ in C must avoid T_1.

Matroids

Let V be a finite set. A matroid basis $\mathcal{B}$ is a non-empty collection of subsets of V with the exchange property: If B_1 and B_2 are in $\mathcal{B}$, then $x \in B_1 - B_2$ implies $\exists y \in B_2 - B_1$ such that $B = B_1 - x \cup y$ is in $\mathcal{B}$.

The independent sets of a matroid consist of the subsets of the basis sets.

Matroid duality: The complements of the sets in a basis form a basis for another matroid.

- The trees on U form the basis of a matroid; the independent sets are graphs on U without cycles.
- The complements of trees on U form the basis of a matroid; the independents sets are complements of connected graphs on U.

Partitioning connected graphs

Let $\mathcal{G}_c$ be the set of connected graphs.
Let $\mathcal{T}$ be the set of trees.
There exists a function $T \to G(T)$ from trees to connected graphs with

$$\mathcal{G}_c = \bigsqcup_{T \in \mathcal{T}} [T, G(T)].$$

Each interval is Boolean: $H \in [T, G(T)]$ iff $T \subseteq H \subseteq G(T)$.

Example of partition

Take $n = 3$.
Four connected graphs T_1, T_2, T_3, K_3.
$G(T_1) = K_3$, while $G(T_2) = T_2$ and $G(T_3) = T_3$.
$\mathcal{G}$ is the disjoint union of $\{T_1, K_3\}, \{T_2\}, \{T_3\}$.

The connected graph partition identity

The weight of graphs is

$$g = \prod_{\{i,j\}} (1 + t_{ij}) = \sum_{G} \prod_{\{i,j\} \in G} t_{ij} .$$

The weight of connected graphs is

$$c = \sum_{G_c} \prod_{\{i,j\} \in G_c} t_{ij} = \sum_{T} \prod_{\{i,j\} \in T} t_{ij} \sum_{H \in [T, G(T)]} \prod_{\{i,j\} \in H \setminus T} t_{ij} .$$

The connected graph identity is

$$c = \sum_{G_c} \prod_{\{i,j\} \in G_c} t_{ij} = \sum_{T} \prod_{\{i,j\} \in T} t_{ij} \prod_{\{i,j\} \in G(T) \setminus T} (1 + t_{ij}).$$

The tree bound

$-1 \leq t_{ij} \leq 0.$

$$|\sum_{G_c} \prod_{\{i,j\} \in G_c} t_{ij}| \leq \sum_{T} \prod_{\{i,j\} \in T} |t_{ij}|.$$

The Kotecký-Preiss cluster expansion theorem is a consequence.

Invariants of partitions

The special case of connected graph identity when $t_{ij} = t$ is

$$\sum_{G_c} t^{|G_c|} = t^{n-1} \sum_{T} (1+t)^{|G(T)\setminus T|} = t^{n-1} h(1+t).$$

Let $r = \binom{n}{2} - (n-1)$. The h-vector polynomial is

$$h(s) = \sum_{i=0}^{r} h_i s^{r-i} = \sum_{T} s^{|G(T)\setminus T|}.$$

$(h_0, h_1, h_2, \ldots, h_r)$ is the h-vector.

- h_i is the number of sets in the partition with 2^{r-i} elements.
- $h_0 = 1$ is the number of partitions with $G(T)$ the complete graph.
- h_r is the number of partitions with $G(T) = T$.

Example of h-vector

- $n = 3$ Four connected graphs
- Weight $t^3 + 3t^2 = t^2(t+3) = t^2((1+t)+2) = t^2 h(1+t)$.
- $h(s) = s + 2$.
- $h_0 = 1,\ h_1 = 2$.
- 1 set where $G(T) = K_3$ is the complete graph. 2 sets where $G(T) = T$.

Simplicial complex interpretation

- V set of vertices.
- Δ set of subsets of V. These are faces (or simplices). Singletons are faces.
- F in Δ, $F' \subseteq \Delta$ imply $F' \in \Delta$.
- Δ is a complex (or simplicial complex).
- The dimension of F is $|F| - 1$.
- A maximal face is called a facet.
- A complex all of whose facets have the same dimension is said to be pure.

The complex of complements of connected graphs

- U is a fixed vertex set of the graphs.
- V is the set of all edges $\{i, j\}$, the vertex set of the complex.
- The complex Δ is the set of all complements of connected graphs.
- A facet is the complement of a tree.
- The complex is pure.
- The dimension of each facet is $r - 1 = \binom{n}{2} - (n-1) - 1 = \binom{n}{2} - n$.

Shelling

A shelling of a pure complex is a sequence $F_1, F_2, \ldots, F_t$ of the facets. For every $i < j$ there is a $k < j$ and an x in F_j such that

- $x \notin F_i$.
- $F_j \setminus \{x\} \subseteq F_k$.

Let Δ_i be the subcomplex of Δ induced by the first i facets in the shelling. Let

$$R(F_i) = \{y \in F_i \mid F_i \setminus \{y\} \in \Delta_{i-1}\}.$$

Note that $R(F_1) = \emptyset$.

- $R(F_i)$ is the minimal new face that is introduced when adding facet F_i.
- $R(F_i) = F_i$ when the only thing needed to be introduced is the facet F_i itself.

f-vectors and h-vectors

Let r be the number of points in a facet (the dimension plus 1).
The face polynomial is

$$f(t) = \sum_{i=0}^{r} f_i t^{r-i} = \sum_{F} t^{r-|F|}.$$

The f-vector is $(f_0, f_1, f_2, \ldots, f_r)$. The number f_i is the number of faces spanned by i vertices.

The shelling polynomial is

$$h(s) = \sum_{i=0}^{r} h_i s^{r-i} = \sum_{\hat{F}} s^{|\hat{F} \setminus R(\hat{F})|}.$$

The h-vector is $(h_0, h_1, h_2, \ldots, h_r)$. The number h_i is the number of facets $\hat{F}$ for which the restriction $R(\hat{F})$ has i points.
h_i is the number of sets in the partition with 2^{r-i} elements.

Topology of complexes

Theorem. The shelling polynomial is independent of the shelling: $f(t) = h(1+t)$.
Proof:

$$h(1+t) = \sum_{\hat{F}} (1+t)^{|\hat{F} \setminus R(\hat{F})|} = \sum_{\hat{F}} \sum_{R(\hat{F}) \subseteq F \subseteq \hat{F}} t^{|\hat{F}|-|F|} = f(t).$$

Theorem. For a shellable complex the reduced homology is zero except in highest dimension. In highest dimension it has $h_r = h(0) = f(-1)$ generators. This is the number of facets F in a shelling for which $R(F) = F$, that is, the only new face introduced is the facet itself.

One-dimensional complexes

$f(t) = t^2 + Vt + E$.
$f(-1) = h(0) = 1 - V + E = 1 - \chi = 1 - h_0 + h_1$.
Shellable $f(-1) = h(0) = 1 - 1 + h_1 = h_1$.

- A segment. $F = \{a, b\}, G = \{b, c\}, H = \{c, d\}$. Then $f(t) = t^2 + 4t + 3$, so $f_0 = 1$ and $f_1 = 4$ and $f_2 = 3$. Also $h(s) = s^2 + 2s$, so $h_0 = 1, h_1 = 2, h_2 = 0$. The sets in the partition are $[\emptyset, F]$, $[\{c\}, G]$, $[\{d\}, H]$. There is one with 4 faces and 2 with two faces.
- A triangle $F = \{a, b\}, G = \{b, c\}, H = \{c, a\}$. Then $f(t) = t^2 + 3t + 3$, so $f_0 = 1$ and $f_1 = 3$ and $f_2 = 3$. Also $h(s) = s^2 + s + 1$, so $h_0 = 1, h_1 = 1, h_2 = 1$. The sets in the partition are $[\emptyset, F]$, $[\{c\}, G]$,$[G, G]$. There is one with 4 faces, one with 2 faces, and one with 1 face. The top dimensional homology has one generator.
- K_4 with 6 facets. $f(t) = t^2 + 4t + 6$. $h(s) = (s-1)^2 + 4(s-1) + 6 = s^2 + 2s + 3$. The top homology has three generators.

Two-dimensional complexes

$f(t) = t^3 + Vt^2 + Et + F$.
$f(-1) = h(0) = -1 + V - E + F = -1 + \chi = -1 + h_0 - h_1 + h_2$.
Shellable: $f(-1) = h(0) = -1 + 1 - 0 + h_2 = h_2$.

- The square. $F = \{a, b, c\}$, $G = \{a, c, d\}$. Then $f(t) = t^3 + 4t^2 + 5t + 2$, so $f_0 = 1$ and $f_1 = 4$ and $f_2 = 5$ and $f_3 = 2$. Also $h(s) = s^3 + s^2$, so $h_0 = 1, h_1 = 1, h_2 = 0, h_3 = 0$. The partitions are $[\emptyset, F]$, $[\{c\}, G]$. These have 8 faces and 4 faces. There is 1 with eight faces and 1 with 4 faces and 0 with two faces and 0 with one face.
- The 2-sphere (surface of a cube). $f(t) = t^3 + 8t^2 + 18t + 12$. $h(s) = f(s-1) = s^3 + 5s^2 + 5s + 1$. The top homology has one generator.
- The annulus. $f(t) = t^3 + 16t^2 + 32t + 16$. $h(s) = s^3 + 13s^2 + 3s - 1$. Not shellable.

Complement of connected graphs example

Consider connected graphs with $n = 4$ vertices. The vertex set for the complex has $\binom{4}{2} = 6$ edges.
The number of connected graphs is

$$1 + 6 + (3 + 12) + (4 + 12) = 1 + 6 + 15 + 16 = 38.$$

The complex has 38 faces and 16 facets.
$f(t) = t^3 + 6t + 15t^2 + 16$.
$h(s) = s^3 + 3s^2 + 6s + 6$.
The complex has 6 vertices and all $\binom{6}{2} = 15$ possible edges.
There are $\binom{6}{2}$ three point subsets, which correspond to the $4 + 12 = 16$ trees and 4 triangles.
The complex has 16 out of the 20 possible facets. These correspond to 12 linear trees and 4 triangles. trees.
The homology in top dimension has 6 generators.

Matroid complexes

A pure complex is a matroid complex if its facets are the basis of a matroid. (The faces are the independent sets of the matroid.)

- Matroid basis exchange property: $x \in F_1 - F_2$ implies $\exists y \in F_2 - F_1$ such that $F = F_1 - x \cup y$ is a facet.
- Complements of trees satisfy the matroid basis exchange property.
- The complex of complements of connected graphs is a matroid complex.

Many shellings—many partitions

1. The complex of complements of connected graphs is a matroid complex.
2. Every matroid complex has a shelling.
   - Arbitrarily linearly order the vertices of the complex.
   - Linearly order the facets with lexicographic ordering.
   - This gives a shelling.
3. Each shelling gives a Boolean partition.
4. The h-vector that enumerates the number of elements of the partition of each size does not depend on the shelling.