Trees versus Connected Graphs II

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Isaac Newton Institute, May 15, 2008

Outline

Given sum over connected graphs:

$$\sum_{G_c} \prod_{\{i,j\} \in G_c} t_{ij}.$$

Express as sum over trees

$$\sum_{T} \mathrm{wt}(T).$$

How do trees emerge? From the distributive law!

The distributive law

The product over the base of the sum over the fibers is the sum over sections of the product over the base.

$$\prod_{i \in B} \sum_{k \in F_i} a_{ik} = \sum_{s} \prod_{i \in B} a_{i\,s(i)}$$

Here $s : B \to \bigcup_i F_i$ satisfies $s(i) \in F_i$.

Example: FOIL

$$\begin{array}{cc} a & c \\ b & d \\ \downarrow & \downarrow \\ 1 & 2 \end{array}$$

$F : 1 \mapsto a, 2 \mapsto c$
$O : 1 \mapsto a, 2 \mapsto d$
$I : 1 \mapsto b, 2 \mapsto c$
$L : 1 \mapsto b, 2 \mapsto d$

$$(a+b)(c+d) = ac + ad + bc + bd$$

Example: Interactions in physics

Potential energy between pairs of particles:
$0 \leq v_{ij} \leq +\infty$
Pairs of particles have probability factors:
$0 \leq \exp(-\beta v_{ij}) = 1 + t_{ij} \leq 1.$

Convert product to graph sum:

$$g = \prod_{\{i,j\}} (1 + t_{ij}) = \sum_{G} \prod_{\{i,j\} \in G} t_{ij}.$$

Here the base is all edges, the fiber is always $\{0, 1\}$. A section is a function from edges to $\{0, 1\}$, that is, the edges of a graph.

$-1 \leq t_{ij} \leq 0.$

Example: The product rule

$$\prod_{i\in B} f_i(x+h) = \prod_{i\in B}(f_i(x) + f_i'(x)h + \epsilon_i)$$

$$\prod_{i\in B} f_i(x+h) = \sum_{R+S+T=B} \prod_{i\in R} f_i(x) \prod_{j\in S} f_j'(x)h \prod_{k\in T} \epsilon_k$$

$$\prod_{i\in B} f_i(x+h) = \prod_{i\in B} f_i(x) + \sum_{|S|=1} \prod_{i\in B\setminus S} f_i(x) \prod_{j\in S} f_j'(x)h + \epsilon$$

The derivative is

$$\sum_{|S|=1} \prod_{i\in B\setminus S} f_i(x) \prod_{j\in S} f_j'(x) = \sum_{j} \prod_{i\neq j} f_i(x) f_j'(x)$$

Example: Fubini's theorem

$$\int f_1(s)\,ds \cdots \int f_n(x)\,ds = \int \cdots \int f(s_1) \cdots f(s_n)\,ds_1 \cdots ds_n.$$

$$\prod_i \int f_i(s)\,ds = \int \cdots \int \prod_i f(s_i)\,ds_1 \cdots ds_n.$$

The connected graph sum

$-1 \leq t_{ij} \leq 0$
Goal: Estimate the connected graph sum

$$\sum_{G_c} \prod_{\{i,j\} \in E(G_c)} t_{ij}.$$

Estimate by tree sum:

$$\sum_{T} \prod_{\{i,j\} \in E(T)} |t_{ij}|.$$

Can we do better?

Rooted trees as functions

- vertex set U
- root $r \in U$.
- $\tau : U \setminus \{r\} \to U$

The orbit of each point in U ends in r.

Connected graph identity

Take $-1 \leq t_{ij} \leq 0$.
This connected graph identity is a variant of an identity of Abdesselam and Rivasseau. Here $\mathbf{s}$ is a family of integration variables in $[0, 1]$.

$$\sum_{G_c} \prod_{\{i,j\} \in G_c} t_{ij} = \sum_{\tau} c(\tau) W(\tau) \int I(\tau, \mathbf{s})\, d\mathbf{s}.$$

$$c(\tau) = \prod_{i \neq r} t_{i\tau(i)}.$$

$0 \leq W(\tau) \leq 1$
$0 \leq I(\tau, \mathbf{s}) \leq 1.$

Tree weights

- Withinlayer factor:

$$W(\tau) = \prod_{\{i,j\} | i \sim j \text{ mod } \tau} (1 + t_{ij}),$$

  where the product is over two-element subsets $\{i, j\}$, and $i \sim j$ mod τ means that i and j are at the same distance from the root in the tree τ.

- Interlayer factor (random):

$$I(\tau, \mathbf{s}) = \prod_{i \neq r} \prod_{j \leftarrow i \text{ mod } \tau,\ j \neq \tau(i)} (1 + s_i t_{ij}),$$

  where $j \leftarrow i$ mod τ means that j is one tree distance unit closer to the root than the tree distance of i from the root.

For each vertex $i \neq r$ there is a uniform integration variable s_i in the interval $[0, 1]$ associated with the tree edge $\{i, \tau(i)\}$.

Fix distances from the root

Decompose U into disjoint non-empty subsets.
$\Delta = (U_0, U_1, \ldots U_h)$
$U_0 = \{r\}$.
Let $\mathcal{G}_c(\Delta)$ consist of all connected graphs G_c so the set of points in G_c a graph distance m from the root is U_m.

$$\sum_{G_c} \prod_{\{i,j\}} t_{ij} = \sum_{\Delta} \sum_{G_c \in \mathcal{G}_c(\Delta)} \prod_{\{i,j\} \in G_c} t_{ij}.$$

1 Distributive law: Resummation

$$\sum_{G_c \in \mathcal{G}_c(\Delta)} \prod_{\{i,j\} \in G_c} t_{ij} = W_\Delta I_\Delta,$$

Withinlayer factor:

$$W_\Delta = \prod_{i \sim j} (1 + t_{ij})$$

$i \sim j$ means $i \neq j$ and i, j are at the same graph distance from the root.

Interlayer factor:

$$I_\Delta = \prod_{i \neq r} [\prod_{j \leftarrow i} (1 + t_{ij}) - 1].$$

$k \leftarrow i$ means the graph distance of k from the root is one less than the graph distance of i from the root.

2 Fundamental theorem of calculus: Product rule

Interlayer factor:

$$I_\Delta = \prod_{i\neq r}[\prod_{j\leftarrow i}(1+t_{ij})-1].$$

$k \leftarrow i$ means the graph distance of k from the root is one less than the graph distance of i from the root.

FTC for each $i \neq r$:

$$\prod_{j\leftarrow i}(1+t_{ij})-1 = \int_0^1 \frac{d}{ds}\prod_{j\leftarrow i}(1+st_{ij})\,ds = \sum_{k\leftarrow i}\int_0^1 t_{ik}\prod_{j\leftarrow i,\, j\neq k}(1+st_{ij})\,ds.$$

3 Distributive law: Trees emerge

The distributive law gives

$$\begin{aligned} I_\Delta &= \prod_{i \neq r} \sum_{k \leftarrow i} \left[\int_0^1 t_{ik} \prod_{j \leftarrow i,\, j \neq k} (1 + s t_{ij})\, ds \right] \\ &= \sum_{\tau} \prod_{i \neq r} \left[\int_0^1 t_{i\tau(i)} \prod_{j \leftarrow i,\, j \neq \tau(i)} (1 + s t_{ij})\, ds \right]. \end{aligned}$$

$\tau : U \setminus \{r\} \to U$
For $i \neq r$ the value $\tau(i) \leftarrow i$.

4 Fubini's theorem

Let $\mathcal{T}(\Delta)$ be the tree functions that respect decomposition Δ.

$$I_\Delta = \sum_{\tau \in \mathcal{T}(\Delta)} \prod_{i \neq r} \left[\int_0^1 t_{i\tau(i)} \prod_{j \leftarrow i,\, j \neq \tau(i)} (1 + s t_{ij})\, ds \right].$$

$$I_\Delta = \sum_{\tau \in \mathcal{T}(\Delta)} \int_0^1 \cdots \int_0^1 \prod_{i \neq r} \left[t_{i\tau(i)} \prod_{j \leftarrow i,\, j \neq \tau(i)} (1 + s_i t_{ij})\, ds_i \right].$$

Equivalently

$$I_\Delta = \sum_{\tau \in \mathcal{T}(\Delta)} c(\tau) \int I(\tau, \mathbf{s})\, d\mathbf{s}.$$

$$c(\tau) = \prod_{i \neq r} t_{i\tau(i)}.$$

$$I(\tau, \mathbf{s}) = \prod_{i \neq r} \prod_{j \leftarrow i \text{ mod } \tau,\, j \neq \tau(i)} (1 + s_i t_{ij}),$$

Conclusion

$$\sum_{G_c \in \mathcal{G}_c(\Delta)} \prod_{\{i,j\} \in G_c} t_{ij} = \sum_{\tau \in \mathcal{T}(\Delta)} c(\tau) W(\tau) \int I(\tau, \mathbf{s}) \, d\mathbf{s}.$$

Sum over Δ.

$$\sum_{G_c} \prod_{\{i,j\} \in G_c} t_{ij} = \sum_{\tau} c(\tau) W(\tau) \int I(\tau, \mathbf{s}) \, d\mathbf{s}.$$

Tree bound

Suppose that $-1 \leq t_{ij} \leq 0$. Then

$$|\sum_{G_c} \prod_{\{i,j\} \in G_c} t_{ij}| \leq \sum_{\tau} c^*(\tau).$$

Here

$$c^*(\tau) = \prod_{i \neq r} |t_{i\tau(i)}|.$$

Recursive structure

$$c^*(\tau_r) = \prod_{\tau(j)=r} |t_{rj}| c^*(\tau_j).$$

Enriched tree bound

An approach to Fernández-Procacci bounds.
Suppose that $-1 \leq t_{ij} \leq 0$. Then

$$|\sum_{G_c} \prod_{\{i,j\} \in G_c} t_{ij}| \leq \sum_{\tau} C^*(\tau) = \sum_{\tau} c^*(\tau) Q(\tau). \tag{1}$$

The fiber weight is

$$Q(\tau) = \prod_{\tau(i)=\tau(j)} (1 + t_{ij}). \tag{2}$$

Recursive structure

$$C^*(\tau_r) = \left[\prod_{\tau(j)=r} |t_{rj}| \prod_{\tau(i)=\tau(j)=r} (1 + t_{ij}) \right] \prod_{\tau(j)=r} C^*(\tau_j).$$

Conclusion for physics

- $-1 \leq t_{ij} \leq 0$
- $|t_{ij}|$ measures the strength of repulsive interaction.
- The tree bound works when many interactions are weak, which happens at long distances.
- $0 \leq 1 + t_{ij} \leq 1$
- The enriched tree bound has additional factors $1 + t_{ij}$. is l when the interactions are large, which happens at short distances.
- As Fernández-Procacci show, such improved bounds give a better grasp of the physics.