Linear Operators Preserving Stability and the Lee-Yang Program

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Based on joint work with

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Main Theme:

Zero loci/non-vanishing properties of (multivariate) polynomials & their dynamics under linear operators

Rota's Philosophy

G.-C. Rota: *"The one contribution of mine that I hope will be remembered has consisted in just pointing out that all sorts of problems of combinatorics* [statistical mechanics, matrix theory...] *can be viewed as problems on location of the zeros of certain polynomials and in giving these zeros a combinatorial interpretation."*

Some Motivating Examples

1. Negative Correlations
2. Monomer-Dimer Systems
3. Phase Transitions

Two main tools in statistical mechanics

- Cluster expansions
  Mayer expansions, Pirogov-Sinai theory...
- Correlation inequalities:
  Positive correlations/dependence
  Harris, FKG...
  Negative correlations/dependence
  ?!...

1. Negative Correlations

$[n] = \{1, \ldots, n\}$, $\mathcal{P}_n = \left\{\text{prob. meas. on } 2^{[n]}\right\}$, $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$. The partition function

$$Z_\mu(\mathbf{z}) = \sum_{S \subseteq [n]} \mu(S)\mathbf{z}^S, \quad \mathbf{z}^S = \prod_{i \in S} z_i.$$

Pairwise Negative Correlations : for $i \neq j$

$$\mu(\{S : i, j \in S\}) \leq \mu(\{S : i \in S\})\mu(\{S : j \in S\})$$

$$\Updownarrow$$

$$Z_\mu(\mathbf{1})\frac{\partial^2}{\partial z_i \partial z_j} Z_\mu(\mathbf{1}) \leq \frac{\partial}{\partial z_i} Z_\mu(\mathbf{1}) \frac{\partial}{\partial z_j} Z_\mu(\mathbf{1}).$$

Negative Association (NA): for up-sets $\mathcal{A}, \mathcal{B}$ with disjoint support

$$\mu(\mathcal{A} \cap \mathcal{B}) \leq \mu(\mathcal{A})\mu(\mathcal{B})$$

$$\Updownarrow$$

$$\int fg d\mu \leq \int f d\mu \int g d\mu$$

for all increasing f, g depending on disjoint sets of coordinates.

Question: When do we have NA(+)?

Rayleigh (R): $\forall \mathbf{z} \in \mathbb{R}^n_+$

$$Z_\mu(\mathbf{z})\frac{\partial^2}{\partial z_i \partial z_j} Z_\mu(\mathbf{z}) \leq \frac{\partial}{\partial z_i} Z_\mu(\mathbf{z}) \frac{\partial}{\partial z_j} Z_\mu(\mathbf{z}).$$

Question: Does R $\Rightarrow$ NA?...

Strongly Rayleigh (SR): $\forall \mathbf{z} \in \mathbb{R}^n$

$$Z_\mu(\mathbf{z})\frac{\partial^2}{\partial z_i \partial z_j} Z_\mu(\mathbf{z}) \leq \frac{\partial}{\partial z_i} Z_\mu(\mathbf{z}) \frac{\partial}{\partial z_j} Z_\mu(\mathbf{z}).$$

Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, $\mathcal{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$.

Definition. $f \in \mathbb{K}[z_1, \ldots, z_n]$ is ($\mathcal{H}$-)stable if $f(\mathbf{z}) \neq 0$ $\forall \mathbf{z} \in \mathcal{H}^n$, real stable (RS) if in addition $\mathbb{K} = \mathbb{R}$.

Theorem [Brändén]. μ SR $\Leftrightarrow$ Z_μ RS.

This allows to use powerful methods from the theory of uni- & multivariate polynomials (Gårding hyperbolicity, Grace-Walsh-Szegö coincidence theorem).

Theorem [B., Brändén, Liggett]. μ SR $\Rightarrow$ μ CNA+.

Theorem [B., Brändén, Liggett]. The *symmetric exclusion process* on 2^E, $|E| \leq \aleph_0$, preserves the SR property.

2. Monomer-Dimer Systems

A matching M in a graph $G = (V, E)$ is a subset of E s.t. no two edges in M have a common vertex

Theorem [Heilmann-Lieb, 1972].
Suppose $\lambda_{ij} \geq 0$ for all $ij \in E$. If $\mathrm{Re}(z_k) > 0$, $k \in [n]$, then

$$\sum_{\text{matchings } M} \prod_{ij \in M} \lambda_{ij} z_i z_j \neq 0$$

Applications of the univariate version: asymptotic normality for uniform matchings in simple graphs...

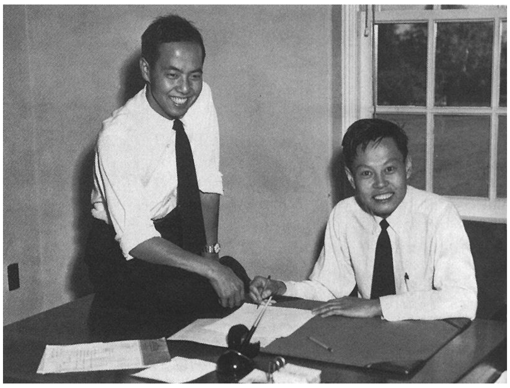
3. Phase Transitions

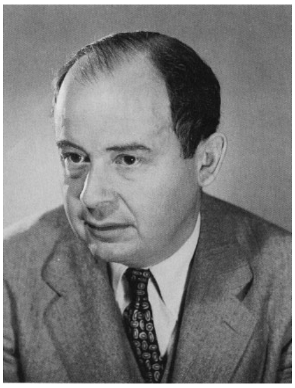
occur when an observable quantity (magnetization, resistivity...) depends non-analytically on some control parameter (temperature, magnetic field...)

Ising model: Let $G = (V, E)$, $\Omega = \{-1, 1\}^V$, $\omega = (\sigma_i)_{i \in V} \in \Omega$, $\mathcal{H}_\omega = -\sum_{ij \in E} J_{ij} \sigma_i \sigma_j - h \sum_{i \in V} \sigma_i$ and $\beta = T^{-1}$. The partition function

$$Z(h) = \sum_{\omega \in \Omega} e^{-\beta \mathcal{H}_\omega}$$

Theorem [Lee-Yang, 1952].
All zeros of $Z(h)$ are on the imaginary axis.
Thus a phase transition occurs only at $h = 0$.

 asked around

  ... 

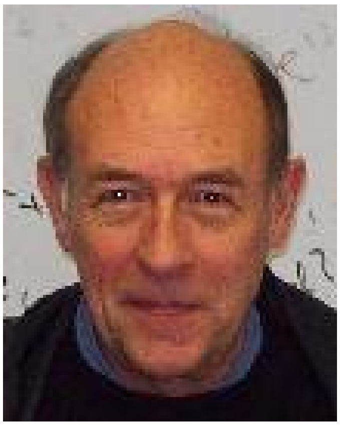
Multivariate versions: $n = |V|$, $[n] = \{1, \ldots, n\}$

$$\mathcal{H}_\omega = -\sum_{1\le i<j\le n} J_{ij}\sigma_i\sigma_j - \sum_{i=1}^{n} h_i\sigma_i$$

Theorem [Lee-Yang, 1952].
If $\mathrm{Re}(h_k) > 0$, $k \in [n]$, then $Z(h_1, \ldots, h_n) \neq 0$.
"Circle Theorem". If (a_{ij}) is a Hermitian $n \times n$ matrix with $|a_{ij}| \le 1$ and $|z_k| < 1$, $k \in [n]$, then

$$\sum_{S\subseteq[n]} \prod_{k\in S} z_k \prod_{i\in S} \prod_{j\notin S} a_{ij} \neq 0.$$

The Lee-Yang theorem: a failure?...



I have called this beautiful result a failure because, while it has important applications in physics, it remains at this time isolated in mathematics.
One might think of a connection with zeta functions. But the miracle has not happened [yet]. *One still does not know what to do with the circle theorem*.

D. Ruelle, Gibbs lecture 1988

Update

- Hinkkanen... (complex analysis)
- Ruelle, Sokal, Scott, Choe, Oxley, Wagner, Bollobás, Riordan... (combinatorics)
- Newman, Conrey, Farmer, Cardon... (analytic number theory)
- Dobrushin, Biskup, Borgs, Chayes, Kleinwaks, Kotecky... (statistical mechanics)
- Cardy, Kesten, Grimmett, Schramm, Smirnov, Werner, Kenyon, Okounkov... (percolation theory)

The Problem(s)

• Many known generalizations of the Lee-Yang theorem: Asano, Fisher, Newman, Ruelle, Lieb-Sokal...

• Common theme: the key step is to show that certain linear operators on multivariate polynomials preserve the property of being non-vanishing whenever all variables are in a prescribed set $\Omega \subset \mathbb{C}$

Definition

A polynomial $f \in \mathbb{C}[z_1, \dots, z_n]$ is **Ω-stable** if $z := (z_1, \dots, z_n) \in \Omega^n \Longrightarrow f(z) \neq 0$

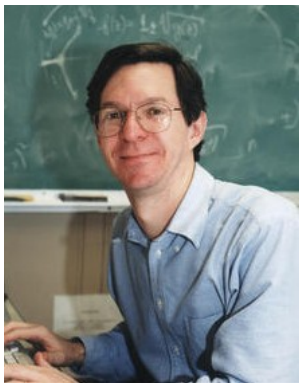
Yet more general: $\Omega_1 \times \cdots \times \Omega_n$-stability

Most general notion: Ω-stability, $\Omega \subset \mathbb{C}^n$

Problem(s)

Given $V \subseteq \mathbb{C}[z_1, \dots, z_n]$ with $\dim V \leq \infty$ characterize all linear $T : V \to \mathbb{C}[z_1, \dots, z_n]$ that preserve Ω-stability, i.e., s.t. $T(f)$ is Ω-stable or $\equiv 0$ whenever $f \in V$ is Ω-stable

"Soft" and "hard" theorems



(Very) roughly: "soft" theorems are constraint-free while "hard" theorems involve constraints of various kinds, e.g. the max-degree of a graph (Sokal et al...)

Notations

$\mathbf{z} = (z_1, \ldots, z_n), \mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{C}^n$

$\mathbf{z} \cdot \mathbf{w} = z_1 w_1 + \ldots + z_n w_n,$
$\mathbf{zw} = (z_1 w_1, \ldots, z_n w_n),$

$\mathbf{z}^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}, \ \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n,$

$\binom{\beta}{\alpha} := \binom{\beta_1}{\alpha_1} \cdots \binom{\beta_n}{\alpha_n}, \ \alpha \leq \beta \in \mathbb{N}^n,$

$\alpha! = \alpha_1! \cdots \alpha_n!, \ (1^n) = (1, \ldots, 1) \in \mathbb{N}^n.$

$\mathcal{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$, $\mathcal{R} = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$

For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $\kappa = (\kappa_1, \dots, \kappa_n) \in \mathbb{N}^n$ let $\mathbb{K}_\kappa[z_1, \dots, z_n]$ be the $\mathbb{K}$-space of polynomials of degree at most κ_i in z_i, $1 \le i \le n$.

The Laguerre-Pólya class LP_n of entire functions in n variables consists of uniform limits on compacts in $\mathbb{C}^n$ of real $\mathcal{H}$-stable n-variate polynomials.

The Hermite-Biehler class HB_n of entire functions in n variables consists of uniform limits of (complex) $\mathcal{H}$-stable polynomials in n variables.

Let $\Omega \subset \mathbb{C}$ ($\Omega = \mathcal{H}$, $\mathcal{R}$ or $\mathbb{D}$ are main interest). Characterize all linear operators

(1) $T : \mathbb{C}_\kappa[z_1, \dots, z_n] \to \mathbb{C}[z_1, \dots, z_n]$ that preserve Ω-stability for fixed $\kappa \in \mathbb{N}^n$
("hard-core"/algebraic characterization)

(2) $T : \mathbb{C}[z_1, \dots, z_n] \to \mathbb{C}[z_1, \dots, z_n]$ that preserve Ω-stability
("soft-core"/transcendental characterization)

Definition

If $\kappa \in \mathbb{N}^n$ and $T : \mathbb{C}_\kappa[z_1, \ldots, z_n] \to \mathbb{C}[z_1, \ldots, z_n]$ then its algebraic symbol w.r.t. $\mathcal{R}$ or $\mathbb{D}$ is the $2n$-variate polynomial

$$T[(\mathbf{1} + \mathbf{zw})^\kappa] = \sum_{\alpha \le \kappa} \binom{\kappa}{\alpha} T(\mathbf{z}^\alpha)\mathbf{w}^\alpha.$$

The algebraic symbol w.r.t. $\mathcal{H}$ is

$$T[(\mathbf{z} + \mathbf{w})^\kappa] = \sum_{\alpha \le \kappa} \binom{\kappa}{\alpha} T(\mathbf{z}^\alpha)\mathbf{w}^{\kappa - \alpha}.$$

"Hard-core"/algebraic characterizations

Theorem [B., Brändén, 2008].

Let $\kappa \in \mathbb{N}^n$ and $T : \mathbb{C}_\kappa[z_1, \ldots, z_n] \to \mathbb{C}[z_1, \ldots, z_n]$ be a linear operator. If $\Omega = \mathcal{R}$, $\mathbb{D}$ or $\mathcal{H}$ then T preserves Ω-stability iff either

- $T(f) = \alpha(f)P$, where α is a linear functional and P is a fixed Ω-stable polynomial, or
- The algebraic symbol of T w.r.t. Ω is Ω-stable in $2n$ variables.

Definition

If $T : \mathbb{C}[z_1, \ldots, z_n] \to \mathbb{C}[z_1, \ldots, z_n]$ its transcendental symbol w.r.t. $\mathcal{R}$ or $\mathbb{D}$ is the formal power series in $2n$ variables

$$T[e^{\mathbf{z}\cdot\mathbf{w}}] := \sum_{\alpha\in\mathbb{N}^n} T(\mathbf{z}^\alpha)\mathbf{w}^\alpha/\alpha!$$

The transcendental symbol w.r.t. $\mathcal{H}$ is

$$T[e^{-\mathbf{z}\cdot\mathbf{w}}] := \sum_{\alpha\in\mathbb{N}^n} (-1)^\alpha T(\mathbf{z}^\alpha)\mathbf{w}^\alpha/\alpha!$$

"Soft-core"/transcendental characterization

Theorem[B., Brändén, 2008].

Let $T : \mathbb{C}[z_1, \ldots, z_n] \to \mathbb{C}[z_1, \ldots, z_n]$ be a linear operator. Then T preserves $\mathcal{H}$-stability iff either

- $T(f) = \alpha(f)P$, where α is a linear functional and P is a fixed $\mathcal{H}$-stable polynomial, or
- $T[e^{-\mathbf{z}\cdot\mathbf{w}}] \in HB_{2n}$.

Remark. Note that

$$e^{-\mathbf{z}\cdot\mathbf{w}} = \lim_{k\to\infty}\left(1 - \frac{z_1w_1}{k}\right)^k \cdots \left(1 - \frac{z_nw_n}{k}\right)^k.$$

Corollary [B-B, 2008].
If $T : \mathbb{R}[z_1, \ldots, z_n] \to \mathbb{R}[z_1, \ldots, z_n]$ is a linear operator then T preserves real stability ($= \mathcal{H}$-stability) iff either

- $T(f) = \alpha(f)P + \beta(f)Q$, where α, β are linear functionals and P, Q are real stable polynomials such that either $P + iQ$ or $Q + iP$ is real stable, or
- $T[e^{-\mathbf{z}\cdot\mathbf{w}}] \in LP_{2n}$, or
- $T[e^{\mathbf{z}\cdot\mathbf{w}}] \in LP_{2n}$.

Note: If $n = 1$ and $f \in \mathbb{R}[z]$ then f is real stable iff $f^{-1}(0) \in \mathbb{R}$ so one gets a characterization of all linear preservers of real-rootedness:

Corollary [B-B, 2006].
If $T : \mathbb{R}[z] \to \mathbb{R}[z]$ is a linear operator then T preserves real-rootedness iff either

- $T(f) = \alpha(f)P + \beta(f)Q$, where α, β are linear functionals and P, Q are real-rooted polynomials whose zeros interlace, or
- $T[e^{-\mathbf{z}\cdot\mathbf{w}}] \in LP_2$, or
- $T[e^{\mathbf{z}\cdot\mathbf{w}}] \in LP_2$.

This solves a long-standing problem going back to Pólya-Schur (Hermite, Laguerre...)

Examples

Soft-core: Fix f, define $T(g) = f(\frac{\partial}{\partial \mathbf{z}})g$. Then $T[e^{\mathbf{z}\cdot\mathbf{w}}] = f(\mathbf{w})e^{\mathbf{z}\cdot\mathbf{w}} \implies$ Lieb-Sokal's $\mathcal{R}$-stability theorem/extension of Hermite-Poulain-Jensen thm ("soft" Lieb-Sokal lemma: "$z_1 \to \frac{\partial}{\partial z_2}$")

Hard-core: The algebraic symbol of

$$T_d := \frac{1}{d!}\sum_{k=0}^{d} \frac{\partial^d}{\partial z_1^k \partial z_2^{d-k}}$$

viewed as $T_d : \mathbb{C}_{(d,d)}[z_1, z_2] \to \mathbb{C}_{(d,d)}[z_1, z_2]$ is

$T_d[(1 + z_1 w_1)^d (1 + z_2 w_2)^d] =$

$$\frac{1}{d!}\sum_{k=0}^{d} \frac{d!}{(d-k)!}\frac{d!}{k!} w_1^k w_2^{d-k}(1+z_1w_1)^{d-k}(1+z_2w_2)^k$$

$= (w_1 w_2)^d (w_1^{-1} + w_2^{-1} + z_1 + z_2)^d$

($\Rightarrow$ "hard" Lieb-Sokal lemma: " $z_1^k \to (d-k)! \left(\frac{\partial}{\partial z_2}\right)^k$ ")

Heilmann-Lieb: define a linear operator

$$\text{MAP}\left(\sum_{\alpha} a(\alpha)\mathbf{z}^{\alpha}\right) = \sum_{\alpha\colon \alpha_i \le 1\, \forall i} a(\alpha)\mathbf{z}^{\alpha}.$$

Then MAP$[e^{\mathbf{z}\cdot\mathbf{w}}] = (1 + z_1 w_1)\cdots(1 + z_n w_n)$ so MAP preserves $\mathcal{R}$-stability [Choe-Oxley-Sokal-Wagner, 2004]$\Longrightarrow$ Heilmann-Lieb theorem

Lee-Yang (1st proof): define a linear operator

$$\text{MOD}\left(\sum_\alpha a(\alpha)\mathbf{z}^\alpha\right) = \sum_\alpha a(\alpha)\mathbf{z}^{\alpha \text{ mod } 2}.$$

Then $\text{MOD}[(\mathbf{1}+\mathbf{zw})^\kappa] =$

$$2^{-n}(\mathbf{1}+\mathbf{w})^\kappa(\mathbf{1}+\mathbf{z})^{[n]}\prod_{i=1}^n\left[1+\left(\frac{1-w_i}{1+w_i}\right)^{\kappa_i}\frac{1-z_i}{1+z_i}\right]$$

so MOD preserves $\mathcal{R}$-stability [Choe-Oxley-Sokal-Wagner, 2004]$\Longrightarrow$ Lee-Yang theorem.

Schur-Hadamard product: Set $\kappa = (1^n)$, fix a $\mathbb{D}$-stable multi-affine $g(\mathbf{z}) = \sum_{S\subseteq[n]} b(S)\mathbf{z}^S$ and define a linear operator T on $\mathbb{C}_{(1^n)}[z_1,\ldots,z_n]$ by

$$T\left(\sum_S a(S)\mathbf{z}^S\right) = \sum_S a(S)b(S)\mathbf{z}^S.$$

Then $T[(\mathbf{1}+\mathbf{zw})^\kappa] = \sum_S w^S b(S)\mathbf{z}^S = g(\mathbf{zw})$ is $\mathbb{D}$-stable $\Rightarrow$ Hinkkanen's composition theorem $\Rightarrow$

Lee-Yang (2nd proof): if $a \in \mathbb{D}$ then $1+az+\bar{a}w+zw$ is $\mathbb{D}$-stable. Let

$$f_{ij}(z) = (1+a_{ij}z_i+\overline{a_{ij}}z_j+z_iz_j)\prod_{k\in[n]\setminus\{i,j\}}(1+z_k)$$

and take the Schur-Hadamard product $\diamond$ of all f_{ij}'s, $1\le i<j\le n$, to get the $\mathbb{D}$-stability of

$$f_{12}\diamond\cdots\diamond f_{(n-1)n} = \sum_{S\subseteq[n]}\mathbf{z}^S\prod_{i\in S}\prod_{j\notin S}a_{ij}.$$

Multivariate (Grace) apolarity theory (quest by Rota, Hinkkanen, cf. also Ruelle's Dyson-type lemmas)

Master composition theorem(s) [B-B, 2008]
Suppose that $f, g \in \mathbb{C}[z_1, \ldots, w_n]$ are of the form

$$f(z, w) = \sum_{\alpha} \binom{\kappa}{\alpha} P_\alpha(w) z^\alpha,$$

$$g(z, w) = \sum_{\alpha} \binom{\kappa}{\alpha} Q_\alpha(z) w^\alpha.$$

If f, g are Ω-stable, where $\Omega = \mathcal{R}$ or $\mathbb{D}$, then so is

$$(f \star g)(z, w) := \sum_{\alpha} \binom{\kappa}{\alpha} P_\alpha(w) Q_\alpha(z)$$

unless it is $\equiv 0$.

Proof. Take two linear operators T, S with symbols f, g respectively. If T, S preserve Ω-stability then so does $S \circ T$. Its symbol is precisely $f \star g$. $\square$

This generalizes all (univariate) Schur-Maló-Szegö composition/convolution theorems and the multivariate ones in [B-B, 2006] based on the Weyl product. It leads to a multivariate Grace apolarity theorem.

Summary

The gist of it

• Given T(n-variate) let $T(z^\alpha w^\beta) = T(z^\alpha)w^\beta$ for $\alpha, \beta \in \mathbb{N}^n$ and extend T linearly to T($2n$-variate)

• Define appropriate $2n$-variate symbols for T

• Conclusion 1. Stable/Lee-Yang polynomials are "self-generating": linear preservers of n-variate stable polynomials are induced by $2n$-variate stable polynomials via symbol maps

• Conclusion 2. Multivariate ideology ("Sokalization") = right framework (also for $n = 1$: special case of $n \geq 1$)

Proof ideas: sufficiency (algebraic/"hard-core")

1. The multi-affine case: $\kappa = (1^n)$.

Suppose $T[(\mathbf{1}+\mathbf{zw})^\kappa]$ and $f(\mathbf{z})$ are $\mathcal{R}$-stable. Then

$$\sum_{S \subseteq [n]} T[\mathbf{z}^S]\mathbf{w}^S f(\mathbf{v}) \text{ is } \mathcal{R}\text{-stable}$$

Apply the Lieb-Sokal lemma with $w_i \to \partial/\partial v_i \implies$

$$\sum_{S \subseteq [n]} T[\mathbf{z}^S] f^{(S)}(\mathbf{v}) \text{ is } \mathcal{R}\text{-stable or } \equiv 0$$

Apply Hurwitz' theorem with $v_i \to 0^+ \implies$

$$T(f)(\mathbf{z}) = \sum_{S \subseteq [n]} T[\mathbf{z}^S] f^{(S)}(\mathbf{0}) \text{ is } \mathcal{R}\text{-stable or } \equiv 0.$$

2. The general case: $\kappa \in \mathbb{N}^n$.

Step 1. Let $\mathbb{C}_\kappa^{\mathrm{MA}}$ be the space of multi-affine polynomials in the variables $\{z_{ij} : 1 \le i \le n, 1 \le j \le \kappa_i\}$. Define a projection and a polarization operator

$$\Pi_\kappa^\downarrow : \mathbb{C}_\kappa^{\mathrm{MA}} \to \mathbb{C}_\kappa[z_1, \dots, z_n]$$

$$\Pi_\kappa^\uparrow : \mathbb{C}_\kappa[z_1, \dots, z_n] \to \mathbb{C}_\kappa^{\mathrm{MA}}$$

as follows: let $e_k(x_1, \dots, x_n)$ be the kth elementary symmetric polynomial, set

$$\Pi^{\downarrow}_{\kappa}(z_{ij}) = z_i, \quad \Pi^{\uparrow}_{\kappa}(z_i^{\alpha_i}) = \binom{\kappa_i}{\alpha_i}^{-1} e_{\alpha_i}(z_{i1}, \dots, z_{i\kappa_i})$$

and extend linearly. Clearly, $\Pi^{\downarrow}_{\kappa} \circ \Pi^{\uparrow}_{\kappa} = \mathrm{Id}$.

Step 2: reduction to the multi-affine case. Let $\gamma \in \mathbb{N}^n$ and $T : \mathbb{C}_{\kappa}[z_1, \dots, z_n] \to \mathbb{C}_{\gamma}[z_1, \dots, z_n]$ be a linear operator. Define the polarization of T to be the linear operator $\Pi(T) : \mathbb{C}^{\mathrm{MA}}_{\kappa} \to \mathbb{C}^{\mathrm{MA}}_{\gamma}$ given by

$$\Pi(T) = \Pi^{\uparrow}_{\gamma} \circ T \circ \Pi^{\downarrow}_{\kappa}.$$

Conversely, $T = \Pi^{\downarrow}_{\gamma} \circ \Pi(T) \circ \Pi^{\uparrow}_{\kappa}$. Now $\Pi^{\downarrow}_{\gamma}$ preserves $\mathcal{R}$-stability. By the Grace-Walsh-Szegö coincidence theorem, $\Pi^{\uparrow}_{\gamma}$ and $\Pi^{\uparrow}_{\kappa}$ preserve $\mathcal{R}$-stability. Thus T and $\Pi(T)$ preserve $\mathcal{R}$-stability simultaneously.

Lemma [B-B, 2008].
The symbol of the polarization $\Pi(T)$ is the polarization of the symbol of T.

It follows that the symbols of T and $\Pi(T)$ are $\mathcal{R}$-stable simultaneously. □

Sufficiency: transcendental/"soft-core" case

From the algebraic characterization we deduce that if $\mathbf{m} = (m, \ldots, m) \in \mathbb{N}^n$ then

$$F_m(\mathbf{z}, \mathbf{w}) := \sum_{\alpha \le \mathbf{m}} \binom{\mathbf{m}}{\alpha} T(\mathbf{z}^\alpha)\mathbf{w}^\alpha$$

is $\mathcal{R}$-stable hence (replace w_i with w_i/m) so is

$$G_m(\mathbf{z}, \mathbf{w}) = \sum_{\alpha \le \mathbf{m}} \mathbf{m}^{-\alpha} \binom{\mathbf{m}}{\alpha} T(\mathbf{z}^\alpha)\mathbf{w}^\alpha.$$

Now

$$\lim_{m \to \infty} \mathbf{m}^{-\alpha} \binom{\mathbf{m}}{\alpha} = 1/\alpha!$$

Thus, as formal power series we have

$$\lim_{m \to \infty} G_m(\mathbf{z}, \mathbf{w}) = T[e^{\mathbf{z} \cdot \mathbf{w}}] = \sum_{\alpha} T(\mathbf{z}^\alpha)\mathbf{w}^\alpha/\alpha!$$

Theorem [B-B, 2008].
$\{G_m(\mathbf{z}, \mathbf{w})\}_{m=0}^{\infty}$ is uniformly bounded on any compact subset of $\mathbb{C}^{2n}$.

Multivariate Szász principles

Lemma [O. Szász, 1943].
Let $f(z) = 1 + a_1 z + \cdots + a_d z^d \in \mathbb{C}[z]$ be $\mathcal{R}$-stable.
If $r \geq 0$ and $|z| \leq r$ then

$$|f(z)| \leq \exp\left(|a_1| r + 3|a_1|^2 r^2 + 3|a_2| r^2\right).$$

Theorem [B-B, 2008].
As before $\mathbf{z} = (z_1, \ldots, z_n)$. Suppose

$$f(\mathbf{z}) = 1 + \sum_{|\beta|>0} a(\beta)\mathbf{z}^\beta \in \mathbb{C}[z_1, \ldots, z_n]$$

is $\mathcal{R}$-stable and let

$$\begin{aligned} B &= 2^{n-1}\frac{\sqrt{2e^2 - e}}{e-1} \\ C &= 6e^2\left(\sum_{|\alpha|=1} |a(\alpha)|\right)^2 + 4e^2 \sum_{|\alpha|=2} |a(\alpha)|. \end{aligned}$$

If $r \geq 0$ and $|z_i| \leq r$, $1 \leq i \leq n$, then

$$|f(\mathbf{z})| \leq B e^{Cr^2}.$$

Open problems and further directions

For $n = 1$ the problem of characterizing all linear operators preserving the property of having all zeros in Ω (i.e., $\mathbb{C} \setminus \Omega$-stability) is still open for:

- Open circular domains

- Half-lines, e.g. $\mathbb{R}_{\leq 0}$; intervals, e.g. $[0, 1]$ (work by e.g. Brenti, Wagner, Saff...)

- Sectors (work by e.g. Weisner, Marden...)

- Strips (work by e.g. de Bruijn-Springer, Pólya...)

In progress: generalizations to entire functions and Fischer-Fock spaces...