CLUSTER (MAYER) EXPANSIONS

THEIR RESUMMATION

M. Zahradnik
Prague

Motivation, historical remarks

$$\log Z(\Lambda) = \sum_{\varphi \subset \Lambda} w_\varphi = h|\Lambda| \pm \varepsilon|\partial\Lambda| \ldots$$

$$Z(\Lambda) = \sum_{\{\Gamma_i\}} \prod_i w_{\Gamma_i}$$

Γ_i: "polymers"

w_{Γ_i} (complex) weights

φ "clusters" $\equiv$ conglomerates of polymers

absolute convergence ?

K.P. criterion: Estimate

$$a_\Gamma \stackrel{def}{=} \sum_{\varphi \text{ "touches" } \Gamma} w_\varphi$$

If exists b_Γ such that

$$\sum_{\Gamma' \text{ touches } \Gamma} |w_{\Gamma'}| e^{b_{\Gamma'}} < b_\Gamma$$

then $|a_\Gamma| \leq b_\Gamma$

MODELS WITH REPULSION

$$Z = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{(\Gamma_1,\dots,\Gamma_N)} \prod_{i=1}^{N} w_{\Gamma_i} \prod_{i<j} \left(1 + F(\Gamma_i, \Gamma_j)\right)$$

$$= \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{(\Gamma_1,\dots,\Gamma_N)} \prod_i w_{\Gamma_i} \sum F_G$$

G graph on $(1, 2, \dots, N)$

$$F_G = \prod_{(i,j) \in G} F(\Gamma_i, \Gamma_j)$$

$F(\Gamma, \Gamma') \in (-1, 0)$ soft repulsion

$\in \{-1, 0\}$ hard repulsion

THEOREM 0

$$Z = \exp\left(\sum_{M=1}^{\infty} \frac{1}{M!} \sum_{(\Gamma_1,\dots,\Gamma_M)} \prod_i w_{\Gamma_i} \sum_{G \text{ connected}} F_G \right)$$

Notation: $\varphi = (\Gamma_1, \dots, \Gamma_M) \,\&\, G \equiv$ cluster

$$w_\varphi = \prod_i w_{\Gamma_i} F_G$$

RESUMMATION $\equiv$ bracketing of $\sum_G F_G$

many cancellations in this sum!

e.g. $\sum_T \left(\sum_{G \to T} F_G \right)$

EXAMPLE

$$1+x = 1+x+\sum_{N=2}^{\infty} \frac{1}{N!} x^{N} (1-1)^{\binom{N}{2}} =$$

$$= \exp\left(\sum_{M=1}^{\infty} \frac{x^{M}}{M!} \underbrace{\sum_{\substack{G \text{ connected on} \\ (1,2,\dots,M)}} (-1)^{|G|}}_{=(-1)^{M-1}(M-1)!} \right)$$

hence

$$1+x = \exp\left(-\sum_{\circlearrowleft} \frac{(-x)^{M}}{M!} \right)$$

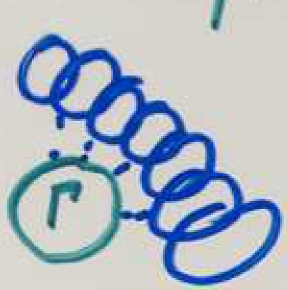
sum over all cyclic perturbations of $(1,2,\dots,M)$.

MORE ON K-P. CRITERION

Put $$a_{\Gamma} = \sum_{\varphi} F_{\Gamma,\varphi}\, F_{\varphi}\, w_{\varphi}$$

"sum over all clusters touching Γ"

φ



$$F_{\Gamma,\varphi} = \prod_{i} \left(1+F(\Gamma,\Gamma_i)\right) - 1 = -1$$ in hard repul. case

$$\varphi = (\Gamma_1,\dots,\Gamma_M)$$

$$F_{\varphi} = \sum_{G \text{ connected}} F_G$$

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who is 'responsible' for touching?

AUXILIARY (t,Γ) model $t \in [0,1]$

$F(\Gamma,\Gamma')$ replaced by $t\,F(\Gamma,\Gamma')$

$[t,\Gamma]$ model

$w_{\Gamma'}$ replaced by $(1+tF(\Gamma,\Gamma'))w_{\Gamma'}$

THEN

$$\frac{\partial}{\partial t} a_\Gamma^{(t,\Gamma)} = \sum_{\Gamma'} \underbrace{F(\Gamma,\Gamma')}_{\substack{-1 \text{ in the hard} \\ \text{rep. case}}} w_{\Gamma'}\, e^{a_{\Gamma'}^{[t,\Gamma]}}$$

RESUMMATION (my favourite one)

Graphs G $\to$ left trees $\mathcal{T}$

DEFINITION Left tree $\mathcal{T}$ on $(1,2,\dots,m)$

$\equiv M$ defined recursively by decomposing of M into a singleton $\{1\}$ (the "chief") and a disjoint decomposition of

$M \setminus \{1\}$ into subtrees $\mathcal{T}_i$ sitting on M_i where $\{M_i\}$ is a decomposition of $M \setminus \{1\}$ DENOTE by $L(m)$ No of left trees

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$$L(m) = \sum_{\text{all decompositions } \{m_i\} \text{ of } m-1} \prod_i L(m_i)$$

SAME ~~as~~ for number of cycles!

$$C(m) = \sum_{\text{all}\dots} \prod_i C(m_i)$$

Hence $(m-1)!$ left trees on M

$(\Gamma_1, \dots, \Gamma_M)$ cluster $\left(\text{i.e. } \sum_G F_G \neq 0\right)$

<u>THEOREM</u> $\displaystyle\sum_G F_G = \sum_{\mathcal{T}} F_{\mathcal{T}}$

$F_{\mathcal{T}}$ defined recursively

$\mathcal{T} = \{1 \,\&\, \{\mathcal{T}_j\}\}$

$$F_{\mathcal{T}} = \prod_j \left(F_{1,\mathcal{T}_j} \cdot F_{\mathcal{T}_j}\right)$$

$$F_{1,\mathcal{T}_j} = \prod_{\Gamma \in \mathcal{T}_j} \left(1 + F(\Gamma_1, \Gamma)\right) - 1 \qquad = -1$$

$(1-1)^{\ell} - 1 = -1$

$$\Rightarrow Z = \exp\left(\sum_{\varphi = (\Gamma_1, \dots, \Gamma_M)} \sum_{\substack{\mathcal{T} \text{ left tree} \\ \text{on } (1,2,\dots,M)}} F_{\mathcal{T}} w_{\varphi}\right)$$

COROLLARY: if all $w_\Gamma = - z_\Gamma$

then $Z = \exp\left(- \sum_\varphi \alpha_\varphi z_\varphi\right)$ with positive α_φ

possibly $0 = \exp(-\infty)$

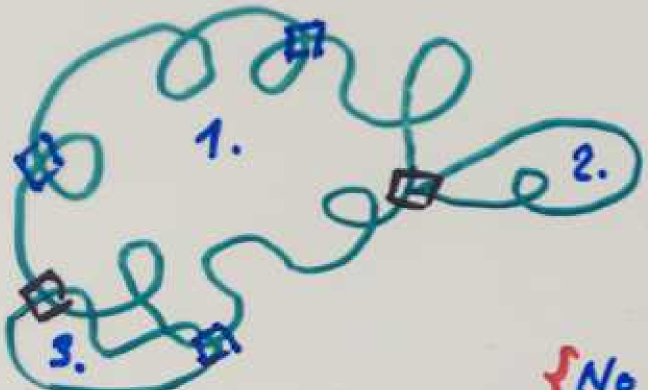
MORE REFINED RESUMMATIONS FOR SPECIAL CLASSES OF MODELS

A) Hardcore by intersection

$F(\Gamma,\Gamma') = -1$ iff $\operatorname{supp}\Gamma \cap \operatorname{supp}\Gamma' \neq \emptyset$

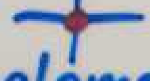
suitable for models with cycles (Laplacian determinants, Kramers-Wannier-Onsager-Ising) huge cancellations in sums, remainder can be organized as sum over 'cactuses' (cycles of cycles of...) leads to exact solutions of some models

B) Models in $\mathbb{Z}^d$, $d \geq 3$ where polymers are 'circles' (closed paths-walks)



(self) interaction of polymers through decorations (multiple visits of the same point)

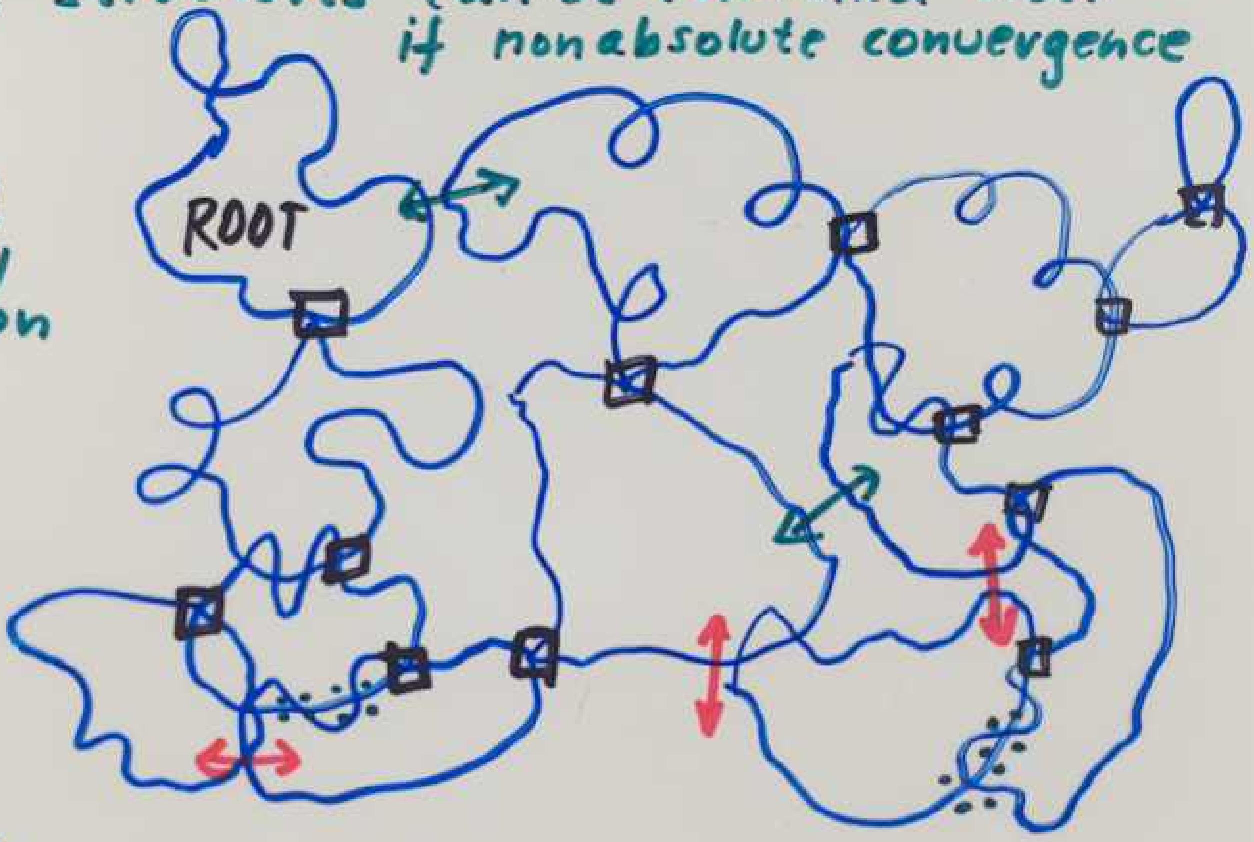
$$w_{\{P_i\}} = (-\varepsilon)^{\{\text{No of decorations}\}} \cdot \text{usual path measure}$$

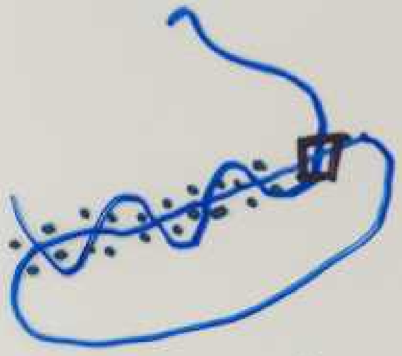
Similar models appear when studying partition functions of Laplacian gaussians with added potentials like x^4 or $-\log(1-x^4)$. Summing over all pairings of  diagrams represented as conglomerates of weakly interacting circles

IDEA: summation over hierarchically (tree) ordered structures can be controlled even if nonabsolute convergence

$\longleftrightarrow$ allowed additional interaction

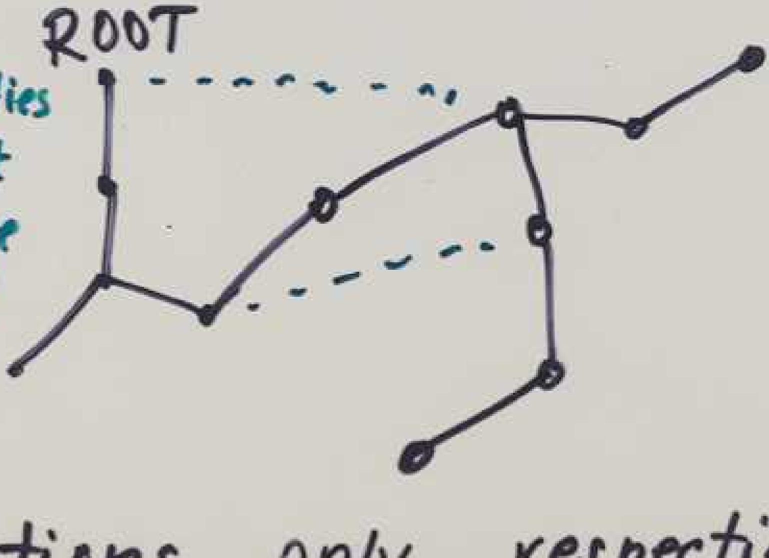
$\longleftrightarrow$ forbidden





Pay $(-\varepsilon)$ for each junction

order points, if more possibilities for junction select the first possible one, other just adding factor $(1-\varepsilon)$



Interactions <u>only</u> respecting tree hierarchy

IDEA: Organize the series as (they would be series) of

rooted trees of circles

with usual path integral measure $|\frac{1}{6}|^{|P|}$ and with additional activities $-\varepsilon$ at junction points of circles and possibly also at points of self interactions of circles

Other (repulsive) interactions between circles allowed only in 'hierarchical' order given by the tree. Find estimates

$$\sum_{\text{clusters } \mathcal{C} \text{ containing circle } C} w_{\mathcal{C}} = \xi\, w_C$$

ξ something like $(1-\varepsilon')^{|C|}$

or $(1-\varepsilon')^{|C|} - 1$