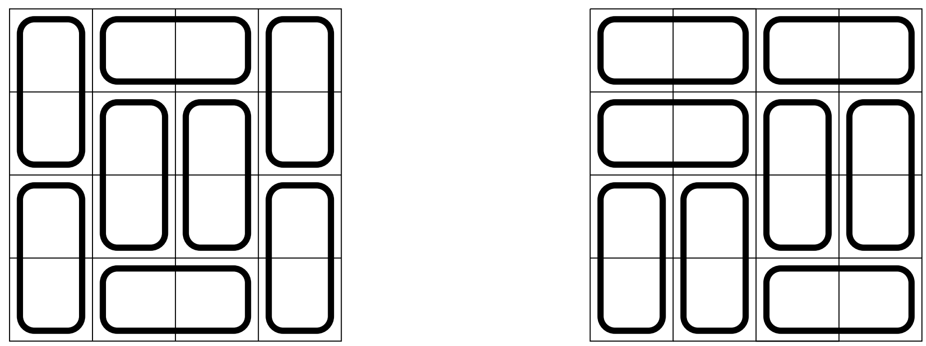
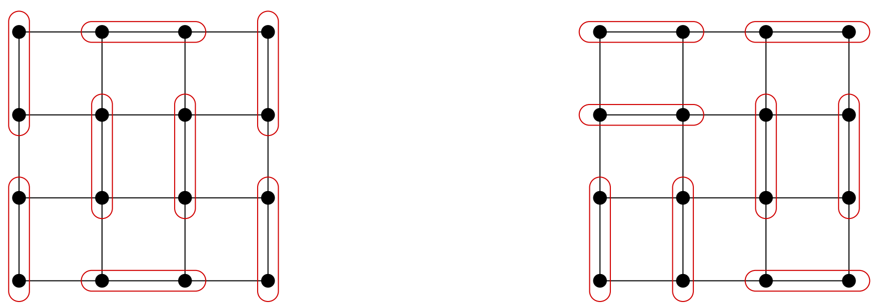
MONOMER CORRELATIONS ON THE SQUARE LATTICE

Mihai Ciucu

Indiana University
Department of Mathematics, Bloomington, IN 47405, USA

Domino tilings





Domino tilings are equivalent to **perfect matchings**.

$\mathrm{M}(G) :=$ # perfect matchings of graph G.

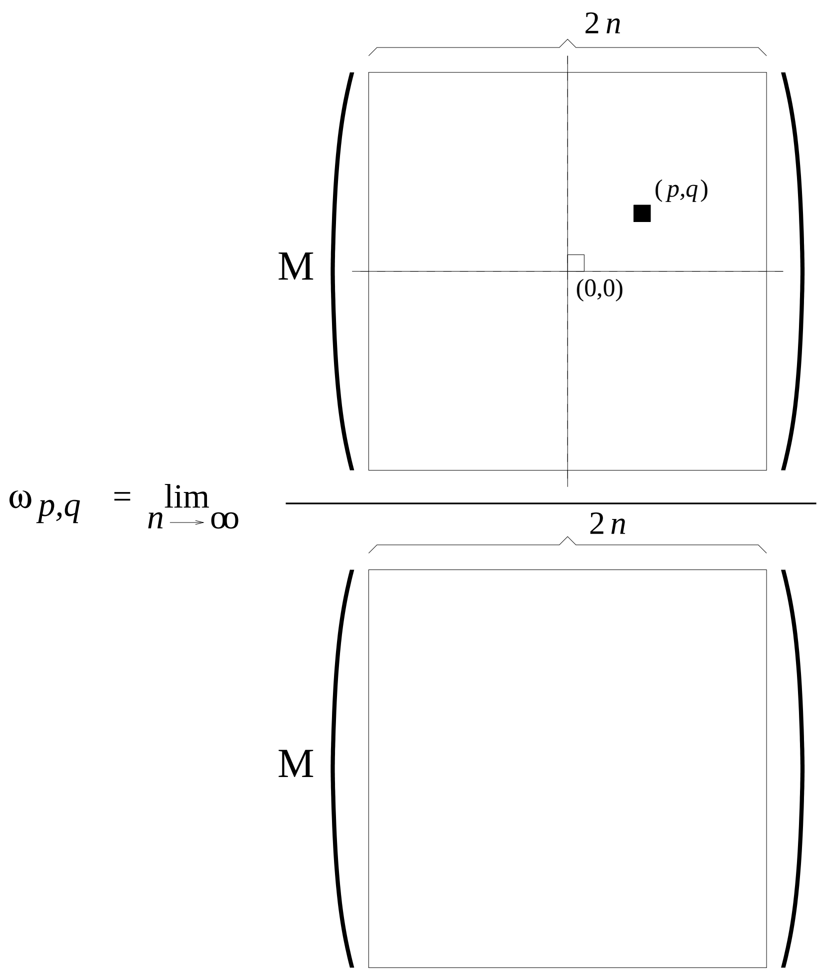
Let G_n:=$n \times n$ grid graph.

Dimer problem on square lattice (1936): $\mathrm{M}(G_{2n}) =?$

Theorem (Kasteleyn, and independently Temperley and Fisher, 1961).

$$\mathrm{M}(G_{2n}) = 2^{2n^2} \prod_{j=1}^{n} \prod_{k=1}^{n} \left(\cos^2 \frac{j\pi}{2n+1} + \cos^2 \frac{k\pi}{2n+1} \right).$$

Monomer-monomer correlation in a sea of dimers



Fisher and Stephenson (1963): conjectured (from exact data) that

$$\omega_{2p+1,0} \sim c \frac{1}{\sqrt{\mathrm{d}((2p+1,0),(0,0))}}$$

$$\omega_{p+1,p} \sim c \frac{1}{\sqrt{\mathrm{d}((p+1,p),(0,0))}}$$

as $p \to \infty$, with *same* c.

Based on this, they conjectured that $\omega_{p,q}$ is *rotationally invariant* for $p_k/q_k \to s$, $k \to \infty$, over all slopes s.

This still stands open.

It's analogous to the rotational invariance of the spin-spin correlation in Ising model.

Emergence of Coulomb's law

In previous work we:

- Generalized the problem to an arbitrary finite collection of holes of any sizes

- For large classes we proved that asymptotics of joint correlation function is given by Coulomb's law:

White unit holes charge +1, blacks −1, for larger holes define charge q additively. Then

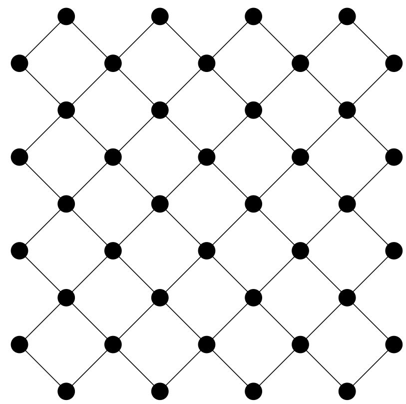
$$\omega(O_1, \ldots, O_n) \sim C \prod_{1 \le i < j \le n} \mathrm{d}(O_i, O_j)^{\frac{1}{2}\,\mathrm{q}(O_i)\,\mathrm{q}(O_j)}$$

for large separations between the holes.

- This casts the Fisher-Stephenson conjecture in the more general light of Coulomb interactions:

For two opposite color monomers, the exponent is $\frac{1}{2}(1)(-1)$, and above formula specializes to the FS conjecture.

The Aztec diamond graph



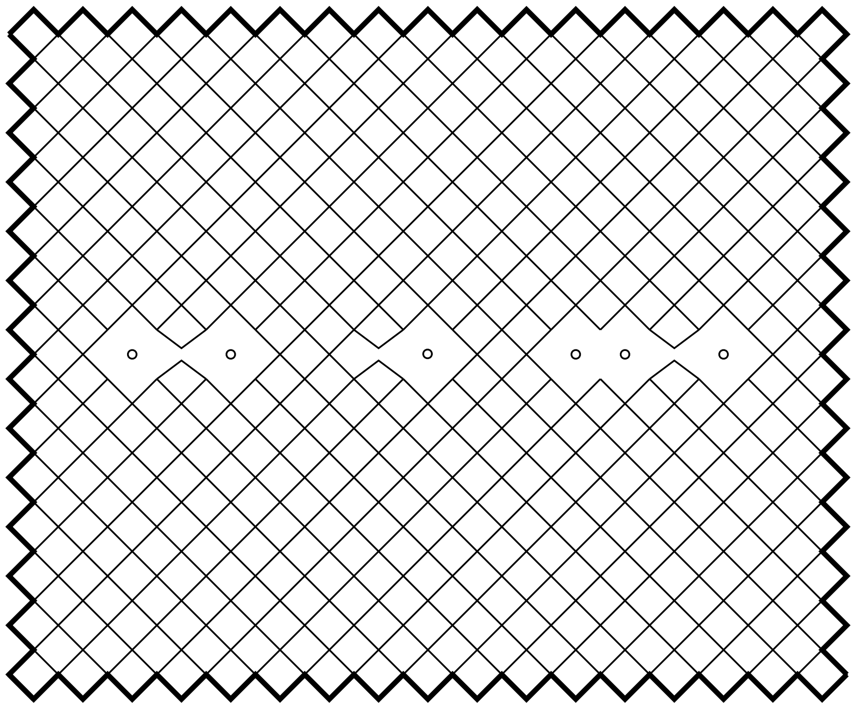
The Aztec diamond AD_4.

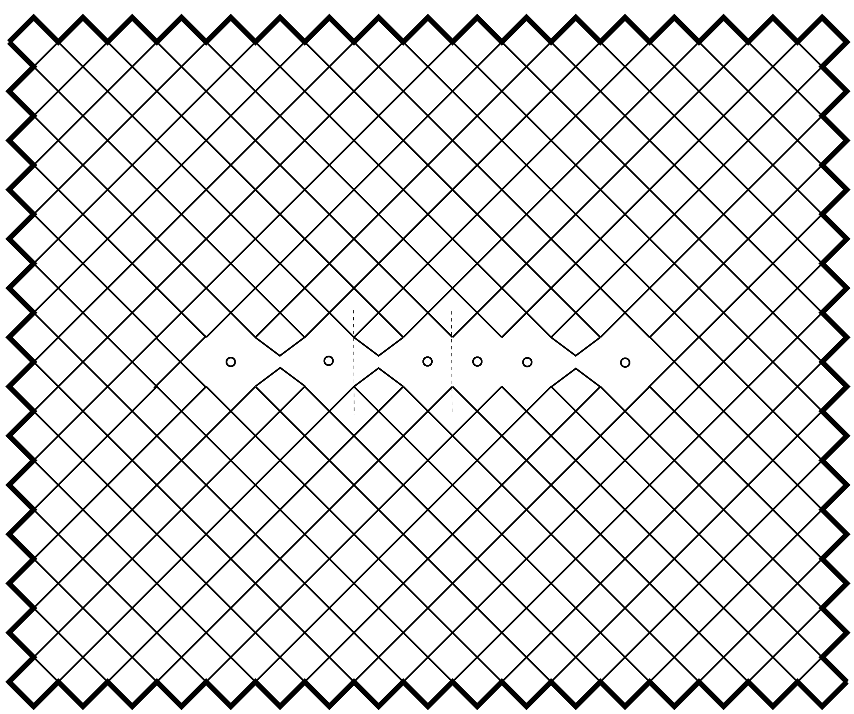
Theorem (Elkies, Kuperberg, Larsen, and Propp, 1992).

$$\mathrm{M}(AD_n) = 2^{n(n+1)/2}.$$

Alternative definition: via Aztec rectangles

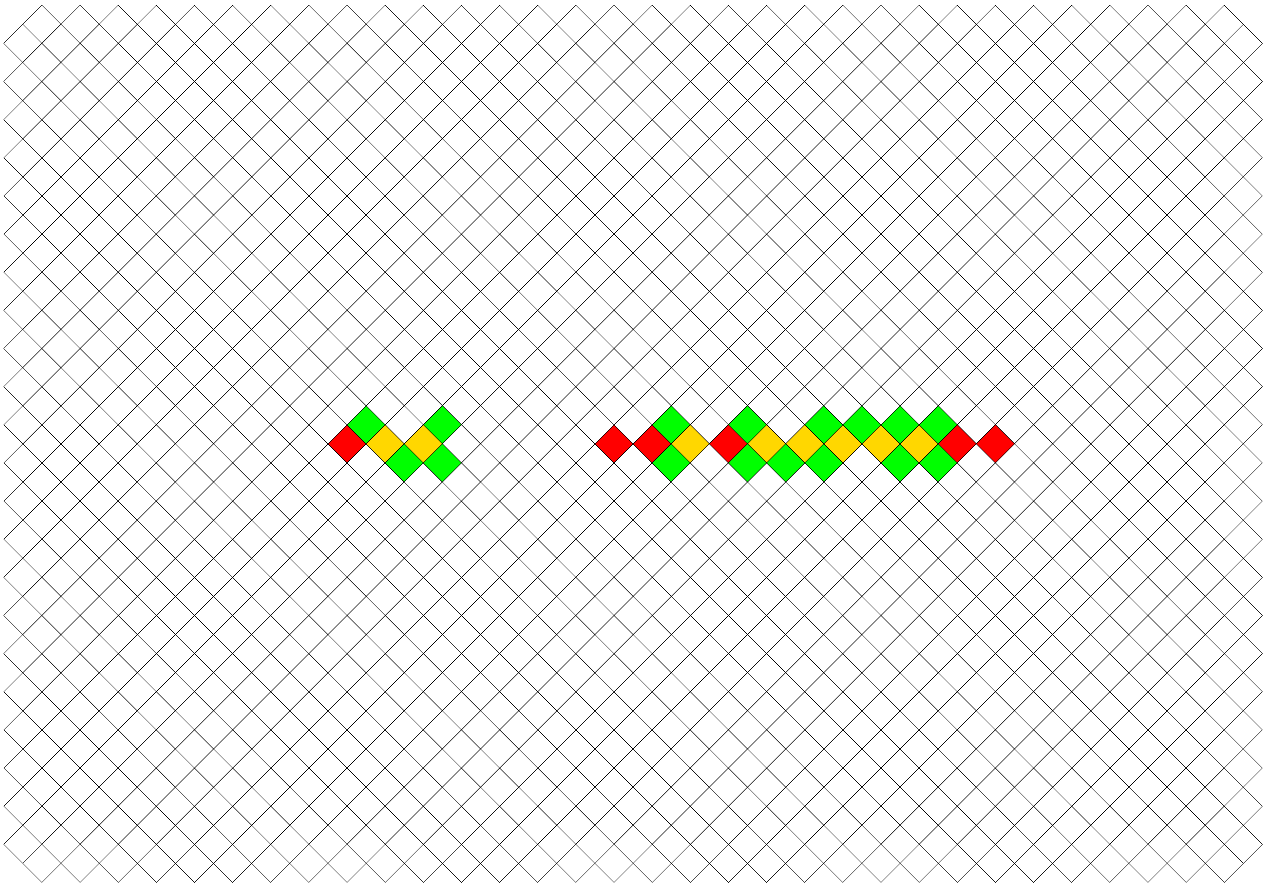
Defect clusters consisting of holes and separations





Define $\tilde{\omega}$ by limits of ratios of dimer coverings of these.

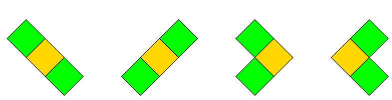
The dual picture: holes in domino tilings



A defect cluster is equivalent to **averaging over holes like this** (over 4^s such holes, if there are s separation defects in the defect cluster).

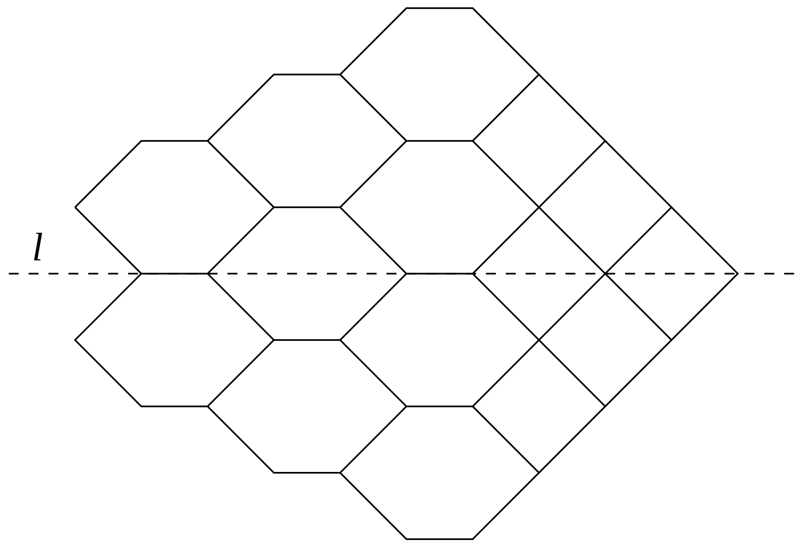
Key

red: hole
gold: point of separation
green: induced by dimers covering points of separation
The union of gold and green needs to split into disjoint "I"'s and "L"'s:

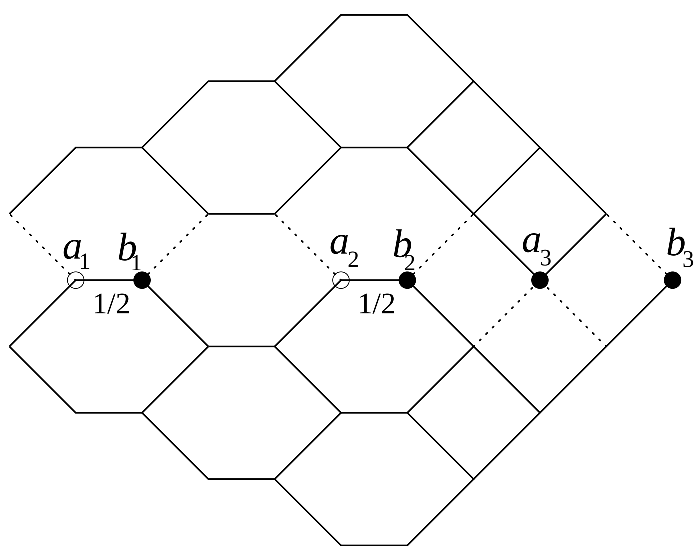


Factorization Theorem (C., 1997)

G planar, weighted, bipartite, symmetric graph



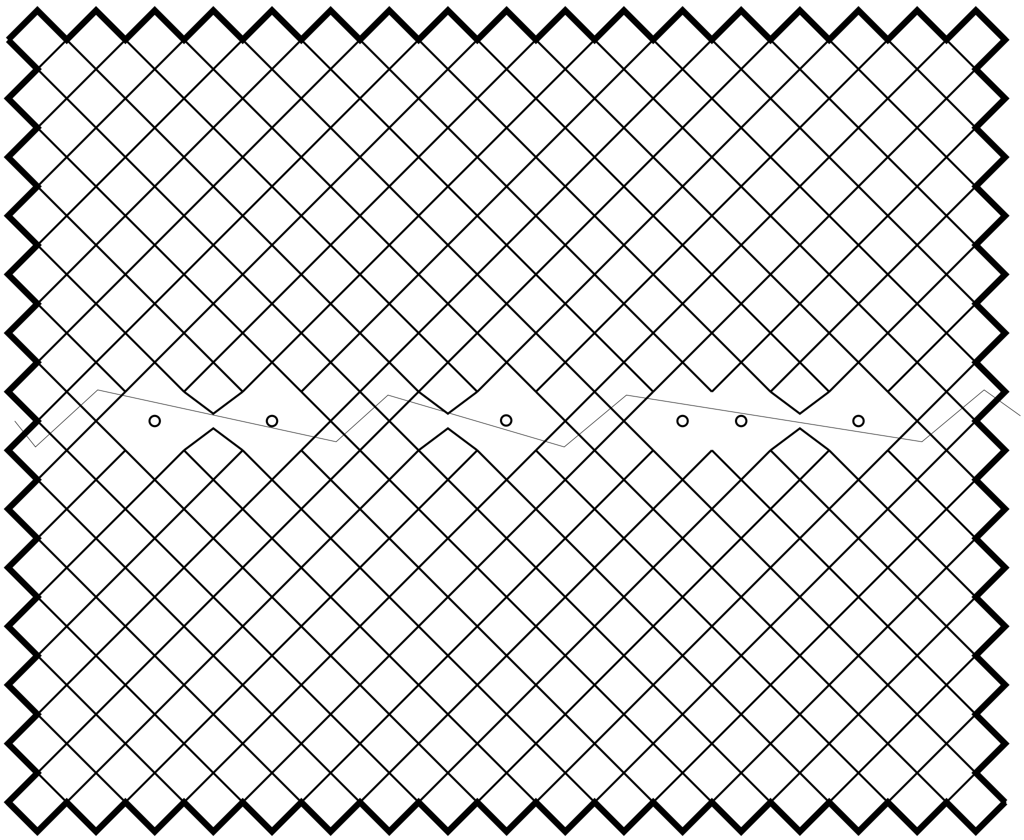
There is a specific way to cut G into G^+, G^-

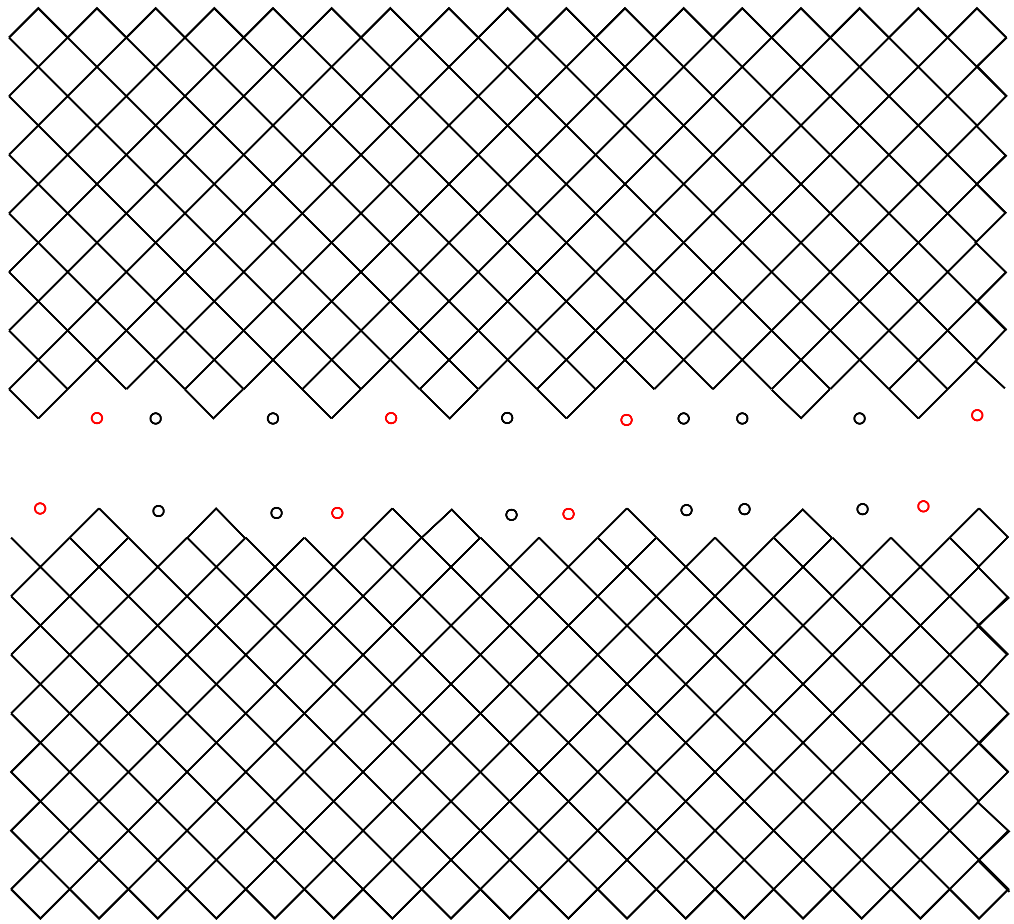


so that we have:

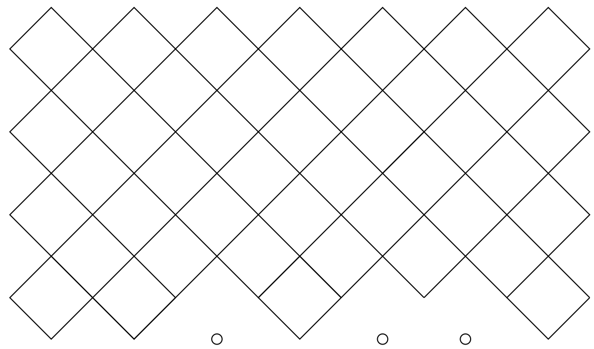
$$\mathrm{M}(G) = 2^{\mathrm{w}(G)}\, \mathrm{M}(G^+)\, \mathrm{M}(G^-).$$

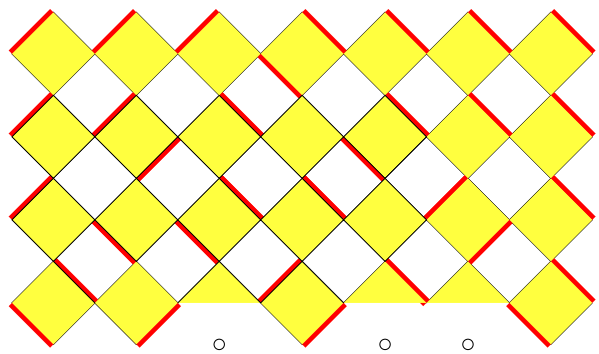
Apply factorization theorem:





Both connected components are “Aztec rectangles with missing vertices along a side” — their number of perfect matchings turns out to have a simple product formula.





$$\begin{array}{ccccccc}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 1 & 0 \\
1 & 0 & -1 & 1 & 0 & -1 & 1
\end{array}$$

We have:

$$\begin{aligned} M(AR(m,n;S)) &= \sum_{A \in ASM(m,n;S)} 2^{N_+(A)} \\ &= 2^m \sum_{A \in ASM(m,n;S)} 2^{N_-(A)}. \end{aligned}$$

Monotone triangle:

• all rows are strictly increasing

• the numbers are non-decreasing in the polar directions $+60°$ and $-60°$.

• $s(T)$ = the number of elements of T that are strictly between their neighbors in the row below

• weight every monotone triangle T by $2^{s(T)}$

• $f(t_1, \ldots, t_m)$ denotes total weight of monotone triangles with bottom row $\{t_1, \ldots, t_m\}$

We have

$$\sum_{A \in ASM(m,n;S)} 2^{N_-(A)} = f(t_1, \ldots, t_m).$$

0	0	0	1	0	0	0
0	1	0	-1	1	0	0
0	0	1	0	-1	1	0
1	0	-1	1	0	-1	1

$$\begin{array}{ccccccc} & & & 4 & & & \\ & & 2 & & 5 & & \\ & 2 & & 3 & & 6 & \\ 1 & & 2 & & 4 & & 7 \end{array}$$

0	0	0	1	0	0	0
0	1	0	0	1	0	0
0	1	1	0	0	1	0
1	1	0	1	0	0	1

By a result of Mills Robbins and Rumsey (1983) we have

$$f(t_1, \ldots, t_m) = \frac{2^{m(m-1)/2}}{0!\,1! \cdots (m-1)!} \prod_{1 \le i < j \le m} (t_j - t_i).$$

Consequences of the product formula

- Get exact correlations of arbitrary collinear holes along the diagonal of the square lattice
- In the limit of large separations Coulomb's law emerges again
- Can determine asymptotics of correlation for arbitrary defect clusters along diagonal direction.

$\bar{\omega}$: obtained by normalizing in a different way in the definition of $\tilde{\omega}$

As $R \to \infty$ we can verify

Strong Superposition Principle (collinear defect clusters)

$$\bar{\omega}(D_1(Rx_1), \ldots, D_n(Rx_n)) \sim \prod_{1 \le i \le n} \bar{\omega}(D_i) \prod_{1 \le i < j \le n} \mathrm{d}(D_i(Rx_i), D_j(Rx_j))^{\frac{1}{2}\,\mathrm{q}(D_i)\,\mathrm{q}(D_j)},$$

where d is the Euclidean distance.

Special case: Two unit holes on the diagonal.

When squares have opposite color we have by Hartwig (1966) that

$$\omega(\lozenge, \blacklozenge) \sim B \; \mathrm{d}(\lozenge, \blacklozenge)^{-1/2},$$

with $B = 0.98...$

By definition $\bar{\omega}(\lozenge) = \bar{\omega}(\blacklozenge) = \sqrt{B}$.
Also expect $\bar{\omega}(\lozenge, \blacklozenge) = \omega(\lozenge, \blacklozenge)$.
Then

$$\bar{\omega}(\lozenge, \blacklozenge) \sim \bar{\omega}(\lozenge)^2 \; \mathrm{d}(\lozenge, \blacklozenge)^{-1/2}.$$

Claim.

$$\bar{\omega}(\lozenge, \lozenge) \sim \bar{\omega}(\lozenge)^2 \; \mathrm{d}(\lozenge, \lozenge)^{1/2}.$$

Justification.

$$\frac{\bar{\omega}(\lozenge, \lozenge)}{\bar{\omega}(\lozenge\lozenge)} = \frac{\tilde{\omega}(\lozenge, \lozenge)}{\tilde{\omega}(\lozenge\lozenge)} \sim A \, \mathrm{d}(\lozenge, \lozenge)^{1/2}$$

We should have

$$\bar{\omega}(\lozenge\lozenge, \blacklozenge\blacklozenge) \sim \bar{\omega}(\lozenge\lozenge)^2 \, \mathrm{d}(\lozenge\lozenge, \lozenge\lozenge)^{-2}$$

and

$$\bar{\omega}(\lozenge\lozenge, \blacklozenge\blacklozenge) = \hat{\omega}(\lozenge\lozenge, \blacklozenge\blacklozenge).$$

Latter is $\sim \frac{1}{\pi^2} \, \mathrm{d}(\lozenge\lozenge, \blacklozenge\blacklozenge)^{-2}$ by Kenyon's determinant formula.
And indeed: $B = \frac{A}{\pi}$.

Moessner and Sondhi (2002), based on Monte Carlo simulation, predicted Coulomb repulsion between same color monomers in large squares on the square lattice, but with multiplicative constant increasing with the size of the system.

Above shows this is not the case — in fact we get *same* multiplicative constant as for monomers of opposite colors.

Conjecture. Same is true for arbitrary, not necessary collinear defect clusters.

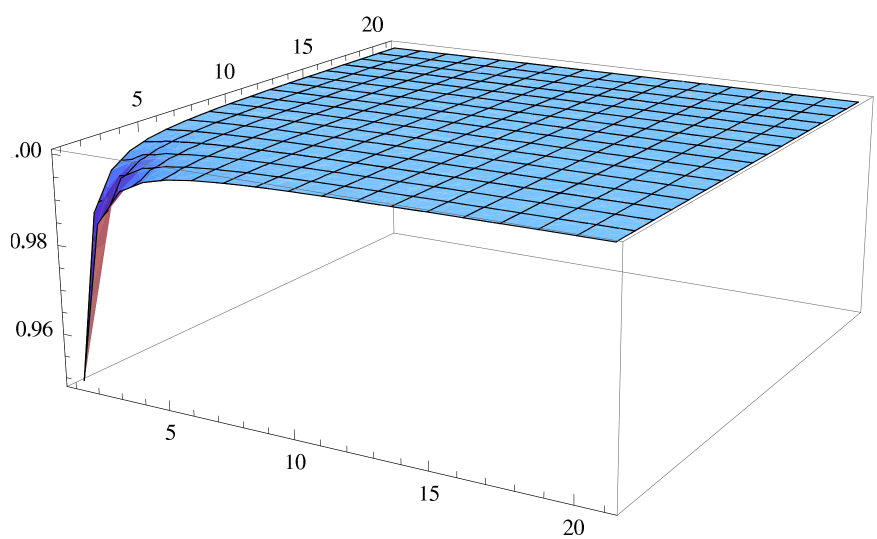
Remark. We proved the analog of this conjecture for a large class of hole defects on the triangular lattice.

Correlation inequalities

$$\bar{\omega}(m_1, m_2, m_3) \leq \frac{\bar{\omega}(m_1, m_2)\bar{\omega}(m_1, m_3)\bar{\omega}(m_2, m_3)}{\bar{\omega}(m_1)\bar{\omega}(m_2)\bar{\omega}(m_3)}$$

Form is reminiscent of GHS inequality for spin correlations:

$$\langle\sigma_i\sigma_j\sigma_k\rangle - \langle\sigma_i\sigma_j\rangle\langle\sigma_k\rangle - \langle\sigma_i\sigma_k\rangle\langle\sigma_j\rangle - \langle\sigma_j\sigma_k\rangle\langle\sigma_i\rangle + 2\langle\sigma_i\rangle\langle\sigma_j\rangle\langle\sigma_k\rangle \leq 0$$



Can be proved using our exact product formulas and error bounds for the Barnes G-function $G(n) = 0!1!\cdots(n-2)!$

Some questions

1. Is this inequality true for any three same-color monomers (not necessarily collinear)?

2. What about more than 3 monomers?

3. What if not all monomers have same color?

4. Larger holes?

Remarks about doing the calculations

- Exact “raw” formula

- Can be interpreted as “gas” of alternating atoms filling unaffected sites, with likes repelling Coulomb style, unlikes non-interacting

- This brings about Coulomb interaction between affected sites (likes repell, unlikes attract)

- Compare ratios of configurations differing by unit dislocation of single affected site, or switching two adjacent affected sites of opposite types.

- Instances of “exactness”: ratio of correlations of nearby configurations **exactly equals** ratio of SP contributions (in general they are only asymptotically equal)

The key fact that allows this solution:

That with the right averaging, correlations at finite distance have simple explicit expressions.