On enstrophy dissipation in 2D turbulence

Anna L. Mazzucato - Penn State University

Milton Lopes, Helena Nussenzveig Lopes - UNICAMP

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Motivation

• Reconcile Kraichnan - Batchelor theory of $2D$ *decaying* turbulence with properties of solutions to the $2D$ Euler equations.

• Regular Euler flows conserve energy and enstrophy *exactly*.

Energy dissipation $\sim 2\nu\Omega \to 0$ as $\nu \to 0$.

Enstrophy dissipation should not vanish as $\nu \to 0$ to balance growth of vorticity gradients, i.e., palinstrophy (Batchelor 1969).

• Dynamics dominated by small, coherent vortices (McWilliams 1984).

• *Assume* that turbulent flows modeled by *irregular* (weak) Euler solutions (Onsager 1949, Majda 1993) that are dissipative $\Rightarrow$ **Do such solutions exist?**

• Prior results by Eyink (2001). Independent results from ours by Tran & Dritschel, with bound on dissipation $\nu\langle|\nabla\omega|^2\rangle \sim \ln(Re)^{-1/2}$.

2D Euler Flows

Vorticity $\omega = \operatorname{curl} u$-velocity u formulation to 2D Euler in $\mathbb{R}^2$:

$$\partial_t \omega + u \cdot \nabla \omega = 0, \tag{1a}$$

$$u(x) = K * \omega(x) = \int_{\mathbb{R}^2} K(x-y)\,\omega(y)\,dy, \tag{1b}$$

where $K(x) \equiv \dfrac{x^\perp}{2\pi|x|^2}$ is the Biot-Savart kernel.

(1a) is a transport equation for $\omega \Rightarrow$ if u is regular,

$$\omega(\Phi(x,t),t) = \omega(x,0), \quad \frac{d\Phi}{dt}(x,t) = u(\Phi(x,t),t).$$

$\Rightarrow$ Regular flows conserve spatial mean of $f(\omega)$ ($f \in C^2$).

In particular conserve enstrophy $\Omega(t) = \dfrac{1}{2}\|\omega(t)\|_{L^2}^2$.

Function spaces

- $L^p(\mathbb{R}^2) =: L^p$, $1 \leq p < \infty$ Lebesgue space of pth integrable functions, $L^\infty(\mathbb{R}^2) =: L^\infty$ space of (essentially) bounded functions.

- $W^{s,p}(\mathbb{R}^2) =: W^{s,p}$, $s \in \mathbb{Z}_+$, $1 \leq p \leq \infty$, Sobolev space of functions with s derivatives in L^p. Set $W^{0,p} = L^p$, and $W^{s,2} =: H^s$ and $H^{-s} := (H^s)^*$.

- $B^s_{p,\infty}(\mathbb{R}^2) =: B^s_{p,\infty}$, $s \in \mathbb{R}$, Besov space of *distributions* with:

$$\|f\|_{B^s_{0,\infty}} := \|S_0 f\|_{L^2} + \sup_{j \in \mathbb{Z}_+} 2^{js} \|\Delta_j f\|_{L^2} < \infty,$$

where S_0 is a low-pass filter and Δ_j is a band filter at scale 2^j.

In particular, $B^\alpha_{\infty,\infty} \equiv C^\alpha$, the space of α-Hölder continuous functions.

Energy spectrum

• If $\omega \in L^2$, then $u \in H^1 \Rightarrow E(\kappa) = o(\kappa^{-3})$ satisfies upper bound of Kraichnan spectrum.

• $E(\kappa) \sim \kappa^{-1} \|\Delta_j u\|_{L^2}^2$ with $\kappa \sim 2^j$ (Costantin).

• If $\omega \in B^0_{2,\infty}$, then $u \in B^1_{2,\infty} \Rightarrow E(\kappa) \sim \kappa^{-3}$.

• $L^2 \subset B^0_{2,\infty}$, so choosing $B^0_{2,\infty}$ data allows for infinite enstrophy but preserves Kraichnan spectrum up to log factors (Eyink).

• Kraichnan spectrum studied in simulations, with incomplete agreement (Sulem et al. 1988, Benzi et al. 1989, Dmitruk-Montgomery 2005)

Weak solutions to 2D Euler

Consider *localized* initial vorticity with finite enstrophy, $\omega_0 \in L^p_c$, $p \geq 2$ ("c" for compact support) $\Rightarrow$

$\omega \in L^\infty([0,T); L^p)$ is an Euler weak solution with data ω_0 if

$$\int_0^T \int_{\mathbb{R}^2} \varphi_t \omega + \nabla\varphi \cdot u\,\omega \; dxdt + \int_{\mathbb{R}^2} \varphi(x,0)\omega_0(x) \; dx = 0, \quad \forall \phi \in C_c^\infty.$$

Uniqueness only for nearly bounded vorticity (Yudovich, Vishik).

Existence for measures ($\omega_0 \in (\mathcal{B}M_{c,+} + L^1_c) \cap H^{-1}_{\text{loc}}$), e.g. vortex sheets (Delort, Majda, Schochet, Vecchi-Wu), using a *different* weak formulation ($\omega \in L^p$, $p < 4/3$, $u\,\omega \notin L^1_{\text{loc}}$).

Enstrophy dissipation

If vorticity ω is bounded, then flow of u is well-defined (i.e. log-Lipschitz) $\Rightarrow$ no enstrophy dissipation is possible.

One approach to reconcile observed behavior of the enstrophy spectrum is to account for the dependence of the entrophy on Re in the spectrum (Dritschel-Tran-Scott 2007).

Another approach is to consider unbounded vorticity, corresponding to *irregular* velocities, and study how these velocities transport enstrophy (Eyink).

If u is a vector field with only Sobolev regularity, then its flow is not defined everywhere, but it is defined a.e. $\Rightarrow$

renormalized solutions of transport equations (DiPerna- P. L. Lions). Apply to linear advection of scalar quantities.

Renormalized solutions to transport equations

Key idea is to regularize the quantity transported, not the flow.

Definition. $u \in L^1([0,T], W^{1,1}_{\text{loc}})$, $\operatorname{div} u = 0$, $\omega \in L^\infty([0,T], L^0)$.
ω is a renormalized solution to $\partial_t \omega + u \cdot \nabla \omega = 0$ if

$$\partial_t \beta(\omega) + u \cdot \nabla \beta(\omega) = 0, \qquad a.e.,$$

for all β in admissible class $\mathcal{A} = \{\beta \in C^1 \cap L^\infty, \beta \equiv 0 \text{ near } 0\}$.

• Renormalized solutions are unique given u. Distribution function is exactly preserved $\Rightarrow$ enstrophy $\Omega(t) = \Omega(0)$, if $\Omega(0)$ finite (initial vorticity $\omega_0 \in L^p$, $p \geq 2$) , since then weak solution of 2D Euler is renormalized solution of transport equation.

• If $p > 2$, can enlarge $\mathcal{A}$ to include $\beta(s) = s^2 \Rightarrow$ enstrophy density $\vartheta(x,t) = |\omega(x,t)|^2/2$ is exactly transported by u (Eyink).

Enstrophy dissipation cont.

Enstrophy dissipation only possible for vorticities in L^2 or below.

• Finite-enstrophy case, $\omega^0 \in L^2_c$: dissipation may arise from irregular transport of enstrophy density ϑ.

Need to define non-linear term $u \cdot \nabla \vartheta$ in transport equation (u unbounded). No cancellation between $\vartheta = \omega^2$ and u (cancellation between u and ω by antisymmetry of Biot- Savart Op.).

• Infinite-enstrophy case, $\omega^0 \in B^0_{2,\infty}$: define non-trivial enstrophy defects as limit of source terms in local balance equation after regularization (Eyink).

Regularization by vanishing viscosity and smooth averaging. These methods do not commute in general with respect to enstrophy.

Two Regularizations

Viscosity solutions: $\omega = \lim_{\nu \to 0} \omega_\nu$

ω_ν solution of 2D Navier-Stokes equations with $\omega_\nu(0) = \omega_0$:

$$\partial_t \omega_\nu + u_\nu \cdot \nabla \omega_\nu - \nu \Delta \omega_\nu = 0,$$
$$u_\nu = K * \omega_\nu,$$

Smooth averaging: $\omega = \lim_{\epsilon \to 0} \omega_\epsilon$

where $$\begin{cases} \omega_\epsilon = j_\epsilon * \omega, \\ u_\epsilon = j_\epsilon * u. \end{cases}$$

$j_\epsilon(x) = \epsilon^{-n} j(\epsilon |x|)$ standard, radially symmetric smoothing kernel.

Regularized enstrophy density $\theta_\nu = \dfrac{1}{2}\omega_\nu^2$ and $\theta_\epsilon = \frac{1}{2}\omega_\epsilon^2$ satisfies balance equations with non-trivial flux terms.

Enstrophy defects

Transport enstrophy defect: let ω any weak solution to Euler $\Rightarrow \vartheta_\epsilon(x,t)$ satisfies:

$$\partial_t \vartheta_\epsilon + \text{div}\,[u_\epsilon \vartheta_\epsilon + \omega_\epsilon\,(j_\epsilon * (u\omega) - u_\epsilon \omega_\epsilon)] = -Z_\epsilon(\omega), \qquad (2)$$

Measure dissipation due to irregular transport :

$$Z^T(\omega) := \lim_{\epsilon \to 0} Z_\epsilon(\omega) = \lim_{\epsilon \to 0} [-\nabla \omega_\epsilon \cdot ((u\omega)_\epsilon - u_\epsilon \omega_\epsilon)]$$

Viscous enstrophy defect: let ω viscosity solution $\Rightarrow \vartheta_\nu(x,t)$ satisfies:

$$\partial_t \vartheta_\nu + u_\nu \cdot \nabla \vartheta_\nu - \nu \Delta \vartheta_\nu = -Z^\nu(\omega_\nu), \qquad (3)$$

Measure dissipation due to viscosity:

$$Z^V(\omega) \equiv \lim_{\nu \to 0} Z^\nu(\omega) = \lim_{\nu \to 0} \nu |\nabla \omega_\nu|^2 \geq 0,$$

• Compare with energy defect $D(u)$ in 3D $\Rightarrow$

Onsager's conjecture: no energy dissipation for Euler solutions in C^α, $\alpha > 1/3$ (Eyink 1994, under stronger Hölder condition).

• $D(u) \geq 0$, if $u \in L^3([0,T], L^3)$ (Duchon-Robert 2000)
$D(u) = 0$, if $u \in C([0,T), B^\alpha_{3,\infty})$, $\alpha > 1/3$ (Constantin-E-Titi 1994). Recent refinement of this result (Cheskidov-Constantin-Friedlander-Shvydkoy 2008).

• $\exists$ finite-energy solutions to 3D Euler with $D(u) > 0$ (Scheffer 1993, Shnirelman 2000, De Lellis-Szekelyhidi 2007).

• Rigorous estimates on energy spectrum known (Doering-Titi 1995, Doering-Gibbon 2002).

• In 2D, we have solutions to 2D Euler with infinite enstrophy such that $Z^V(\omega) > 0$, but have no examples for which $Z^T(\omega) > 0$.

Initial data with finite enstrophy

If $\omega_0 \in L^p_c(\mathbb{R}^2)$, $p \geq 2$, total enstrophy is conserved (flow is measure-preserving). If $p > 2$, enstrophy density $\vartheta = \frac{1}{2}|\omega|^2$ transported by u (DiPerna-Lions).

If $\omega_0 \in L^2_c$, ϑ may not satisfy the advection equation:

$$\vartheta_t + \mathrm{div}(u\,\vartheta) = 0.$$

Non-zero enstrophy defects may exist solely from irregular transport.

Simple energy estimates on $2D$ Navier-Stokes imply

$$Z^V(\omega) = \lim_{\nu \to 0_+} \nu \|\omega\|^2_{L^2} \equiv 0.$$

To control transport of θ need more regularity on ω.

Pick ω_0 in $L^2_c \log L^{1/4}$, logarithmic refinement of $L^2 \Rightarrow$ norm invariant under measure-preserving flows, and $u\,\theta \in L^1_{\text{loc}}$.

Theorem 1. *If $\omega \in L^\infty([0,T); L^2(\log L)^{1/4}$ is a viscosity solution of 2D Euler in L^2, then*

$$\partial_t(\theta) + \text{div}(u\,\theta) = 0,$$

where $u = K * \omega$,

Theorem 2. *If $\omega \in L^\infty([0,T); L^2(\log L)^{1/4} \cap L^1$ is a weak solution of 2D Euler, then $Z^T(\omega)$ exists (as a distribution).*

If ω is, in addition, a viscosity solution, then $Z^T(\omega) \equiv 0$.

- Theorem 1 is nearly optimal $\Rightarrow$ there exist a *function* $\omega \in L^2 \log L^{1/6} \supset L^2(\log L)^{1/4}$, such that $u := K * \omega$ is *unbounded*, and $u\,\vartheta = \frac{1}{2}(K * \omega)\,\omega^2$ cannot be defined as distribution.

  An example is given by

$$\omega(x) = \begin{cases} \frac{1}{|x|\,|\log|x||^{\alpha}}, & \text{if } x \in D(0, 1/3) \cap \{x_2 > 0\}; \\ 0, & \text{otherwise,} \end{cases}$$

  with $1/2 < \alpha < 1$.

- Euler solutions in L^2 such that $Z^T(\omega) > 0$ would give agreement with Kraichnan-Batchelor and indicate that non-linear interactions are responsible for enstrophy dissipation at very high Re, if enstrophy is finite.

Eyink's conjecture (infinite-enstrophy data)

Consider initial vorticity ω_0 in the Besov space $B^0_{2,\infty} \supset L^p_c$, $p \geq 2$

$\Rightarrow$ velocity $u_0 = K * \omega_0$ has energy spectrum $E(\kappa) \sim \kappa^{-3}$.

If upper bound on spectrum as $\nu \to 0$ given by Kraichnan spectrum:

$$\sup_{\nu>0} \|\omega^\nu\|_{L^2([0,T],B^0_{2,\infty})} < C,$$

then viscosity solutions $\omega = \lim_{\nu\to 0} \omega^\nu$ exist.

Conjecture : Let ω be a viscosity solution with data $\omega_0 \in B^0_{2,\infty}(\mathbb{R}^2)$. Then:

$$Z^T(\omega), Z^V(\omega) \geq 0, \qquad \text{in } \mathcal{D}'.$$

Moreover there exist initial data for which $Z(\omega) > 0$.

Enstrophy dissipative solutions

Construct dissipative solution in $B^0_{2,\infty}$ as a steady vortex.

Let $\omega_0(x) = \tilde{\omega}(|x|) = \phi(|x|)\dfrac{1}{|x|}$, ϕ cut-off function for $|x| \geq 1$. Set

$$u_0(x) = r^{-2}\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \int_0^r s\tilde{\omega}(s)\, ds.$$

$\omega_0(x)$ is an exact steady solution to 2D Euler $\Rightarrow u_0(x) \perp \nabla\omega_0(x)$.

If ω_ν is solution to 2D heat equation with initial data ω_0, similar formula give solution to 2D Navier-Stokes.

Compute: $\quad Z^T(\omega_0) = 0, \qquad Z^V(\omega_0) = \dfrac{4\pi^3}{t}\delta_o, \qquad t > 0.$

Strictly dissipative solutions exist, but enstrophy defects *depends* on the regularization.

Conclusions

- Proved total enstrophy exactly conserved in the limit $\nu \to 0$ under sole assumption that enstrophy initially finite (Tran & Dritschel assume vorticity bounded).

- Given indication that dissipation may still occur when enstrophy finite, by irregular transport of thdensity. When enstrophy infinite, viscosity alone can sustain non-zero dissipation.

- Partially established Eyink's conjecture by constructing explicit dissipative Euler solutions.

  Difficult to show $B^0_{2,\infty}$-solutions genericaly dissipative, since Besov norm not invariant under measure-preserving flows.

- Natural extension to forced 2D Navier-Stokes using long-time averages $\Rightarrow$ enstrophy balance holds as $\nu \to 0$ for damped-driven Navier-Stokes (Constantin-Ramos 2007).