Scaling law of fractal-generated turbulence and its derivation from a new scaling group of the multi-point correlation equation

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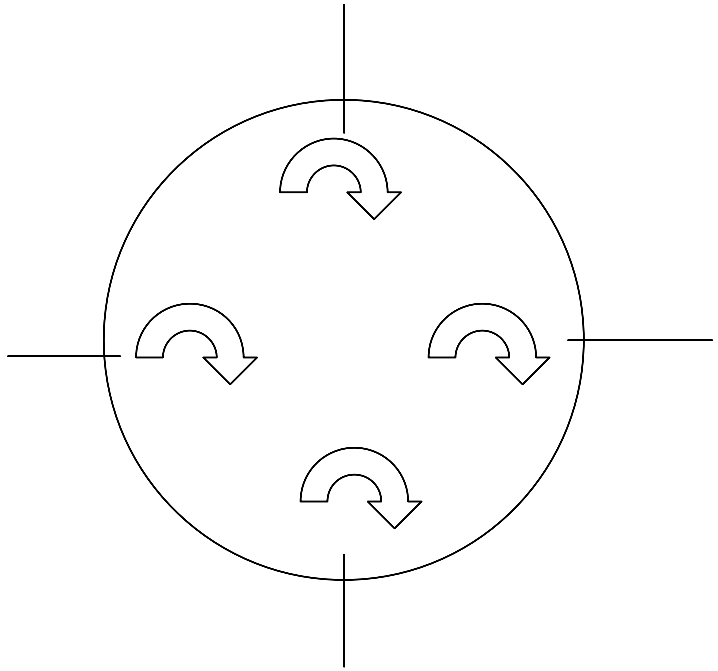
Content

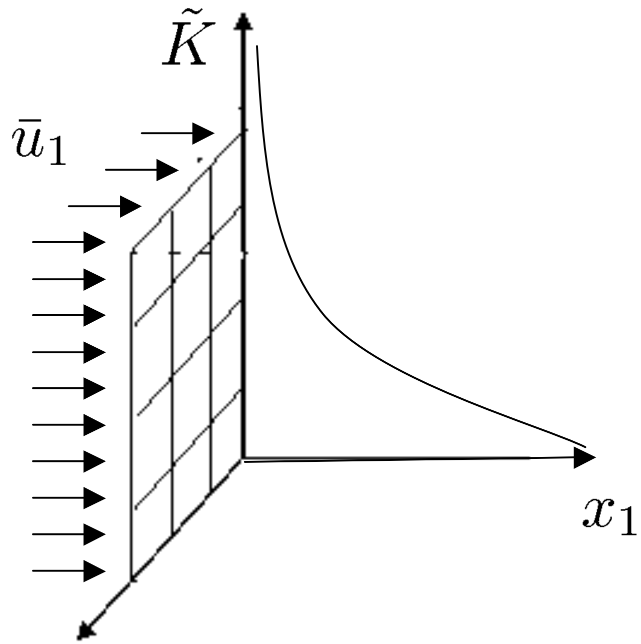
- Definition of test cases
- Hierarchy of multi-point equations
- Symmetries of the Euler and Navier-Stokes equation
- Multi-point correlation equations of homogeneous turbulence and new symmetries
- Turbulent scaling laws
- Summary and Conclusions

Two test cases

- Temporally decaying turbulence



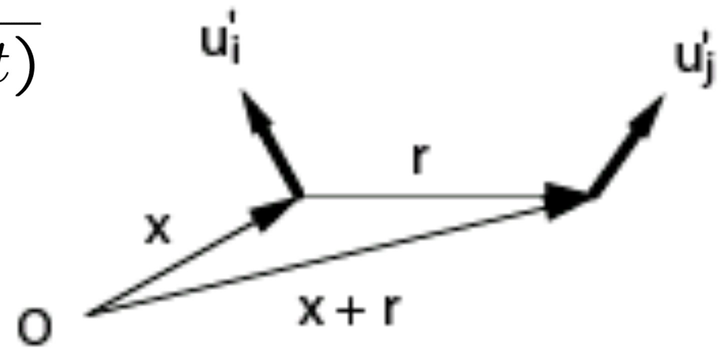
Spatially decaying turbulence



What are their related symmetries and scaling laws?
What are their differences due to turbulence generation?

- Definition of the two-point correlation tensor

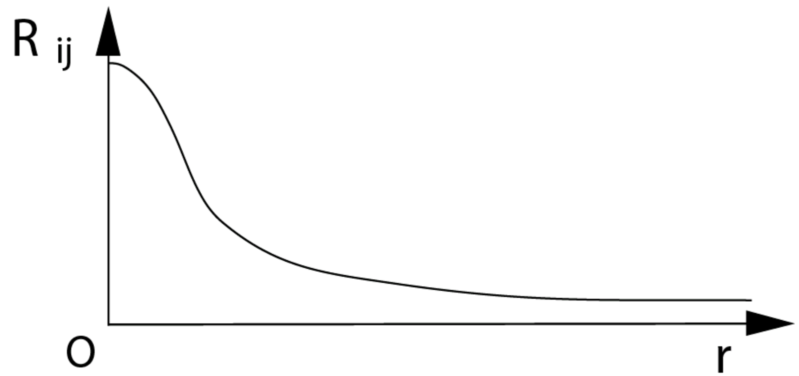
$$R_{ij}(\boldsymbol{x},\boldsymbol{r},t) = \overline{u_i'(\boldsymbol{x},t)\,u_j'(\boldsymbol{x}+\boldsymbol{r},t)}$$



- One-Point limit

$$\overline{u_i'u_j'} = \lim_{r\to 0} R_{ij}(\boldsymbol{x},\boldsymbol{r},t)$$

- Sketch of the correlation function



- Note: R_{ij} contains one additional dimension due to $\boldsymbol{r}$

Multi-point correlation equation

$$
\begin{aligned}
\mathcal{S}_{i_{\{n+1\}}} = \frac{\partial R_{i_{\{n+1\}}}}{\partial t} + \sum_{l=0}^{n} \Bigg[& \bar{u}_{k_{(l)}}(\boldsymbol{x}_{(l)}) \frac{\partial R_{i_{\{n+1\}}}}{\partial x_{k_{(l)}}} + R_{i_{\{n+1\}}[i_{(l)} \mapsto k_{(l)}]} \frac{\partial \bar{u}_{i_{(l)}}(\boldsymbol{x}_{(l)})}{\partial x_{k_{(l)}}} \\
& + \frac{\partial P_{i_{\{n\}}[l]}}{\partial x_{i_{(l)}}} - \nu \frac{\partial^2 R_{i_{\{n+1\}}}}{\partial x_{k_{(l)}} \partial x_{k_{(l)}}} - R_{i_{\{n\}}[i_{(l)} \mapsto \emptyset]} \frac{\partial \overline{u_{i_{(l)}} u_{k_{(l)}}}(\boldsymbol{x}_{(l)})}{\partial x_{k_{(l)}}} \\
& + \frac{\partial R_{i_{\{n+2\}}[i_{(n+1)} \mapsto k_{(l)}][\boldsymbol{x}_{(n+1)} \mapsto \boldsymbol{x}_{(l)}]}}{\partial x_{k_{(l)}}} + 2\Omega_k e_{i_{(l)}km} R_{i_{\{n+1\}}[i_{(l)} \mapsto m]} \Bigg] = 0
\end{aligned}
$$

- **Important notes:**
  - **Infinite series of correlations equations beginning with two-point correlation equation, etc.**
  - **Each correlation equation has three more dimensions**
  - **Homogenous turbulence limit: linear PDE system!!**

Goal: Similarity solution to the infinite series of correlation equations for the two cases!

Symmetries of the Euler and Na-St equations

- Scaling of space:

$$T_{s_1}: \quad t^* = t \quad , \quad \boldsymbol{x}^* = \mathrm{e}^{a_1}\boldsymbol{x} \quad , \quad \boldsymbol{u}^* = \mathrm{e}^{a_1}\boldsymbol{u} \quad , \quad p^* = \mathrm{e}^{2a_1}p$$

$$\mathsf{X}_{s_1} = x_i\frac{\partial}{\partial x_i} + u_j\frac{\partial}{\partial u_j} + 2p\frac{\partial}{\partial p},$$

- Scaling of time:

$$T_{s_2}: \quad t^* = \mathrm{e}^{a_2}t \quad , \quad \boldsymbol{x}^* = \boldsymbol{x} \quad , \quad \boldsymbol{u}^* = \mathrm{e}^{-a_2}\boldsymbol{u} \quad , \quad p^* = \mathrm{e}^{-2a_2}p$$

$$\mathsf{X}_{s_2} = t\frac{\partial}{\partial t} - u_i\frac{\partial}{\partial u_i} - 2p\frac{\partial}{\partial p},$$

- Other known symmetries are: rotation, Galilean, translation in space and time, ...

Scaling of Two- and Multi-point equations

- Scaling of space from NS und Euler eqn:

$$t^* = t,\ r_i^* = \mathrm{e}^{a_1} r_i,\ R_{ij}^* = \mathrm{e}^{2a_1} R_{ij},\ \overline{u_i p}^* = \mathrm{e}^{3a_1} \overline{u_i p} \cdots$$

- Scaling of time from NS und Euler eqn :

$$t^* = \mathrm{e}^{a_2} t,\ r_i^* = r_i,\ R_{ij}^* = \mathrm{e}^{-2a_2} R_{ij},\ \overline{u_i p}^* = \mathrm{e}^{-3a_2} \overline{u_i p} \cdots$$

Additional scaling group in the two- and multi-point equation for homogeneous turbulence due to linearity of the infinite system

$$t^* = t,\ r_i^* = r_i,\ R_{ij}^* = \mathrm{e}^{a_s} R_{ij},\ \overline{u_i p}^* = \mathrm{e}^{a_s} \overline{u_i p} \cdots$$

Invariant solution and symmetry breaking

- Invariant solution condition leads to self-similarity (*here infinitesimal form*)

$$\boxed{\frac{dt}{a_2 t + a_4} = \frac{dr_i}{a_1 r_i} = \frac{dR_{ij}}{[2(a_1 - a_2) + a_s]R_{ij}} = \cdots}$$

- Symmetry breaking
  - Breaking scaling of space:

$$a_1 = 0: \quad R^\star_{ij} = e^{2(\overbrace{a_1}^{=0} - a_2) + a_s} R_{ij}; \quad t^\star = e^{a_2} t; \quad \mathbf{r}^\star = e^{\overbrace{a_1}^{=0}} \mathbf{r}; \ldots$$

  - Breaking scaling of time:

$$a_2 = 0: \quad R^\star_{ij} = e^{2(a_1 - \overbrace{a_2}^{=0}) + a_s} R_{ij}; \quad t^\star = e^{\overbrace{a_2}^{=0}} t; \quad \mathbf{r}^\star = e^{a_1} \mathbf{r}; \ldots$$

- Classical turbulence quantities characterize the flow

  - Turbulent kinetic energy

$$\overline{K}(t) = \frac{1}{2} \lim_{\mathbf{r}=0} R_{kk}(\mathbf{r}, t)$$

  - Integral length-scale

$$\ell_t(t) = \frac{1}{\overline{K}} \int R_{kk}(\mathbf{r}, t) d\mathbf{r}$$

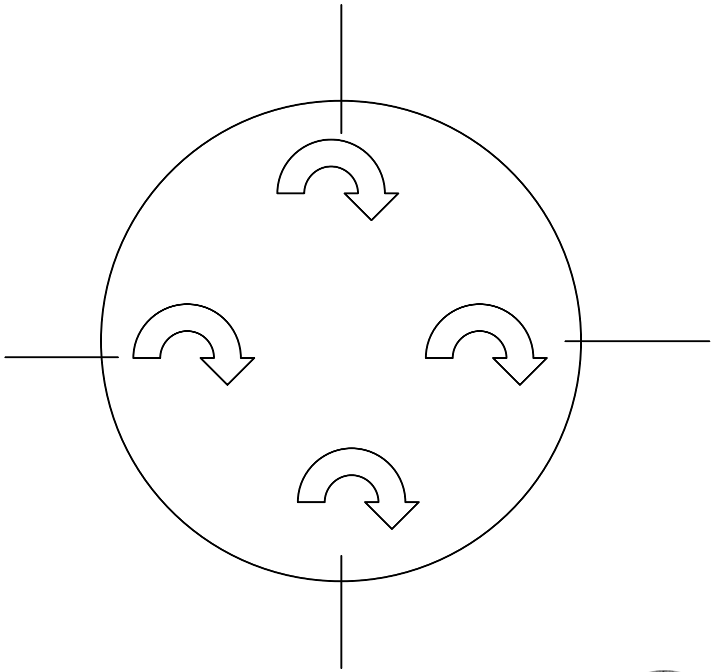
Case1: Temporally decaying turbulence

- Two-point correlation equations:

$$\frac{\partial R_{ij}}{\partial t} - \frac{\partial \overline{pu_j}}{\partial r_i} + \frac{\partial \overline{up_i}}{\partial r_j} - \frac{\partial}{\partial r_k}\left[R_{(ik)j} - R_{i(jk)}\right] = 0,$$

- Assumptions:
  - No mean flow
  - No spatial gradient
  - In case of isotropy:
    von Kármán-Howarth equation

$$\frac{\partial \mathcal{F}}{\partial t} = \frac{1}{r^4}\frac{\partial}{\partial r}\left[r^4\left(\mathcal{K} + 2\nu\frac{\partial \mathcal{F}}{\partial r}\right)\right]$$



Case1: Temporally decaying turbulence

fdy

„Regular" decaying turbulence

- Characteristic variable: **only from classical groups**
  - Birkhoff integral, Loitsyansky integral, … $a_1 \neq a_2 \neq 0$
- Scaling symmetries

$$R_{ij}^{\star} = e^{2(a_1 - a_2)} R_{ij}; \quad t^{\star} = e^{a_2} t; \quad \mathbf{r}^{\star} = e^{a_1} \mathbf{r}; \ldots$$

- Invariant solution

$$R_{ij} = t^{-n} \tilde{R}_{ij}(r/t^m) \quad \Longrightarrow \quad \boxed{\begin{array}{c} \overline{\mathbf{K}} \sim t^{-n} \\ \ell_t \sim t^m \end{array}} \qquad m = f(n)$$

TUD

Case1: Temporally decaying turbulence

fdy

Constant length scale ℓ_t decaying turbulence

- Characteristic variable: fixed box length

$$\ell_t = const. \Longrightarrow a_1 = 0$$

- Scaling symmetries:

$$R^\star_{ij} = e^{2(\overbrace{a_1}^{=0} - a_2)} R_{ij}; \quad t^\star = e^{a_2} t; \quad \mathbf{r}^\star = e^{\overbrace{a_1}^{=0}} \mathbf{r}; \ldots$$

- Invariant solution:

$$R_{ij} = t^{-2} \tilde{R}_{ij}(\mathbf{r}) \quad \Longrightarrow \quad \boxed{\begin{array}{c} \overline{\mathbf{K}} \sim t^{-2} \\ \ell_t \sim const. \end{array}}$$

TUD

Case1: Temporally decaying turbulence

Constant time scale τ decaying turbulence

- Characteristic variable

$$\tau = const. \quad \rightarrow \quad a_2 = 0$$

- Scaling symmetries

$$R^{\star}_{ij} = e^{2(a_1 - \overbrace{a_2}^{=0})} R_{ij}; \quad t^{\star} = e^{\overbrace{a_2}^{=0}} t; \quad \mathbf{r}^{\star} = e^{a_1}\mathbf{r}; \ldots$$

- Invariant solution:

$$R_{ij} = e^{-2t/t_0} \tilde{R}_{ij}(r/e^{t/t_0}) \quad \Longrightarrow \quad \boxed{\begin{array}{c} \overline{\mathbf{K}} \sim e^{-2t/t_0} \\ \ell_t \sim e^{t/t_0} \end{array}}$$

TUD

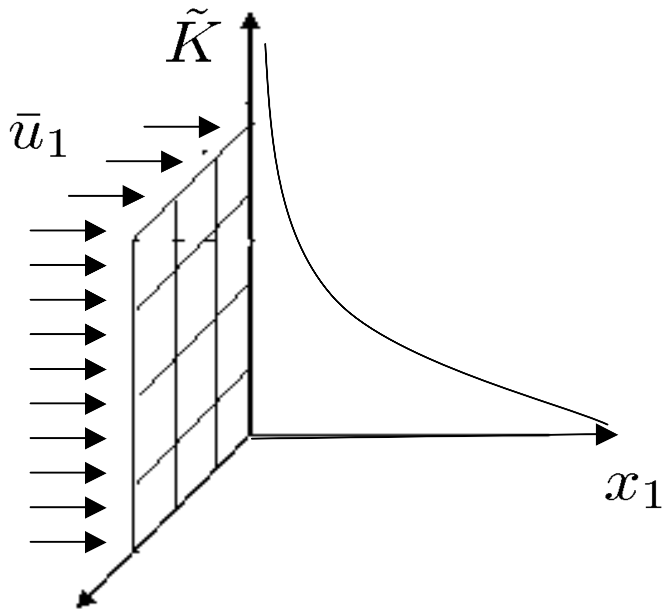
Case 2: Spatially decaying turbulence

- Two-point correlation equations

$$\overline{u}_1 \frac{\partial R_{ij}}{\partial x_1} - \frac{\partial \overline{pu_j}}{\partial r_i} + \frac{\partial \overline{up_i}}{\partial r_j} - \frac{\partial}{\partial r_k} \left[R_{(ik)j} - R_{i(jk)} \right] = 0$$

- Assumptions:
  - Mean flow $\overline{u}_1 = const.$
  - No diffusion terms: $\frac{\partial}{\partial x_1} = 0$,

    except for convection.



- Definition: $\Gamma = \frac{\ell_t \varepsilon}{\overline{\mathbf{K}}^{3/2}}$

Case 2: Spatially decaying turbulence

New Key point for this flow.

- x_1 behaves like „time“

$$\tilde{t} = \frac{x_1}{\overline{u}_1}$$

- Scaling symmetries:

$$R_{ij}^{\star} = e^{2(a_1 - a_2) + a_s} R_{ij}, \qquad \underbrace{x_1^{\star} = e^{a_2} x_1,}_{\text{„Scaling of time“}} \qquad \mathbf{r}^{\star} = e^{a_1} \mathbf{r}, \qquad \ldots$$

- Symmetry breaking of time $a_2 = 0$ acts on space variable.

Case 2: Spatially decaying turbulence

„Regular“ wind tunnel turbulence

- Characteristic variable
  - „spatial Birkhoff“, „spatial Loitsyansky‘s integral“: $a_1 \neq a_2 \neq 0$
- Scaling symmetries:

$$R_{ij}^{\star} = e^{2(a_1 - a_2) + a_s} R_{ij}, \quad x_1^{\star} = e^{a_2} x_1, \quad \mathbf{r}^{\star} = e^{a_1} \mathbf{r}, \quad \ldots$$

- Invariant solution:

$$R_{ij} = {x_1}^{-n} \tilde{R}_{ij}(x_1 / r^m) \quad \Longrightarrow \quad \boxed{\begin{array}{c} \overline{\mathbf{K}} \sim {x_1}^{-n} \\ \ell_t \sim {x_1}^{m} \\ \Gamma \sim const. \end{array}}$$

Constant correlation length scale

- Characteristic variable
  - Constant integral length scale: $\ell_t = const, \Longrightarrow \quad a_1 = 0$
- Scaling symmetries:

$$R^{\star}_{ij} = e^{2(\overbrace{a_1}^{=0} - a_2) + a_s} R_{ij}; \quad {x_1}^{\star} = e^{a_2} x_1; \quad \mathbf{r}^{\star} = e^{\overbrace{a_1}^{=0}} \mathbf{r}; \ldots$$

- Invariant solution:

$$R_{ij} = {x_1}^{-2+\gamma} \tilde{R}_{ij}(\mathbf{r}) \quad \Longrightarrow \quad \boxed{\begin{array}{c} \overline{\mathbf{K}} \sim {x_1}^{-2+\gamma} \\ \ell_t \sim const \\ \Gamma \sim x_1^{-1/2\gamma} \end{array}}$$

Constant „time scale“

- Characteristic variable

  Characteristic length scale such as grid spacing

  - Constant „time scale“: $\tau = \dfrac{a}{\overline{u}_1}$: $\quad a_2 = 0$

- Scaling symmetries:

$$R^\star_{ij} = e^{2(a_1 - \overbrace{a_2}^{=0}) + a_s} R_{ij}; \quad x^\star_1 = e^{\overbrace{a_2}^{=0}} x_1; \quad \mathbf{r}^\star = e^{a_1}\mathbf{r}; \ldots$$

- Invariant solution:

$$R_{ij} = e^{-2x_1/a} \tilde{R}_{ij}(r/e^{x_1/a}) \quad \Longrightarrow \quad \boxed{\begin{aligned} \overline{\mathbf{K}} &\sim e^{-(2+\gamma)x_1/a} \\ \ell_t &\sim e^{x_1/a} \\ \Gamma &\sim e^{(2+\gamma/2)x_1/a} \end{aligned}}$$

Case 2: Spatially decaying turbulence

fdy

Constant space and „time scale“

- Characteristic variable
  - All space and time scales are broken: $a_1 = a_2 = 0$
  - **multiple length scales imposed - fractal generated turb.**
- Scaling symmetries:

Remaining scaling group

$$R^\star_{ij} = e^{2\overbrace{(a_1 - a_2)}^{=0} + a_s} R_{ij}; \quad x^\star_1 = e^{\overbrace{a_2}^{=0}} x_1; \quad \mathbf{r}^\star = e^{\overbrace{a_1}^{=0}} \mathbf{r}; \ldots$$

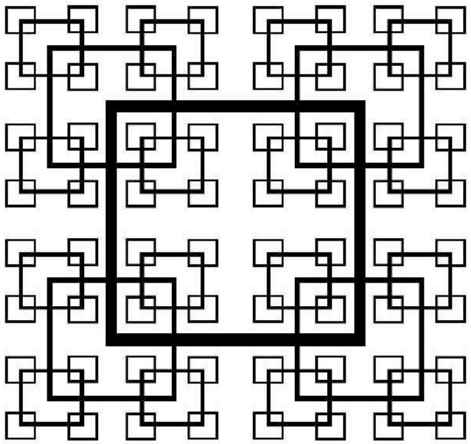
- Invariant solution:

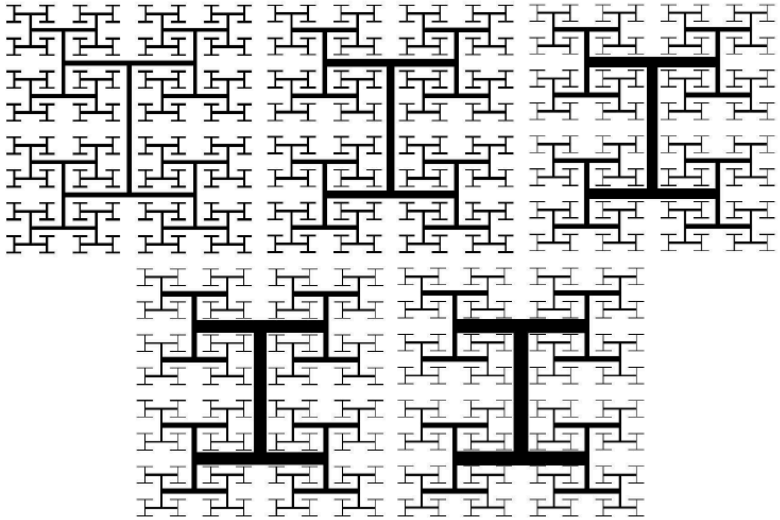
$$R_{ij} = e^{-x_1/a} \tilde{R}_{ij}(r) \quad \Longrightarrow \quad \boxed{\begin{array}{l} \overline{\mathbf{K}} \sim e^{-x_1/a} \\ \ell_t \sim const. \\ \Gamma \sim e^{\frac{1}{2} x_1/a} \end{array}}$$

TUD

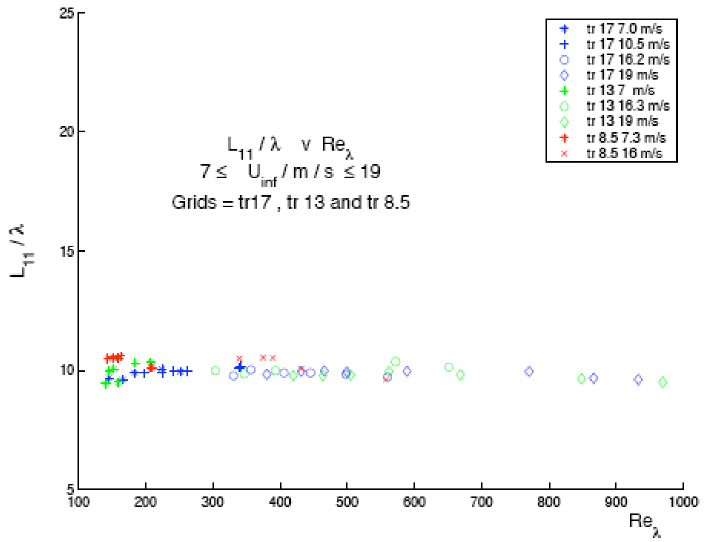
Fractal generated turbulence (Vassilicos et al.)

- Different grids and grid series



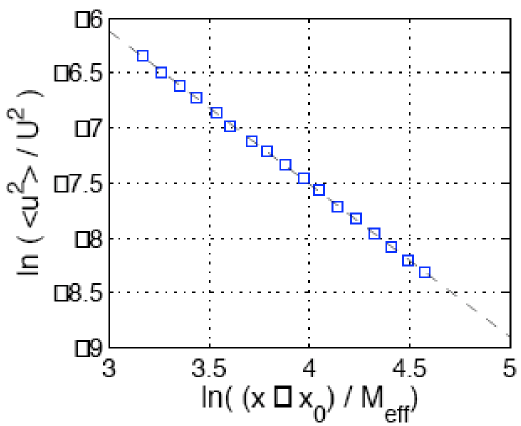


- Constant length scale

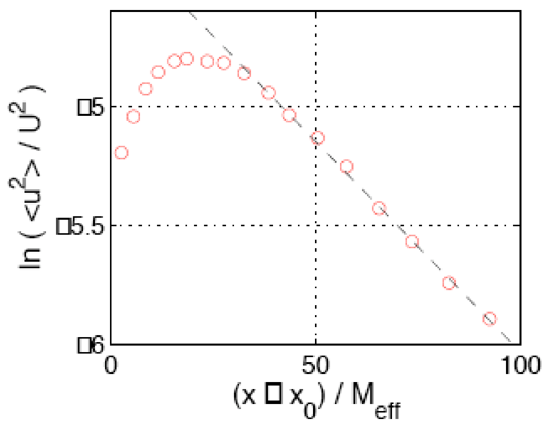


- Turbulence decay

Algebraic decay



Exponential decay



Summary and open questions

- A new scaling group for homogeneous turbulence has been discovered
- Lie group methods allowed to establish a variety of new scaling laws from the new and classical symmetries of multi-point eqation
- Experimental observation of some scaling laws are still to be shown
- Open questions for wind tunnel turbulence
  - How can all scaling laws be established experimentally?
  - What are their bounds?
- Open question for temporally decaying turbulence
  - How can a exponential decay be established experimentally?

Thanks for your attention!