Fractal Superconductivity near Localization Threshold

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Plan of the talk

1. Motivation from experiments
2. Hard-gap insulator due to electron pairing on localized states
3. BCS-like theory for critical eigenstates
4. Superconductivity with pseudogap and S-I transition
Example: Disorder-driven S-I transition in TiN thin films


Specific Features of Direct SIT:

- **Insulating behaviour of the $R(T)$ separatrix**
- On insulating side of SIT, low-temperature resistivity is activated: $R(T) \sim \exp(T_0/T)$
- Crossover to VRH at higher temperatures
- Seen in TiN, InO, Be (extra thin) – all are amorphous, with low electron density

There are other types of SC suppression by disorder!
Superconductivity v/s Localization

• Granular systems with Coulomb interaction
  K.Efetov 1980 et al “Bosonic mechanism”

• Coulomb-induced suppression of Tc in uniform films “Fermionic mechanism”
  A.Finkelstein 1987 et al

• Competition of Cooper pairing and localization (no Coulomb)
  Imry-Strongin, Ma-Lee, Kotliar-Kapitulnik, Bulaevsky-Sadovsky(mid-80’s)
  Ghosal, Randeria, Trivedi 2001

There will be no grains and no Coulomb in this talk!
Experimental puzzle: Localized Cooper pairs

D. Shahar & Z. Ovadyahu
amorphous InO 1992

V. Gantmakher et al
InO
D. Shahar et al
InO
T. Baturina et al
TiN
Strongly insulating $\text{InO}$ and nearly-critical $\text{TiN}$

$d = 20 \text{ nm}$
$T_0 = 15 \text{ K}$
$R_0 = 20 \text{ k}\Omega$

$I_2$: $T_0 = 0.38 \text{ K}$
$R_0 = 20 \text{ k}\Omega$

What is the charge quantum? Is it the same on left and on right?
Giant magnetoresistance near SIT
(Samdanmurthy et al, PRL 92, 107005 (2004))

FIG. 2: $\rho$ versus $B$ isotherms (a) at a low $B$ range for film Maf at $T$'s = 0.24, 0.60, 0.78, 0.98 and 1.13 K (b) at a large $B$ range for sample Nalc at $T$'s 0.07, 0.16, 0.35, 0.62 and 1.00 K. The critical point of the $B$-driven SIT, $B_c$, is indicated by the vertical arrow.
Phase Diagram

- Mott-law Insulator
- Hard Gap Insulator
- Metal
  - $B_{MI}$
- Superconductor (SC)
- $B_C$
Theoretical model

Simplest BCS attraction model, but for localized electrons

\[ H = H_0 - g \int d^3r \, \Psi_{\uparrow}^\dagger \Psi_{\downarrow}^\dagger \Psi_{\downarrow} \Psi_{\uparrow} \]

\[ \Psi = \sum c_j \Psi_j (r) \quad \text{Basis of localized eigenfunctions} \]

M. Ma and P. Lee (1985)  
S-I transition at \( \delta \approx T_c \)
Localization length $L_{\text{loc}}$ is finite but large

\[ H = \sum_{j\sigma} \xi_j c_{j\sigma}^\dagger c_{j\sigma} - \frac{\lambda}{\nu_0} \sum_{i,j,k,l} M_{ijkl} c_{i\uparrow}^\dagger c_{j\downarrow}^\dagger c_{k\downarrow} c_{l\uparrow}, \]

where

\[ M_{ijkl} = \int d\mathbf{r} \psi_i(r) \psi_j(r) \psi_k(r) \psi_l(r) \]

\[ \lambda = g \nu_0 \]

Wavefunction’s fractality

\[ M_{ij} = \int \psi_i^2(r) \psi_j^2(r) d^d r \]

\[ M_j = \int \psi_j^4(r) d^d r \propto L_{\text{loc}}^{-D_2} \]

\[ D_2 = 1.30 \pm 0.05 \]

All other (off-diagonal) terms: beyond BCS MFA
**INSULATING STATE AT LARGE** \( \delta_L = (\nu_0 L_{loc}^3)^{-1} \)

Typical value of superdiagonal matrix element:

\[
\tilde{M} = L_0^{-3}(L_{loc}/L_0)^{-D_2}
\]

where \( L_0 \) is the short-scale cutoff length of the fractal behaviour.

K. Matveev and A. Larkin 1997: Parity gap in ultrasmall grains

\[
\Delta \ll \delta: \quad \text{no many-body correlations}
\]

\[
\Delta_P = \frac{1}{2} \lambda \delta \quad \lambda_R = \lambda/(1 - \lambda \log(\epsilon_0/\delta)).
\]

\[
\Delta_P = \frac{\delta}{2 \ln \frac{\delta}{\Delta}}
\]
Parity gap for Anderson-localized eigenstates

The increase of thermodynamic potential $\Omega$ due to addition of odd electron to the ground-state is

$$\delta \Omega_{\text{oe}} = \xi_{m+1} = \xi_{m+1} - \tilde{\xi}_{m+1} + \tilde{\xi}_{m+1} = \frac{g}{2} M_{m+1} + O(V^{-1})$$

$$\tilde{\xi}_j = \xi_j - \frac{g}{2} M_j$$

Energy of two single-particle excitations due to depairing:

$$2\Delta_P = \xi_{m+1} - \xi_m + g M_m = \frac{g}{2} (M_m + M_{m+1}) + O(V^{-1})$$
Typical value of superdiagonal matrix element:

\[ \bar{M} = L_0^{-3} (L_{loc}/L_0)^{-D_2} \]

where \( L_0 \) is the short-scale cutoff length of the fractal behaviour.

\[ \Delta_P = \frac{\lambda}{2} E_0 \left( \frac{L_0}{L_{loc}} \right)^{D_2} \propto (E_m - E_F)^{\nu D_2} \]

where

\[ E_0 = \frac{1}{\nu_0 L_0^3} \ll E_F \]
Average Density of States

\[ \rho(E) = \sum_{M, \ell} \delta(E - \min(M \pm \ell \lambda_{\ell})) \frac{M \lambda_{\ell}}{2V} \]

no coherence peak!

[Ghosal et al 2001]

effective gap \( \Delta_p \)
P(M) distribution
Activation energy $T_1$ from Shahar-Ovadyahu exp. and fit to theory

FIG. 10: Experimental values of the gap from Ref.34, $T_1$ (boxes) and a fit to the equation (61) with $\nu = 1$, $D_2 = 1.3$. The only fitting parameter was the constant $A = 0.15\lambda E_0$; the data points of Ref.[1] correspond to $E_0 \approx 300K$ at $\lambda \approx 0.2$ extracted from the BCS value of $T_c \approx 3K$ for less disordered samples$^{34}$ and $\omega_D \approx 500K$. The value of $\sigma_c$ was determined from high $T$ data. Application of scaling formulas to the data shown here is justified by the large value of $L_{nc} > 30\AA$ of the most disordered sample shown in this plot. This estimate comes from the analysis using Mott temperature which characterizes the resistivity of even more disordered sample presented in$^{21}$ for intermediate temperatures.
Shahar-Ovadyahu 1992

\( \delta_L \sim \frac{1}{V} L_a^{-3} \sim (n-n_c)^3 \)
Superconductivity at the Localization Threshold: $T_c \gg \delta_L$

Now we will consider the case of Fermi energy very close to the mobility edge: single-electron states are extended but fractal and populate small fraction of the whole volume.

How BCS theory should be modified to account for eigenstate’s fractality?
Mean-Field Eq. for $T_c$

$$\Delta(r) = \int K_T(r, r') \Delta(r') d^{d-1}r'$$  \hspace{1cm} (9)

where kernel $\hat{K}_T$ is equal to

$$K_T(r, r') = \frac{\lambda}{2\nu_0} \sum_{ij} \frac{\tanh \frac{\xi_i}{2T} + \tanh \frac{\xi_j}{2T}}{\xi_i + \xi_j} \psi_i(r)\psi_j(r)\psi_i(r')\psi_j(r')$$  \hspace{1cm} (10)

Standard averaging over space $\Delta(r) \rightarrow \overline{\Delta}$ leads to "Anderson theorem" result: totally incorrect in the present situation.

The reason: critical eigenstates $\psi_j(r)$ are strongly correlated in real 3D space, they fill some small submanifold of the whole space only.
In fact one should define $T_c$ as the divergence temperature of the Cooper ladder

$$C = \left(1 - \hat{K}\right)^{-1}$$

Thus averaging procedure should be applied to $C$ instead of $K$.

We expand $C$ in powers of $K$ and average over disorder realizations. Keeping main sequence of resulting diagramms only, we come to the following equation for determination of $T_c$:

$$\Phi(\xi) = \frac{\lambda}{2} \int \frac{d\xi' \tanh(\xi' / 2T)}{\xi'} M(\xi - \xi')\Phi(\xi')$$

(11)
$$M(\omega) = \mathcal{V} M_{ij} = \int \frac{\psi_i^2(r) \psi_j^2(r)}{d^d r} \text{ for } |\xi_i - \xi_j| = \omega$$

For critical eigenstates

$$L_{loc} \to \infty$$

one finds

$$M(\omega) = \left( \frac{E_0}{\omega} \right)^\gamma$$

where

$$\gamma = 1 - \frac{D_2}{d}$$

is a measure of fractality

Usual "dirty superconductor":

$$M(\omega) = 1 \quad \gamma = 0$$

3D Anderson model: $\gamma = 0.57$
Modified mean-field approximation for critical temperature $T_c$

$$\Delta(\xi) = \frac{\lambda}{2} \int d\zeta \eta(\zeta) M(\xi - \zeta) \Delta(\zeta)$$

$$\eta_i \equiv \eta_{ii} = \xi^{-1}_i \tanh(\xi_i / 2T).$$

$$T^0_c(\lambda, \gamma) = E_0 \lambda^{1/\gamma} C(\gamma)$$

For small $\lambda$ this $T_c$ is higher than BCS value!
The above equation for $T_c$ is equivalent to the neglect in the Hamiltonian "off-diagonal" terms. We employ eigenfunction expansion of the gap function $\Delta(r)$ and use the idea that pairing amplitude

$$F_j = \langle c_{j\uparrow} c_{j\downarrow} \rangle = F(\xi_j)$$

is a smooth function of the bare energy $\xi_j$:

$$F(\xi) = \frac{\Delta(\xi)}{\sqrt{\Delta^2(\xi) + \xi^2}} \tanh \frac{\sqrt{\Delta^2(\xi) + \xi^2}}{2T}$$

where

$$\Delta(\xi) = \lambda \int d\xi' M(\xi - \xi') F(\xi')$$

Then local pairing amplitude:

$$F(r) = \sum_j \psi_j^2 \langle c_{j\uparrow} c_{j\downarrow} \rangle \equiv \sum_j \psi_j^2 F_j$$

fluctuates strong in real space

Volume fraction $\left(\frac{T_c}{E_0}\right)^\gamma \ll 1$
Fluctuations of SC order parameter

With Prob = \( p \ll 1 \) \( \Delta(r) = \Delta \), otherwise \( \Delta(r) = 0 \)

\[
\frac{(\tilde{\Delta}(r))^2}{(\bar{\Delta}(r))^2} = \lambda Q(\gamma) = \frac{Q(\gamma)}{C^\gamma(\gamma)} \left( \frac{T_c}{E_0} \right)^\gamma \ll 1
\]

SC fraction = prefactor \( \approx 1.7 \) for \( \gamma = 0.57 \)

\[
\frac{(\tilde{\Delta}(r))^n}{(\bar{\Delta}(r))^n} \propto \left( \frac{T_c}{E_0} \right)^{1-d_n/d}(n-1)
\]

Higher moments:
Critical temperature $T_c$ is well-defined through the whole system in spite of strong $\Delta(r)$ fluctuations.
Superconductivity with Pseudogap

Now we move Fermi-level into the range of localized eigenstates.

Both local pairing (like in insulator) and collective pairing are present.
Self-consistent "gap equation" in terms of $\Delta(\xi)$:

$$\Delta(\xi) = \lambda \int d\xi' M(\xi - \xi') \frac{\Delta(\xi)}{\sqrt{\Delta^2(\xi) + \xi^2}} \tanh \frac{\sqrt{\Delta^2(\xi) + \xi^2}}{2T}$$

Dimensional analysis of the Mean Field equation:

$$T_c = C(\gamma) E_0 \lambda^{1/\gamma}$$

$$\Delta(\xi = 0, T = 0) = D(\gamma) E_0 \lambda^{1/\gamma}$$

Functions $C(\gamma)$ and $D(\gamma)$ were found numerically:

Now we can relate **collective gap** $\Delta(0)$ and **local pairing gap** $\Delta_P$:

$$\Delta_P = \frac{1}{2D\gamma(\gamma)} \delta_L \left( \frac{\Delta(0)}{\delta_L} \right)^\gamma$$

where $\delta_L = \frac{1}{\nu_0 L^3_{\text{loc}}}$ - typical level spacing inside localization volume.

**Compare with Matveev-Larkin result:**

$$\Delta_P = \frac{\delta}{2 \ln \frac{\delta}{\Lambda}}$$
3D Anderson insulator: $M(\omega)$

No saturation at $\omega < \delta_L$:
$M(\omega) \sim \ln^2 (\delta_L / \omega)$
(Cuevas & Kravtsov 1997)

Superconductivity with $T_c < \delta_L$ is possible

Then “local gap”

$$\Delta_P = \frac{1}{2D^{\gamma(\gamma)}} \delta_L \left( \frac{\Delta(0)}{\delta_L} \right)^\gamma$$

exceeds $T_c$!

This region was not found by Ma&Lee
$T_c$ versus Pseudogap
At $T = T_c$ - almost fully developed gap but no coherence peak

Full 1-particle gap is a sum of insulating and superconductive contributions
Contribution of single-electron states is suppressed by pseudogap.

\[ S^+ = a^+_\mu a^+_\nu, \quad S^- = a_{\mu} a_{\nu}, \quad 2S^z = a^+_\mu a_{\mu} + a^+_\nu a_{\nu} \]

H_{Brs} acts on **Even Sector**:
all states which are 2-filled or empty.
"Pseudospin" approximation

\[ \hat{H} = \sum_{\mu} 2 \xi_{\mu} S_{\mu}^z - g \sum_{\mu, \nu} M_{\mu \nu} S_{\mu}^+ S_{\nu}^- + \sum_{B_{\mu}} (\xi_{\mu} + \frac{g_{\mu}}{2}) \]

\[ \bar{M}_{\mu \nu} = \frac{1}{V} M(\xi_{\mu} - \xi_{\nu}) \]

Effective number of interacting neighbours
Critical temperature in the pseudogap regime

MFA:

\[ \Delta(\xi) = \frac{\lambda}{2} \int d\zeta \eta(\zeta) M(\xi - \zeta) \Delta(\zeta) \]

Take the same

\[ \eta_i \equiv \eta_{ii} = \xi_i^{-1} \tanh(\xi_i / 2T). \]

but replace \( 2T \to T \)

MFA is OK as far as \( Z \sim \nu_0 T_c L_{loc}^d \) is large
Third Scenario

- **Bosonic mechanism**: preformed Cooper pairs + competition Josephson v/s Coulomb – SIT in arrays

- **Fermionic mechanism**: suppressed Cooper attraction, no paring – SMT

- **Pseudospin mechanism**: individually localized pairs
  - SIT in amorphous media, fractal superconductivity
  - SIT occurs at small Z and lead to paired insulator

The origin of the transport gap:

Mobility edge for hopping pairs
S-I Transition

- Hamiltonian of the pseudospin array:

\[
H = 2 \sum_i \xi_i s_i^z - \sum_{ij} M_{ij} (s_i^x s_j^x + s_i^y s_j^y)
\]

\[
Z \sim v_0 T_c L_{loc}^d
\]

At \( Z \ll 1 \) Insulating state is realized: localized pairs

How can we describe quantum phase transition?
S-I transition on Cayley tree example with branching number $q = 3$

Hamiltonian:

$$H = 2 \sum_i \xi_i s_i^z - \sum_{ij} M_{ij} (s_i^x s_j^x + s_i^y s_j^y) \tag{1}$$

Eq. (1) contains random energies $\xi_i$

Full self-consistent equation can be written for distribution functions

$$\Delta(x) = \frac{\lambda}{2} x [\tanh(\beta x)]^{-1} \int_{-W/2}^{W/2} dy \Delta(y) \frac{\tanh(\beta e_+) \tanh(\beta e_-)}{e_+ e_-} \tag{2}$$

Eigenvalues of 2-spin problem

$$e_{\pm} = \frac{1}{2} \left[ x + y \pm \sqrt{(x - y)^2 + M^2} \right]$$
“Phase diagram”
and spectrum of collective modes

\[ T_c(\infty) = W \exp(-1/\lambda) \]

Continuous spectrum: \( N \)

Discrete spectrum: Insulator

Many-body localization?

Superconductor

\[ q_1 \approx \exp(1/\lambda) \]
S-I transition in real space?

- Very strong mesoscopic fluctuations
- Effective coordination number $Z < 1$
- Formation of SC droplets in the I matrix (or vice-verse) is unavoidable
- These droplets are pieces of fractal superconductive state and not the usual SC grains
Major unsolved problems (theor)

• 1. Role of Coulomb enhancement near mobility edge ? (this effect was treated by Finkelstein for metal thin-film case)

• 2. How to include magnetic field into the “fractal” scheme ?

• 3. Transition between pseudogap SC and insulator. Why Cooper pair transport is activated ?
Conclusions

- Pairing of electrons on localized states leads to hard gap and Arrhenius resistivity for 1e transport.
- Pairing on nearly-critical states produces fractal superconductivity with relatively high $T_c$ but very small superconductive density.
- Pseudogap behaviour is expected near S-I transition, with "insulating gap" exceeding $T_c$.
- Insulator appears from Superconductor via First-Order transition.
$T_c$ from 3 different calculations

Modified MFA equation leads to:

$$T_c = (6.5 \pm 0.8) \lambda^{1.77}$$
Qualitative features of “Pseudogap Superconductivity”:

- STM DoS evolution with $T$ - shown for $InOx$
- anomaly in fluctuational Nernst effect: studied by P. Spathis, H. Aubin et al in $InOx$
- Nonconservation of full spectral weight across $T_c$ : found before in underdoped HTSC, not measured yet near SIT
Features of the S-I transitions

Observed in: amorphous InO thin TiN ultrathin Be
(Possible in: Boron-doped Diamond)

- Insulating behaviour of the $R(T)$ separatrix
- On insulating side of SIT, low-temperature resistivity is activated: $R(T) \sim \exp(T_0/T)$
- Crossover to VRH at higher temperatures
- Negative magnetoresistance is seen at high magnetic fields on both sides of SIT
- Positive magnetoresistance at low fields in insulating samples not far from the SIT

No “universal resistance” at the S-I transition