Field theories in three-dimensional space-time

Sergey M. Sergeev

Discipline of Mathematics and Statistics, University of Canberra

Isaac Newton Institute for Mathematical Sciences.
Quantum Integrable Discrete Systems.
Cambridge, 23 of March, 2009

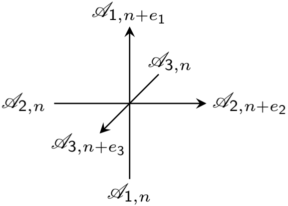
Equations of motion

Let n be a node of simple cubic lattice

$$n = n_1 e_1 + n_2 e_2 + n_3 e_3, \quad n_i \in \mathbb{Z}$$

Field variables reside the edges of lattice

$$\mathscr{A}_{j,n} = \{k_{j,n}, a^{\pm}_{j,n}\} \;:\; k^2_{j,n} = 1 - a^{+}_{j,n} a^{-}_{j,n} \ .$$



Equations of motion are:

$$\begin{cases} k_{2,n+e_2} a^{\pm}_{1,n+e_1} = \ u_1^{\pm 1}\left(k_{3,n} a^{\pm}_{1,n} + u_2^{\mp 1} k_{1,n} a^{\pm}_{2,n} a^{\mp}_{3,n}\right), \\ a^{\pm}_{2,n+e_2} = \left(a^{\pm}_{1,n} a^{\pm}_{3,n} \ - \ u_2^{\mp 1} k_{1,n} k_{3,n} a^{\pm}_{2,n}\right), \\ k_{2,n+e_2} a^{\pm}_{3,n+e_3} = u_3^{\pm 1}\left(k_{1,n} a^{\pm}_{3,n} + u_2^{\mp 1} k_{3,n} a^{\mp}_{1,n} a^{\pm}_{2,n}\right), \end{cases}$$

u_i – parameters

where $k_{1,n+e_1}/k_{1,n} = k_{2,n}/k_{2,n+e_2} = k_{3,n+e_3}/k_{3,n}$.

These equations of motion are:

- Explicit form of implicit Korepanov model [Korepanov]
- A general form of discrete three-wave equations [Doliwa, Santini, Bogdanov, Konopelchenko]
- u_i-generalization of discrete orthogonal net equations [Bobenko, Doliwa et al, Konopelchenko, Schief]
- q-oscillator model [Bazhanov, Mangazeev, Sergeev] – the iteration of q-oscillator maps

$$\mathscr{A}_{j,n} = \mathscr{A}_j \;\rightarrow\; \mathscr{A}'_j = \mathscr{A}_{j,n+e_j}\;, \quad j = 1,2,3\;.$$

Detailed derivation of these equations in the framework of discrete differential geometry, Korepanov equation, and their quantization will be given in Prof. V. V. Bazhanov's talk.

Define "discrete time" for cubic lattice as

$$\tau(n) \;=\; n_1 + n_2 + n_3 \;.$$

Then equations of motion

$$\mathscr{A}_{j,n+e_j} \;\leftarrow\; \mathscr{A}_{j,n}$$

expressing the fields at time $\tau + 1$ in terms of fields at time τ define thus the evolution map.
The collection of fields along initial constant time surface $\tau(n) = \tau_0$ is a set of *independent* variables – initial data for Cauchy problem.

Discrete time Hamiltonian flow

In classics, $k_{j,n}^2 = 1 - a_{j,n}^+ a_{j,n}^-$, the evolution preserves symplectic form

$$\omega \;=\; \sum_{\tau(n)=\tau} \sum_{j=1}^{3} \frac{da_{j,n}^+ \;\wedge\; da_{j,n}^-}{k_{j,n}^2}.$$

Generating function of symplectic map $\tau \to \tau + 1$ is the Lagrangian of the field theory.

In quantum case, define the algebra of observables by

$$\mathscr{A} \ : \quad a^+a^- = 1 - q^{-1}k^2 \ , \quad a^-a^+ = 1 - qk^2 \ , \quad ka^{\pm} = q^{\pm 1}a^{\pm}k$$

Elements of $\mathscr{A}_{j,n}$ along initial constant time surface $\tau(n) = \tau_0$ are independent copies of q-oscillators.

Heisenberg evolution

In quantum case, the evolution is automorphism of algebra of observables, evolution equations become the Heisenberg equations of motion; for fixed representation of algebra of observables the evolution map $\tau \to \tau + 1$ is the conjugation by discrete-time unitary evolution operator

$$\Phi(\tau + 1) \ = \ U \ \Phi(\tau) \, U^{-1}$$

Principal problem of quantum theory is the spectrum of evolution operator (Schrödinger equation) – problem yet unsolved.

... And Fermions!

In what follows, we discuss Classical field theories (however, we think about classical theories as about classical limit of quantum ones).

Few simple principles of classical theories

- Physical criterion of a field theory / statistical mechanics is the reality of its action/energy.
- Second physical criterion is the existence of ground state – *homogeneous*, or *flat* solution of equations of motion.
- In a statistical mechanics the ground state is the absolute minimum of the energy functional for fixed reality regime of fields and fixed boundary conditions.
- An extremum of field-theoretical action corresponds a set of elementary excitations (stationary modes), field-theoretical ground state is the case of zero amplitudes of all elementary excitations.

Step one: Physical regimes

Convenient canonical variables for $\mathscr{A}$ are k and v:

$$k^2 = 1 - a^+ a^- \quad \text{and} \quad v = \frac{a^+}{a^-}$$

Generating function is defined by

$$dG(k;k') = \frac{1}{2} \sum_{j=1}^{3} \big(\log v'_j \, d \log k'_j - \log v_j \, d \log k_j \big) \ ,$$

where v_j, v'_j as functions of k_j, k'_j are defined by equations of motion $\mathscr{A}_j \to \mathscr{A}'_j$, $j = 1, 2, 3$.
There are four regimes when G is manifestly real or purely imaginary:

Regime 1: $k^2 < 0$, $|v| = 1$, Regime 2: $k^2 < 0$, $v \in \mathbb{R}$,

Regime 3: $k^2 > 0$, $|v| = 1$, Regime 4: $k^2 > 0$, $v \in \mathbb{R}$,

Distinction between $k^2 > 0$ and $k^2 < 0$ comes from representation theory of q-oscillators.

Solution for ground state equations $\mathscr{A}_j = \mathscr{A}'_j$,

$$\begin{cases} k_2 a_1^{\pm} = \; u_1^{\pm}\Big(k_3 a_1^{\pm} + u_2^{\mp 1} k_1 a_2^{\pm} a_3^{\mp}\Big), \\ a_2^{\pm} = \Big(a_1^{\pm} a_3^{\pm} \; - \; u_2^{\mp 1} k_1 k_3 a_2^{\pm}\Big), \\ k_2 a_3^{\pm} = u_3^{\pm 1}\Big(k_1 a_3^{\pm} + u_2^{\mp 1} k_3 a_1^{\mp} a_2^{\pm}\Big), \end{cases}$$

essentially depend on spectral parameters u_1, u_2, u_3. Thus, the regimes of ground state are defined by regimes of spectral parameters.

Four physical Regimes correspond to four domains of spectral parameters equivalent to four types of spherical/hyperbolic triangles.

Spectral parameters of Regime 1 are given by

$$u_1 = \mathrm{e}^{\mathrm{i}\phi_1}, \quad u_2 = \mathrm{e}^{\mathrm{i}\phi_2}, \quad u_3 = \mathrm{e}^{\mathrm{i}\phi_3}$$

where $\phi_i > 0$ are sides of spherical triangle. Ground state values of $k_i = k_i'$ are

$$k_1^2 = -\tan^2\frac{\theta_1}{2}, \quad k_2^2 = -\cot^2\frac{\theta_2}{2}, \quad k_3^2 = -\tan^2\frac{\theta_3}{2}$$

where $\theta_i > 0$ are dihedral angles of the spherical triangle

$$\cos\theta_i = \frac{\cos\phi_i - \cos\phi_j\cos\phi_k}{\sin\phi_j\sin\phi_k}$$

In this regime

$$\theta_1 + \theta_2 + \theta_3 > \pi$$

Spectral parameters of Regime 2 are given by

$$u_1 = e^{\phi_1}, \quad u_2 = e^{\phi_2}, \quad u_3 = e^{\phi_3}$$

where $\phi_i > 0$ are hyperbolic sides of a triangle on upper sheet of two-sheets hyperboloid. Ground state values of $k_i = k_i'$ are

$$k_1^2 = -\tan^2\frac{\theta_1}{2}, \quad k_2^2 = -\cot^2\frac{\theta_2}{2}, \quad k_3^2 = -\tan^2\frac{\theta_3}{2}$$

where $\theta_i > 0$ are dihedral angles of the hyperbolic triangle

$$\cos\theta_i = \frac{\cosh\phi_j \cosh\phi_k - \cosh\phi_i}{\sinh\phi_j \sinh\phi_k}$$

In this regime

$$\theta_1 + \theta_2 + \theta_3 < \pi$$

Spectral parameters of Regime 3 are given by

$$u_1 = \mathrm{e}^{\mathrm{i}\phi_1}, \quad u_2 = -\mathrm{e}^{\mathrm{i}\phi_2}, \quad u_3 = \mathrm{e}^{\mathrm{i}\phi_3}$$

where $\phi_i > 0$ are trigonometric sides of a hyperbolic triangle formed by intersection of three planes with time-like normals and one-sheet hyperboloid. Trigonometric sides are defined in motionless frame of reference for each plane. Ground state values of $k_i = k_i'$ are

$$k_1^2 = \coth^2\frac{\theta_1}{2}, \quad k_2^2 = \tanh^2\frac{\theta_2}{2}, \quad k_3^2 = \coth^2\frac{\theta_3}{2}$$

where $\theta_i > 0$ are hyperbolic dihedral angles of the triangle (angles between time-like normals),

$$\cosh\theta_i = \frac{\cos\phi_i + \cos\phi_j\cos\phi_k}{\sin\phi_j\sin\phi_k}$$

Spectral parameters of Regime 4 are given by

$$u_1 = \mathrm{e}^{-\phi_1}, \quad u_2 = -\mathrm{e}^{-\phi_2}, \quad u_3 = \mathrm{e}^{-\phi_3}$$

where $\phi_i > 0$ are hyperbolic sides of a triangle formed by intersection of three planes with space-like normals and one-sheet hyperboloid. Such triangle has no counterpart on two-sheets hyperboloid. Ground state values of $k_i = k'_i$ are

$$k_1^2 = \tanh^2 \frac{\theta_1}{2}, \quad k_2^2 = \coth^2 \frac{\theta_2}{2}, \quad k_3^2 = \tanh^2 \frac{\theta_3}{2}$$

where $\theta_i > 0$ are hyperbolic dihedral angles of the triangle,

$$\cosh \theta_i = \frac{\cosh \phi_i + \cosh \phi_j \cosh \phi_k}{\sinh \phi_j \sinh \phi_k}$$

Step 3: Solitons

Elementary excitations over the ground state are the solitons.

For the number of solitons g let

$$\{X_j, Y_j\}_{j=1,\dots,g} \quad \text{and} \quad f = \{f_j\}_{j=1,\dots,g}$$

be a set of $3g$ *complex* numbers. For given f and $\{X, Y\}$ define

$$F_j = f_j \prod_{k \neq j} \frac{X_j - X_k}{Y_j - X_k}$$

and

$$\Theta(f) = \frac{\det |X_j^{k-1} + F_j Y_j^{k-1}|_{j,k=1,\dots,g}}{\prod_{i>j}(X_i - X_j)}$$

Parameter f_j is an amplitude of j-th soliton, X_j, Y_j parameterize a "wave vector" of soliton plane wave. Number of solitons is additive parameter: if one of $f_j = 0$ then $\Theta(f)$ is just $g-1$-solitons expression.

Let next

$$\omega_j(Q,P) = \frac{(X_j - Q)(Y_j - P)}{(X_j - P)(Y_j - Q)}$$

and for $n = (n_1, n_2, n_3) \in \mathbb{Z}^3$ the oscillating amplitudes be

$$f(n) = \{f_j \omega_j(Q_1,P_1)^{n_1} \omega_j(Q_2,P_2)^{-n_2} \omega_j(Q_3,P_3)^{n_3}\}_{j=1,\dots,g} \ ,$$

where Q_k, P_k, $k = 1,2,3$, are common for all solitons. Finally, let

$$\Theta_n = \Theta(f(n))$$

The last definition (prime form):

$$E(Q,P) = \frac{Q-P}{\sqrt{dQdP}}$$

Comment: the algebraic-geometrical notations are used since

$$f_j \Leftarrow \exp(\mathrm{i} z_j), \quad \omega_j(Q,P) \Leftarrow \exp\left(\mathrm{i}\int_P^Q \gamma_j\right)$$

Equations of motion on cubic lattice have the following solution solution (adapting Korepanov's algebraic-geometry solution, 1993)

$$k_{1:n}^2 = \frac{E(Q_2,Q_3)E(P_2,P_3)}{E(Q_2,P_3)E(P_2,Q_3)}\,\frac{\Theta_n\Theta_{n-e_2+e_3}}{\Theta_{n-e_2}\Theta_{n+e_3}}\ ,$$

$$k_{2:n}^2 = \frac{E(Q_1,Q_3)E(P_1,P_3)}{E(Q_1,P_3)E(P_1,Q_3)}\,\frac{\Theta_{n-e_2}\Theta_{n+e_1-e_2+e_3}}{\Theta_{n+e_1-e_2}\Theta_{n-e_2+e_3}}\ ,$$

$$k_{3:n}^2 = \frac{E(Q_1,Q_2)E(P_1,P_2)}{E(Q_1,P_2)E(P_1,Q_2)}\,\frac{\Theta_n\Theta_{n+e_1-e_2}}{\Theta_{n+e_1}\Theta_{n-e_2}}$$

Ground state is zero soliton solution $\Theta_n = 1$.

- Spherical and hyperbolic triangles are formed by intersection of sphere or hyperboloids and three planes with normals ν_i, $i = 1, 2, 3$.
- Parameters Q_i and P_i are stereographic coordinates of ν_i (Q_i is a point on complex plane tangent to northern pole, P_i is that for southern pole).
- Correspondence to Regimes 1-4 is the following:

$$\text{Regime 1: } P_i Q_i^* = -1 \, ,$$

$$\text{Regime 3: } P_i Q_i^* = 1 \, ,$$

$$\text{Regimes 2,4: } |P_i| = |Q_i| = 1 \, ,$$

where $*$ stands for complex conjugation.

The physical plane waves

$$f(n) = \{f_j \omega_j(Q_1, P_1)^{n_1} \omega_j(Q_2, P_2)^{-n_2} \omega_j(Q_3, P_3)^{n_3}\}_{j=1,\ldots,g} \ ,$$

require the unitarity of exponential frequencies

$$\omega_{X,Y}(Q_i, P_i) = \frac{(X - Q_i)(Y - P_i)}{(X - P_i)(Y - Q_i)}, \quad i = 1, 2, 3 \ .$$

For our four Regimes all <u>three</u> $\omega_{X,Y}(Q_i, P_i)$ are unitary when

Regime 1: $P_i Q_i^* = -1, \quad \text{no } X \text{ and } Y,$

Regime 3: $P_i Q_i^* = 1, \quad |X| = |Y| = 1 \ ,$

Regimes 2,4: $|P_i| = |Q_i| = 1, \quad XY^* = 1 \ .$

For the given Q_i, P_i the exponential frequencies

$$\omega_i = \omega_{X,Y}(Q_i, P_i) = \frac{(X - Q_i)(Y - P_i)}{(X - P_i)(Y - Q_i)}, \quad i = 1, 2, 3 .$$

form a two-dimensional variety – a solution of a single dispersion relation for three $\omega_1, \omega_2, \omega_3$.
The dispersion relation for small momenta p_i,

$$\omega_1 = \mathrm{e}^{\mathrm{i}p_1}, \quad \omega_2 = \mathrm{e}^{-\mathrm{i}p_2}, \quad \omega_3 = \mathrm{e}^{\mathrm{i}p_3}$$

is the following fourth order homogeneous relation:

$$p_1^2 p_2^2 (u_3 - u_3^{-1})^2 + p_1 p_2 p_3^2 (u_1 - u_1^{-1})(u_2 - u_2^{-1})(u_3 + u_3^{-1}) + \text{cyclic prem} = 0$$

Due to the homogeneity, it defines a cone-type surface in momentum space.

The symmetric case $\phi_1 = \phi_2 = \phi_3 = \phi$ for all four Regimes is very illustrative. Dispersion relations becomes

$$p_1^2 p_2^2 + p_1^2 p_3^2 + p_2^2 p_3^2 + 2c p_1 p_2 p_3 (p_1 + p_2 + p_3) = 0$$

where

$$\text{Regime 1: } c = \cos\phi, \quad -1/2 < c < 1,$$

$$\text{Regime 2: } c = \cosh\phi, \quad 1 < c,$$

$$\text{Regime 3: } c = -\cos\phi, \quad -1 < c < -1/2,$$

$$\text{Regime 4: } c = -\cosh\phi, \quad c < -1.$$

Regimes cover the whole real axis of $c \in \mathbb{R}$ except three critical points

$$c = -1\,, \quad c = -1/2 \quad \text{and} \quad c = 1\,.$$

The symmetric dispersion relation is uniformized by substitution

$$p_i = E + \pi_i \ , \quad \pi_1 + \pi_2 + \pi_3 = 0$$

Let

$$\pi^2 = \frac{1}{2}(\pi_1^2 + \pi_2^2 + \pi_3^2), \quad \gamma = \frac{\pi_1\pi_2\pi_3}{\pi^3}, \quad |\gamma| < \sqrt{\frac{4}{27}}$$

Solution of dispersion relation is given by

$$E = \alpha(\gamma)\pi$$

where $\alpha(\gamma)$ is a real solution of

$$(c + \frac{1}{2})\alpha^4 - c\alpha^2 + (c-1)\gamma\alpha + \frac{1}{6} = 0$$

In Regime 1 this equation has no real solutions. In Regimes 3,4 (big anisotropy) this equation has one positive and one negative solutions. In Regime 2 it has two positive and two negative solutions (two tangent cones). In a "static limit" of Regime 2, $c \sim 1$, we have the Lorentz group symmetry:

$$(p_1p_2 + p_1p_3 + p_2p_3)^2 = 0$$

- Regime 1: modular representation of q-oscillators, real external fields, convergent statistical mechanics.
- Regime 2: modular representation of q-oscillators, unitary external fields and unitary R-matrix, well defined quantum field theory.
- Regime 3: unitary Fock-space representations of q-oscillators, unitary external fields and unitary R-matrix, well defined quantum field theory.
- Regime 4: no well defined quantum counterpart. Regime for k_i and external fields correspond to non-unitary Fock-space representations of q-oscillators with real positive matrix elements of R-matrix. However, this is *divergent* statistical mechanics. Convergent statistical mechanics needs non-linear boundary fields.

THANK YOU

Acknowledgements. I would like to thank my colleagues from UC Yvonne Wisbey, Mary Hewett and Judith Ascione for picking up my classes.

Let $\mathscr{A} = \{k, a^{\pm}\}$ stands for the q-oscillator algebra

$$a^+a^- = 1 - q^{-1}k^2 \ , \quad a^-a^+ = 1 - qk^2 \ , \quad ka^{\pm} = q^{\pm 1}a^{\pm}k$$

The automorphism of tensor cube of q-oscillators

$$\mathcal{R}_{123} \ : \quad \mathscr{A}_1 \otimes \mathscr{A}_2 \otimes \mathscr{A}_3 \ \to \ \mathscr{A}_1' \otimes \mathscr{A}_2' \otimes \mathscr{A}_3'$$

defined by

$$\begin{cases} (k_2 a_1^{\pm})' = \ u_1^{\pm}\Big(k_3 a_1^{\pm} + u_2^{\mp 1} k_1 a_2^{\pm} a_3^{\mp}\Big), \\ (a_2^{\pm})' = \Big(a_1^{\pm} a_3^{\pm} \ - \ u_2^{\mp 1} k_1 k_3 a_2^{\pm}\Big), \\ (k_2 a_3^{\pm})' = u_3^{\pm 1}\Big(k_1 a_3^{\pm} + u_2^{\mp 1} k_3 a_1^{\mp} a_2^{\pm}\Big), \end{cases} \qquad u_1, u_2, u_3 \in \mathbb{C}$$

and

$$k_1' k_2' = k_1 k_2 \ , \quad k_2' k_3' = k_2 k_3$$

satisfies the adjoint tetrahedron equation.

Regime of q:

$$q = \mathrm{e}^{\mathrm{i}\pi b^2}\,, \quad \eta = \frac{1}{2}(b + b^{-1}) > 0\,,$$

and spectrum of k:

$$k = -\mathrm{i}\mathrm{e}^{\pi b\sigma}, \quad \sigma \in \mathbb{R}$$

Kernel of constant R-matrix is

$$\langle \sigma_1\sigma_2\sigma_3|R|\sigma_1'\sigma_2'\sigma_3'\rangle \;=\; \delta_{\sigma_1+\sigma_2,\sigma_1'+\sigma_2'}\delta_{\sigma_2+\sigma_3,\sigma_2'+\sigma_3'} \times$$

$$\mathrm{e}^{-\mathrm{i}\pi(\sigma_1\sigma_3-\mathrm{i}\eta(\sigma_1+\sigma_3-\sigma_2'))} \int_{\mathbb{R}} du\, \mathrm{e}^{2\pi\mathrm{i}u(\sigma_2'-\mathrm{i}\eta)}\, \frac{\varphi(u+\frac{\sigma_1'+\sigma_3'+\mathrm{i}\eta}{2})\varphi(u+\frac{-\sigma_1-\sigma_3+\mathrm{i}\eta}{2})}{\varphi(u+\frac{\sigma_1-\sigma_3-\mathrm{i}\eta}{2})\varphi(u+\frac{\sigma_3-\sigma_1-\mathrm{i}\eta}{2})}$$

where φ is the Barns-Faddeev non-compact dilogarithm:

$$\varphi(z) = \exp\left(\frac{1}{4}\int_{\mathbb{R}+\mathrm{i}0} \frac{\mathrm{e}^{-2\mathrm{i}zw}}{\sinh(wb)\sinh(w/b)}\,\frac{dw}{w}\right)\,.$$

- Regime 1: R-matrix is

$$\langle\sigma_1\sigma_2\sigma_3|R(\phi)|\sigma_1'\sigma_2'\sigma_3'\rangle = \mathrm{e}^{-2\eta\phi_2\sigma_2}\langle\sigma_1\sigma_2\sigma_3|R|\sigma_1'\sigma_2'\sigma_3'\rangle\mathrm{e}^{2\eta\phi_1\sigma_1+2\eta\phi_3\sigma_3}$$

where $\phi_i > 0$. Partition function per site is

$$z = \exp\left\{-\frac{4\eta^2}{\pi}\sum_{j=0}^{3}Л(\beta_j)\right\}$$

where

$$Л(\beta) = -\int_0^\beta \log 2\sin x dx$$

is Milnor's Lobachevski function and β_j are linear excesses

$$\beta_0 = \pi - \frac{\phi_1+\phi_2+\phi_3}{2}\,, \quad \beta_j = \pi - \beta_0 + \phi_j, \quad 0 < \beta_j < \pi$$

- Regime 2: Unitary R-matrix

$$\langle\sigma_1\sigma_2\sigma_3|R(\phi)|\sigma_1'\sigma_2'\sigma_3'\rangle = \mathrm{e}^{2\mathrm{i}\eta\phi_2\sigma_2}\langle\sigma_1\sigma_2\sigma_3|R|\sigma_1'\sigma_2'\sigma_3'\rangle\mathrm{e}^{-2\mathrm{i}\eta\phi_1\sigma_1-2\mathrm{i}\eta\phi_3\sigma_3}$$

Spectrum of k:

$$k = q^{1/2+n}, \quad \text{Spectrum}(n) = \mathbb{Z} = \mathbb{Z}_{\geq 0} \ \oplus \ \mathbb{Z}_{<0}$$

Note, $k < 1$ in $\mathbb{Z}_+$ and $k > 1$ in $\mathbb{Z}_-$. Matrix element of constant R-matrix is

$$\begin{aligned} \langle n_1 n_2 n_3 | R | n_1' n_2' n_3' \rangle \ &= \ \delta_{n_1+n_2, n_1'+n_2'} \delta_{n_2+n_3, n_2'+n_3'} \times \\ q^{n_1 n_3 + n_2'} \ \frac{1}{2\pi \mathsf{i}} \oint \ \frac{dz}{z^{n_2'+1}} \ & \frac{(-q^{2+n_1'+n_3'} z; q^2)_\infty (-q^{-n_1-n_3} z; q^2)_\infty}{(-q^{+n_1-n_3} z; q^2)_\infty (-q^{-n_1+n_3} z; q^2)_\infty} \ . \end{aligned}$$

where

$$(z; q^2)_\infty = \prod_{n=0}^{\infty} (1 - z q^{2n})$$

and the integration contour circles all poles of the integrand in the counterclockwise direction. Constant R-matrix has block-diagonal structure with respect to

$$\mathbb{Z}_\pm \otimes \mathbb{Z}_\pm \otimes \mathbb{Z}_\pm$$

- Regime 3: R-matrix in $\mathbb{Z}_- \otimes \mathbb{Z}_+ \otimes \mathbb{Z}_-$,

$$R_{123}(\phi) = \mathrm{e}^{\mathrm{i}(\pi-\phi_2)n_2} R_{123} \mathrm{e}^{\mathrm{i}\phi_1 n_1 + \mathrm{i}\phi_3 n_3}$$

  is the unitary matrix of a quantum field theory.

- Regime 4: R-matrix in $\mathbb{Z}_+ \otimes \mathbb{Z}_- \otimes \mathbb{Z}_+$

$$R_{123}(\phi) = (-)^{n_2} \mathrm{e}^{\phi_2 n_2} R_{123} \mathrm{e}^{-\phi_1 n_1 - \phi_3 n_3}$$

  has real positive matrix elements growing as

$$\langle n_1 n_2 n_3 | R | n_1 n_2 n_3 \rangle \sim q^{n_1 n_2 - n_1 n_3 + n_2 n_3}, \quad |n_i| \to \infty$$