# “Manin” matrices and quantum spin models

Gregorio Falqui

Dipartimento di Matematica e Applicazioni
Università di Milano-Bicocca

March 23rd 2009, QIDS, Cambridge

# Outline

1. Introduction and (brief) overview
   - The Gaudin model
   - ... and their limits
   - Lax matrices of "Gaudin" and "Yangian" type
2. Manin matrices
   - Main properties
   - On Quantum separation of variables

We discuss some properties of Lax (and Transfer) matrices associated with quantum integrable systems. In the present talk we will consider the Lax matrix of the Gaudin system, and namely the problem of what happens when the arbitrary points $z_1, \ldots, z_N$ appearing in the Lax matrix, and in the quadratic Hamiltonians $H_i$ glue together.
(Joint work with A. Chervov and L. Rybnikov, SIGMA 2009)
This will lead us to the discuss - in some more greater detail - the notion of Manin matrix.
(Joint work with A. Chervov , J. Phys. A.: Math. Theor. 41,n.19. May 2008, paper no. 194006, and with A. Chervov and V. Rubtsov, Angers arXiv:0901.0235 *to appear in Adv. Appl. Math.* , and work in progress also with A. Sylantyev)

Our point of view stems from the fact that Lax matrices satisfy special commutation properties, considered by Yu. I. Manin some twenty years ago at the beginning of Quantum Group Theory. They are the commutation properties of matrix elements of linear homomorphisms between polynomial rings; more explicitly they read: 1) elements in the same column commute; 2) commutators of the cross terms are equal: $[M_{ij}, M_{kl}] = [M_{kj}, M_{il}]$ (e.g. $[M_{11}, M_{22}] = [M_{21}, M_{12}]$).
Twofold Main aim : 1) Such matrices (which we call *Manin* matrices for short) behave almost as well as matrices with commutative elements. Namely theorems of linear algebra (e.g., a natural definition of the determinant, the Cayley-Hamilton theorem, the Newton identities and so on and so forth) have a straightforward counterpart in the case of Manin matrices.

2) Such matrices often enter theory of quantum integrable spin systems. For instance, Manin matrices include matrices satisfying the Yang-Baxter relation "RTT=TTR" and the so–called Cartier-Foata matrices.
Idea/Hope: Theorems of linear algebra, after being established for such matrices, have (or might have) various applications to quantum integrable systems and Lie algebras.

The Gaudin model was introduced by M. Gaudin as a spin model related to the Lie algebra $sl_2$, and later generalized to the case of arbitrary semisimple Lie algebras.
The Hamiltonian is

$$H_G = \sum_{a=1}^{dim\ \mathfrak{g}} \sum_{i \neq j} x_a^{(i)} x^{a(j)}, \tag{1}$$

where $\{x_a\},\ a = 1, \dots, \dim \mathfrak{g}$, is an orthonormal basis of $\mathfrak{g}$ with respect to the Killing form (and $x^a$ its dual). These objects are regarded as elements of the polynomial algebra $\mathcal{S}(\mathfrak{g}^*)^{\otimes N}$ in the classical case, and as elements of the universal envelopping algebra $U(\mathfrak{g})^{\otimes N}$ in the quantum case, as

$$x_a^{(i)} = 1 \otimes \cdots \otimes \underbrace{x_a}_{i-th\ factor} \otimes 1 \cdots \otimes 1. \tag{2}$$

Gaudin himself found that the quadratic Hamiltonians

$$H_i = \sum_{k \neq i} \sum_{a=1}^{\dim \mathfrak{g}} \frac{x_a^{(i)} x_a^{(k)}}{z_i - z_k}. \tag{3}$$

provide a set of "constants of the motion" for $H_G$. Later it was shown (Jurco) that - in the classical case - the spectral invariants of the Lax matrix

$$L_G(z) = \sum_{i,a} \frac{x_a^{(i)}}{z - z_i}$$

encode a (basically complete) set of invariant quantities for the corresponding model on an arbitrary simple Lie algebra $\mathfrak{g}$.
Feigin Frenkel and Reshetikhin proved the existence of a large commutative subalgebra $\mathcal{A}(z_1, \dots, z_N) \subset U(\mathfrak{g})^{\otimes N}$ containing $H_i$.
For $\mathfrak{g} = sl_2$, the algebra $\mathcal{A}(z_1, \dots, z_N)$ is generated by $H_i$ and the central elements of $U(sl_2)^{\otimes N}$.

In other cases, the algebra $\mathcal{A}(z_1, \ldots, z_N)$ has also some new generators known as higher Gaudin Hamiltonians. Their explicit construction for $\mathfrak{g} = gl_n$ was obtained in 2004 by D. Talalaev.
Let us we consider the problem of discussing what happens when the arbitrary points $z_1, \ldots, z_N$ appearing in the Lax matrix, and in the (quadratic) Hamiltonians $H_i$ glue together. Our limits of the Gaudin algebras when some of the points $z_1, \ldots, z_N$ glue together are as follows: We keep some points $z_1, \ldots, z_k$ "fixed", and let the remaining $N-k$ points glue to a new point $w$, via

$$z_{k+i} = w + s\, u_i, \quad i = 1, \ldots, N-k, \quad z_i \neq z_j; u_i \neq u_j,\, s \to 0. \quad (4)$$

Limits of the Lax matrix:

- $$L_G(z) \to L_2(z) = \sum_{i=1}^{k} \frac{X_i}{z - z_i} + \frac{\sum_{i=k+1}^{N} X_i}{z - w}, \; s \to 0. \tag{5}$$
- Too *naïve*: the number of Hamiltonians obtained from $L_2$ is not sufficient to yield complete integrability.
- Rescaling: let us introduce a new variable $\tilde{z}$ s.t. $z = w + s\tilde{z}$, and rewrite the Lax matrix

$$L_G = \sum_{i=1}^{k} \frac{X_i}{w + s\tilde{z} - z_i} + \sum_{i=k+1}^{N} \frac{X_i}{w + s\tilde{z} - w - su_i}.$$

Get the Lax matrix

$$L_1(z) = \, \mathrm{Res}_{s=0} L_G(\tilde{z}) = \sum_{i=k+1}^{N} \frac{X_i}{\tilde{z} - u_i}$$

Summing up: to the Lax matrix with generic (distinct) points $z_1, \dots, z_N$, we can associated, to the gluing $\{z_{k+1}, \dots z_N\} \to w$ the following pair of "Lax matrices":

$$L_1(z) = \sum_{i=k+1}^{N} \frac{X_i}{z - u_i}; \qquad L_2(z) = \sum_{i=1}^{k} \frac{X_i}{z - z_i} + \frac{\sum_{i=k+1}^{N} X_i}{z - w}. \quad (6)$$

We can choose the gluing procedure to be explicitly given by, e.g.,

$$z_{k+i} = w + s(z_{k+i} - w), \quad s \in (0,1) \qquad (7)$$

and, using invariance w.r.t. transformation of the spectral parameter $z \to z - w$, trade the matrix $L_1$ of (6) for

$$\tilde{L}_1(z) = \sum_{i=k+1}^{N} \frac{X_i}{z - z_i}.$$

In particular, in the example $N = 5$ and $z_3, z_4, z_5 \to w$, we would associate, to the Lax matrix $L = \sum_{i=1}^{5} \frac{X_i}{z - z_i}$ the two matrices

$$L_1(z) = \frac{X_3}{z - z_3} + \frac{X_4}{z - z_4} + \frac{X_5}{z - z_5},$$

$$L_2(z) = \frac{X_1}{z - z_1} + \frac{X_2}{z - z_2} + \frac{X_3 + X_4 + X_5}{z - w}.$$

The number of independent Hamiltonians gotten in this way is enough to ensure complete integrability of the model.
In some sense we recover integrability by adding one more pole.

## Proposition

*For every choice of $w \in \mathbb{C}$ the family of spectral invariants $\mathcal{H}^{(1)}, \mathcal{H}^{(2)}$ associated with the Lax matrices $L_1$ and $L_2$ satisfy the following properties:*

1. *The elements of $\mathcal{H}^{(1)}, \mathcal{H}^{(2)}$ commute w.r.t. the standard (diagonal) Poisson brackets on $\mathfrak{g}^N$;*
2. *The* dimension *of the Poisson commutative subalgebra $\mathcal{H}_{1,2,w}$ generated by the spectral invariants $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$ respectively associated with the Lax matrices $L_1$ and $L_2$, coincides with that of the spectral invariants associated with the generic Lax matrix $L_G$.*
3. *The physical Hamiltonian $H_G$ lies in $\mathcal{H}_{1,2,w}$.*
4. *Suitable spectral invariants obtained from $L_1$ and $L_2$ commute among themselves also in the quantum case.*

Quantization of the spectral invariants:
Problem: in the quantum Gaudin case (linear $r$ matrix structure)

$$[\mathrm{Tr}L^2(z), \mathrm{Tr}L^4(u)] \neq 0!$$

"Good" (that is, commuting) quantum Hamiltonians (QH) are obtained via the prescription

$$\text{"Det"}(\partial_z - L(z)) = \sum_{i=0}^{n} QH_{n-i}\partial_z^i$$

(Talalaev(04), Chervov-Talalaev 2006).
Question: is there a (possibly natural) simple and manageable algebraic framework for these and related models?

# Some more examples .

Let $\mathcal{K}$ be an associative algebra over $\mathbb{C}$. Let $\Pi \in Mat_n \otimes Mat_n$ be the permutation matrix: $\Pi(a \otimes b) = b \otimes a$, and let $L(z)$ be a matrix with elements in $\mathcal{K}((z))$, and $\overset{1}{L}(z) = L(z) \otimes 1, \overset{2}{L}(u) = 1 \otimes L(u)$. We say that $L(z)$ is of Gaudin type if

$$[L(z) \otimes 1, 1 \otimes L(u)] = [\frac{\Pi}{z-u}, L(z) \otimes 1 + 1 \otimes L(u)],$$

(*linear* $r$-matrix structure, the $r$-matrix being $r = \frac{\Pi}{z-u}$).

Let $K$ be an arbitrary constant matrix, and $n, k \in \mathbb{N}$, and $z_1, \dots, z_k$ arbitrary points in the complex plane. Consider

$$L(z) = K + \sum_{i=1,\dots,k} \frac{1}{z - z_i} \begin{pmatrix} \hat{q}_{1,i} \\ \dots \\ \hat{q}_{n,i} \end{pmatrix} \otimes \begin{pmatrix} \hat{p}_{1,i} & \dots & \hat{p}_{n,i} \end{pmatrix} =$$
$$K + \hat{Q}\ diag(\frac{1}{(z - z_1)}, \dots, \frac{1}{(z - z_k)})\ \hat{P}^t$$

where $\hat{p}_{i,j}, \hat{q}_{i,j},\ i = 1, \dots, n; j = 1, \dots, k$ are standard generators of the standard Heisenberg algebra

$$[\hat{p}_{i,j}, \hat{q}_{k,l}] = \delta_{i,k}\delta_{j,l},\ [\hat{p}_{i,j}, \hat{p}_{k,l}] = [\hat{q}_{i,j}, \hat{q}_{k,l}] = 0,$$

collected in $n \times k$-rectangular matrices $\hat{Q}, \hat{P}$ with elements $\hat{Q}_{i,j} = \hat{q}_{i,k},\ \hat{P}_{i,j} = \hat{p}_{i,j}$.

# Example 2 - standard

Consider $\underbrace{gl_n \oplus \ldots \oplus gl_n}_{N-times}$ and denote by $e^a_{kl}$ the standard basis element from the $a$-th copy of the direct sum $gl_n \oplus \ldots \oplus gl_n$. The standard Lax matrix for the Gaudin system is:

$$L_{gl_n-Gaudin\ \ standard}(z) = \sum_{a=1,\ldots,N} \frac{1}{z - z_a} \begin{pmatrix} e^i_{1,1} & \ldots & e^i_{1,n} \\ \ldots & \ldots & \ldots \\ e^i_{n,1} & \ldots & e^i_{n,n} \end{pmatrix} \quad (8)$$

$L_{gl_n}(z) \in Mat_n \otimes U(gl_n \oplus \ldots \oplus gl_n) \otimes \mathbb{C}(z)$. $z_a$ are a set of arbitrary but distinct complex parameters.

Let $\mathcal{K}$ be an associative algebra over $\mathbb{C}$. Let us call a matrix $T(z)$ with elements in $\mathcal{K}((z))$ a **Lax matrix of Yangian type** if

$$(1 \otimes 1 - \frac{\Pi}{z-u})(T(z) \otimes 1)\,(1 \otimes T(u)) =$$
$$(1 \otimes T(u))\,(T(z) \otimes 1)(1 \otimes 1 - \frac{\Pi}{z-u})$$

Or shortly:

$$R(z-u)\overset{1}{T}(z)\overset{2}{T}(u) = \overset{2}{T}(u)\overset{1}{T}(z)R(z-u).$$

**Examples:** Consider the Heisenberg algebra generated by $\hat{p}_i, \hat{q}_i$, $i = 1, ..., n$ and relations $[\hat{p}_i, \hat{q}_j] = \delta_{i,j}$, $[\hat{p}_i, \hat{p}_j] = [\hat{q}_i, \hat{q}_j] = 0$. Define

$$T_{Toda}(z) = \prod_{i=1,\dots,n} \begin{pmatrix} z - \hat{p}_i & e^{-\hat{q}_i} \\ -e^{\hat{q}_i} & 0 \end{pmatrix} \tag{9}$$

As it is known, this is a limit of the XXX Heisenberg $sl_2$ Transfer matrix,

$$T_{gl_n}(z) = \prod_{i=1,\dots,k} \left( 1_{n\times n} + \frac{1}{z - z_i} \begin{pmatrix} e^i_{1,1} & \dots & e^i_{1,n} \\ \dots & \dots & \dots \\ e^i_{n,1} & \dots & e^i_{n,n} \end{pmatrix} \right) \tag{10}$$

# Manin Matrices

Formally: matrices associated with linear maps between commutative rings.

Operative definition: $M_{ij}$ is (column) Manin if:

- Elements in the same column commute among themselves;
- Commutators of the cross terms in any $2 \times 2$ submatrix are equal:

$$[M_{ij}, M_{kl}] = [M_{kj}, M_{il}] \quad \text{e.g. } [M_{11}, M_{22}] = [M_{21}, M_{12}].$$

# Matrix Notation

As usual, let

$$\overset{1}{M} = M \otimes \mathbf{1}, \ \overset{2}{M} = \mathbf{1} \otimes M.$$

**Proposition**

*A matrix $M$ is a Manin matrix if*

$$[\overset{1}{M}, \overset{2}{M}] = \Pi[\overset{1}{M}, \overset{2}{M}]$$

Semiclassical case (straightforward)

**Proposition**

A Matrix $M$ is a Poisson-Manin matrix iff:

$$\{\overset{1}{M} \overset{\otimes}{,} \overset{2}{M}\} = \Pi\{\overset{1}{M} \overset{\otimes}{,} \overset{2}{M}\}.$$

A matrix with elements in a noncommutative ring $\mathcal{K}$ is called a Cartier-Foata matrix if elements from different *rows* commute with each other.

**Proposition**

*A Cartier-Foata matrix is a Manin matrix.*

The characteristic conditions for Manin matrices are trivially satisfied in this case.

**Proposition**

- $\partial_z - L_{gl_n}(z)$ *is Manin, where* $L_{gl_n-Gaudin}(z)$ *is the Lax matrix for the L ie algebra* $gl_n$ *(as well as its generalization to the affine algebra* $gl_n[t]$*).*
- $e^{-\partial_z} T_{gl_n}(z)$ *is Manin, where* $T_{gl_n}(z)$ *is the Lax (or "transfer") matrix for the Yangian algebra* $Y(gl_n)$*.*

# The determinant of a Manin matrix.

Let $M$ be a Manin matrix. Define the determinant of $M$ by column expansion:

$$det M = det^{column} M = \sum_{\sigma \in S_n} (-1)^{\sigma} \overset{\curvearrowright}{\prod_{i=1,\dots,n}} M_{\sigma(i),i}, \qquad (11)$$

where $S_n$ is the group of permutations of $n$ letters, and the symbol $\curvearrowright$ means that in the product $\prod_{i=1,\dots,n} M_{\sigma(i),i}$ one writes at first the elements from the first column, then from the second column and so on and so forth.

**Prop** The determinant of a Manin matrix does not depend on the order of the columns in the column expansion, i.e.,

$$\forall p \in S_n \quad det^{column} M = \sum_{\sigma \in S_n} (-1)^{\sigma} \overset{\curvearrowright}{\prod_{i=1,\dots,n}} M_{\sigma(p(i)),p(i)} \qquad (12)$$

# Cramer's formula

Let $M$ be a Manin matrix and denote by $M^\vee$ the adjoint matrix defined in the standard way, (i.e. $M^\vee_{kl} = (-1)^{k+l} det^{column}(\widehat{M}_{lk})$ where $\widehat{M}_{lk}$ is the $(n-1) \times (n-1)$ submatrix of $M$ obtained removing the l-th row and the k-th column. Then the same formula as in the commutative case holds true, that is,

$$M^\vee M = det^c(M)\ Id \tag{13}$$

**Remark**: We can consistently define determinant of the minors since any submatrix of a Manin matrixis a Manin matrix.

# An application to the Knizhnik-Zamolodchikov equation

We can give a very simple proof of the formula relating the solutions of KZ to the solutions of the equation defined by the formula:

$$det(\partial_z - L(z))Q(z) = 0.$$

In "general", the standard KZ-equation for $gl_n$, is given by:

$$\left(\partial_z - \sum_{i=1...k} \frac{E^{ab} \otimes \pi_i(e^{(i)}_{ab})}{z - z_i}\right)\Psi(z) = \pi(\partial_z - \kappa L_G(z))\Psi(z) = 0 \quad (14)$$

where $\pi = (V_1 \otimes ... \otimes V_k)$ is a representation of $U(\mathfrak{gl}_n)^{\otimes k}$ and $\Psi(z)$ is a $\mathbb{C}^n \otimes V_1 \otimes ... \otimes V_k$-valued function. $L_G(z)$ is the Lax matrix of the quantum Gaudin system.

### Proposition

*Let $\Psi(z)$ be a solution of the KZ-equation; Then:*

$$\forall i = 1, ..., n \qquad \pi(det(\partial_z - L_G(z)))\Psi_i(z) = 0$$

**Proof** The adjoint matrix $(\partial_z - L_G(z))^{adj}$ exists, so that

$$\begin{aligned}
&\pi(\partial_z - L_{Gaudin}(z))\Psi(z) = 0, \Rightarrow \\
&\pi((\partial_z - L_{Gaudin}(z))^{adj})\pi(\partial_z - L_{Gaudin}(z))\Psi(z) = 0, \\
&\text{hence} \quad \pi(det(\partial_z - L_{Gaudin}(z)))Id \ \ \Psi(z) = 0, \\
&\text{explicitly } \forall i = 1, ..., n \quad \pi(det(\partial_z - L_{Gaudin}(z)))\Psi_i(z) = 0.
\end{aligned} \tag{15}$$

# Further properties of MMs

- The inverse of a Manin matrix $M$ is again Manin.
- Schur's formula for the determinant of block matrices holds:

$$det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = det(A)det(D - CA^{-1}B) = det(D)det(A - BD^{-1}C)$$

.

- The Cayley-Hamilton theorem: $det(t - M)|_{t=M} = 0$ and Netwon identities hold

# Newton Identities

Consider the families of symmetric functions in $n$ variables:

1. $\sigma_k = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} \prod_{p=1,\ldots,n} \lambda_{i_p}$, $i = 1, \ldots, n$, the *elementary* symmetric functions.
2. $\tau_k = \sum_{p=1,\ldots,n} \lambda_p^k$, $k > 0$, the power sums.

In the case of matrices with commuting entries, the family $\{\sigma_i\}, i = 1, \ldots, n$ and $\{\tau_i\}, i = 1, \ldots, n$ are related by the *Newton* identities:

$$(-1)^{k+1} k \sigma_k = \sum_{i=0,\ldots,k-1} (-1)^i \sigma_i \tau_{k-i}. \tag{16}$$

**Theorem**

*The Newton identities between $TrM^k$ and the coefficients of the expansion of $\det(t + M)$ in powers of $t$ hold for Manin matrices..*

# Talalaev's Thm

**Theorem (Talalaev,2004)**

*Let $L(z)$ be the Lax matrix of the $gl_r$-Gaudin model, that is, let $L(z)$ satisfy the $r$-matrix commutation relations*

$$[L(z) \otimes 1, 1 \otimes L(u)] = [\frac{\Pi}{z-u}, L(z) \otimes 1 + 1 \otimes L(u)]. \tag{17}$$

*Consider the differential operator in the variable $z$* $det\left(\partial_z - L(z)\right) = \sum_{i=0,\dots,r} QH_i(z)\partial_z^i$; *Then:*

$$\forall i, j \in 0, \dots, r, \text{ and } u, v \in \mathbb{C}, \quad [QH_i(z)|_{z=u}, QH_j(z)|_{z=v}] = 0. \tag{18}$$

**Idea of the proof**[Ta06]. The quantum determinant in the Yangian (RTT=TTR) case is basically the determinant of the matrix

$$\mathrm{e}^{-\mathsf{h}\partial_z} T(z).$$

In the "semiclassical" limit, we have

$$\mathrm{e}^{-\mathsf{h}\partial_z} T(z) - \mathbf{1} = \mathsf{h}(L(u) - \partial_z) + O(\mathsf{h}^2).$$

# Application: quantization of traces

As recalled above, traces of the powers of the Gaudin Lax matrix do not commute at the quantum level. A good strategy is *not* to consider these quantities, but rather the traces of the powers of the corresponding Manin matrices, i.e.,

$$\mathrm{Tr}\left((\partial_z - \hat{L}(z))^k\right) = \sum_{j=0}^{k} (QTr)_j^k(z)\partial_z^{k-j}, \quad k = 1, \ldots, r. \qquad (19)$$

Thanks to the Newton identities, these objects will commute among each other.

As it is easily seen, there is a recursion relation of the form $QTr^k_{j+1}(z) \simeq QTr^{k+1}_j(z)$, and hence, to obtain the expected number of independent quantities, we can consider simply the coefficients $QTr^k_k$, that is the coefficients of zeroth order of each differential "polynomial" in (19).
These quantities are given by the traces of matrices $\hat{L}_k(z)^{[n]}$, that can be called are "quantum powers" of $L(z)$, defined by the Faà di Bruno formula

$$\hat{L}_k^{[0]}(z) = Id, \qquad \hat{L}_k^{[i]}(z) = \hat{L}_k^{[i-1]}(z)\hat{L}_k(z) - \frac{\partial}{\partial z}(\hat{L}_k^{[i-1]}(z)).$$

# Inversion properties

**Theorem** Let $M$ be a Manin matrix, and assume that a two sided inverse matrix $M^{-1}$ exists (i.e. $M^{-1}M = MM^{-1} = 1$). Then $M^{-1}$ is again a Manin matrix.

Let us show that a theorem of Enriquez Rubtsov, and Babelon Talon about "quantization" of separation relations – follows as a particular case from this theorem.

# BT+ER Theorem

Let $\{\alpha_i, \beta_i\}_{i=1,\ldots,g}$ be a set of quantum "separated" variables, i.e. satisfying the commutation relations

$$[\alpha_i, \alpha_j] = 0, [\beta_i, \beta_j] = 0, [\alpha_i, \beta_j] = f(\alpha_i, \beta_i)\delta_{ij}, i, j, = 1 \ldots, g.$$

satisfying a set of equations (i.e., quantum Jacobi separation relations) of the form

$$\sum_{j=1}^{g} B_j(\alpha_i, \beta_i) H_j + B_0(\alpha_i, \beta_i) = 0, i = 1, \ldots, g, \qquad (20)$$

for a suitable set of quantum Hamiltonians $H_1, \ldots, H_n$. One assumes that the ordering in the expressions $B_a, a = 0, \ldots g$ between $\alpha_i, \beta_i$ has been chosen, and that the operators $H_i$ are, as it is written above, on the right of the $B_j$. Then the statement is that the quantum operators $H_1, \ldots, H_n$ fulfilling (20) commute among themselves).

# Proof via Manin property

The equations (20) can be compactly written, in matrix form, as

$$B \cdot H = -V, \quad \text{with } B_{ij} = R_j(\alpha_i, \beta_i), V_i = B_0(\alpha_i, \beta_i),$$

and thus one is lead to consider the $g \times (g+1)$ matrix

$$A = \begin{pmatrix} V_1 & B_{1,1} & \dots & B_{1,g} \\ \dots & \dots & \dots & \dots \\ V_g & B_{g,1} & \dots & B_{g,g} \end{pmatrix}$$

Thanks to the functional form of the $B_{ij}$'s and of the $V_i$'s, this matrix is a "Cartier-Foata" matrix (i.e., elements form different rows commute among each other), and hence, *a fortiori* a Manin matrix.

Given such a $g \times (g+1)$ Cartier-Foata matrix $A$, we consider the $(g+1) \times (g+1)$ Cartier-Foata (and hence, Manin) matrix:

$$\tilde{A} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ V_1 & B_{1,1} & \dots & B_{1,g} \\ \dots & \dots & \dots & \dots \\ V_g & B_{g,1} & \dots & B_{g,g} \end{pmatrix}$$

Now it is obvious that the solutions $H_i$ of quantum separation equation are the elements of the first column of the inverse of $\tilde{A}$, and namely

$$H_i = (\tilde{A}^{-1})_{i+1,1}, i = 1, \dots, g.$$

Since $\tilde{A}$ is Cartier-Foata, its inverse is Manin, and thus the commutation of the $H_i$'s can be obtained from the inversion theorem for Manin matrices..

# Schur's formula

Consider a Manin matrix $M$ of size $n$, and denote its block as follows:

$$M = \begin{pmatrix} A_{k\times k} & B_{k\times n-k} \\ C_{n-k\times k} & D_{n-k\times n-k} \end{pmatrix} \quad (21)$$

Assume that $M, A, D$ are invertible. Then the same formulas as in the commutative case hold, i.e:

$$det^c(M) = det^c(A)det^c(D - CA^{-1}B) = det^c(D)det^c(A - BD^{-1}C). \quad (22)$$

and the Schur's complements, $D - CA^{-1}B$, $A - BD^{-1}C$ are Manin.

**Application 1:**The Weinstein–Aronszajn formula for Manin matrices. Let $A, B$ be $n \times k$ and $k \times n$ Manin matrices with pairwise commuting elements: $\forall i, j, k, l : [A_{ij}, B_{kl}] = 0$, then

$$det^{col}(1_{n\times n} - AB) = det^{col}(1_{k\times k} - BA). \tag{23}$$

The matrix:

$$\begin{pmatrix} 1_{k\times k} & B \\ A & 1_{n\times n} \end{pmatrix}, \text{ is Manin.}$$

Applying Schur's formula one obtains the result. In particular, for

$$M_n := \mathbf{1}_n - \sum_{\alpha=1}^{k} x_\alpha \otimes y_\beta \quad x_\alpha, \in \mathcal{K}^n, y_\beta \in \mathcal{K}^{*\,n},$$

$det(M_n) = \det(\mathbf{1}_k - S_k)$, where $[S_k]_{\alpha,\beta} = \langle x_\alpha, y_\beta \rangle$.

**Application 2** An identity by Mukhin, Tarasov, Varchenko. [MTV06]. Consider $\mathbb{C}[p_{i,j}, q_{i,j}]$, $i = 1, ..., n; j = 1, ..., k$, endowed with the standard Poisson bracket:
$\{p_{i,j}, q_{k,l}\} = \delta_{i,k}\delta_{j,l}$, $\{p_{i,j}, p_{k,l}\} = \{q_{i,j}, q_{k,l}\} = 0$.
Consider their quantization $\hat{p}_{i,j}, \hat{q}_{i,j}$, $i = 1, ..., n; j = 1, ..., k$, with the relations $[\hat{p}_{i,j}, \hat{q}_{k,l}] = \delta_{i,k}\delta_{j,l}$, $[\hat{p}_{i,j}, \hat{p}_{k,l}] = [\hat{q}_{i,j}, \hat{q}_{k,l}] = 0$.
Collect the variables in $n \times k$-rectangular matrices $Q_{cl}, P_{cl}, \hat{Q}, \hat{P}$, and let $K_1, K_2$ be $n \times n, k \times k$ matrices with elements in $\mathbb{C}$. Let us introduce:

$$L^q(z) = K_1 + \hat{Q}(z - K_2)^{-1}\hat{P}^t, \quad (24)$$

$$L^{cl}(z) = K_1 + Q_{cl}(z - K_2)^{-1}P_{cl}^t \quad (25)$$

These are Lax matrices of Gaudin type.

$$Wick(det(\lambda - L^{cl}(z))) = det(\partial_z - L^q(z)) \tag{26}$$

Here we denote by *Wick* the linear map:
$\mathbb{C}[\lambda, p_{i,j}, q_{i,j}](z) \rightarrow \mathbb{C}[\partial_z, \hat{p}_{i,j}, \hat{q}_{i,j}](z)$, defined as:

$$Wick(f(z)\lambda^a \prod_{ij} q^{c_{ij}} \prod_{ij} p_{ij}^{b_{ij}}) = f(z)\partial_z^a \prod_{ij} \hat{q}_{ij}^{b_{ij}} \prod_{ij} \hat{p}_{ij}^{b_{ij}} \tag{27}$$

"Wick or normal ordering" product (w.r.t. the "dynamical variables" $\{\mathbf{q}, \mathbf{p}\}$ as well as the "spectral" variables $z, \partial_z$).

To show this we consider the following block matrix:

$$MTV = \begin{pmatrix} z - K_2 & \hat{P}^t \\ \hat{Q} & \partial_z - K_1 \end{pmatrix} \tag{28}$$

It is easy to see that $MTV$ is a Manin matrix. Further – as it was observed by Mukhin, Tarasov and Varchenko –, the Lax matrix of the form above (24) appears as the Schur's complement $''D - CA^{-1}B''$ of the matrix MTV:

$$\partial_z - K_1 - \hat{Q}(z - K_2)^{-1}\hat{P}^t = \partial_z - L^q(z).$$

So by Schur's theorem, we get

$$det^c(MTV) = det(z - K_2)det^c(\partial_z - K_2 - \hat{Q}(z - K_2)^{-1}\hat{P}^t). \tag{29}$$

Now remark that in $det^c(MTV)$ all variables $z, \hat{q}_{ij}$ stand on the left of the variables $\partial_z, \hat{p}_{ij}$. This is due to the column expansion of the determinant (e.g., $z, \hat{q}_{ij}$ stand in the first $n$-th columns of MTV).

**A few "No go" Facts** Let $M$ be a Manin matrix with elements in the associative ring $\mathcal{K}$.

**Fact** In general $det(M)$ is not a central element of $\mathcal{K}$. This should be compared with the quantum matrix group $Fun_q(GL_n)$, where $det_q$ is central. The reason why this property does not hold for Manin matrices is that their defining relations are half of those of quantum matrix groups.

**Fact** In general $[TrM^k, TrM^m] \neq 0, \quad [TrM, det(M)] \neq 0$. Actually, for this commutativity property one needs stronger conditions like the Yang-Baxter relation $R\overset{1}{T}\,\overset{2}{T} = \overset{2}{T}\,\overset{1}{T}R$.

**Fact** In general $M^k, k = 2, \ldots,$ is not a Manin matrix nor the sum of two Manin matricesis Manin.

**Fact** Let $M$ be a Manin matrix; then in general $det(e^M) \neq e^{Tr(M)}$, $log(det(M)) \neq Tr(log(M))$.

# Quantum SoV

Let us briefly discuss some results and conjectures about the problem of separation of variables as formulated by E.Sklyanin. The program is not yet completed. The ultimate goal in this framework is to construct coordinates $\alpha_i, \beta_i$ such that a joint eigenfunction of all hamiltonians will be presented as product of functions of one variable:

$$\Psi(\beta_1, \beta_2, ...) = \prod_i \Psi^{1-particle}(\beta_i).$$

We consider this construction at the quantum level, trying to frame in this "Manin matrices" realm some ideas of Sklyanin and others.

Let us remind that, in the classical case, the construction of separated variables for the systems we are considering goes, somewhat algorithmically[1], as follows:

**Step 1** One considers, for a $gl_n$ model, along the Lax matrix $L(z)$, the matrix $M = \lambda - L(z)$ and its classical adjoint $M^\vee$.

**Step 2** One takes a vector $\psi$ by means suitable linear combination of columns (or rows) of $M^\vee$; in the simplest case, one can take $\psi$ to be one of the columns, say the last of $M^\vee$. One seeks for pairs $(\lambda_i, z_i)$ that solve

$$\psi_i = 0, i = 1, \ldots, n.$$

[1] We are herewith sweeping under the rug the problem known as "normalization of the Baker Akhiezer function".

Step 3 To actually solve this problem, one proceeds as follows. Each component $\psi_i$ of $\psi$ is a polynomial of degree at most $n-1$, one can form, out of $\psi$, the matrix $M_\psi$ defined as:

$$[M_\psi]_{j,i} = \mathrm{res}_{\lambda=0}\psi_i\lambda^{n-j-1},\ i,j = 1,\ldots,n.$$

Step 4 The separation coordinates are given by pairs $(\lambda_i, z_i)$ where $z_i$'s are roots of $\mathrm{Det}(M_\psi)$ and $\lambda_i$ are the corresponding values of $\lambda_i$ [2], that can be obtained, e.g., via the Cramer's rule. By construction , the Jacobi separation relations are the equation(s) of the spectral curve, $\mathrm{Det}(\lambda - L(z) = 0$.

[2] In the quadratic $R$-matrix case, actually one has to take as canonical momenta, the logarithms of these $\lambda_i$.

# Quantum case: the Yangian

Let $T(z)$ be a Lax matrix of the Yangian type, so $(1 - e^{-\partial_z} T(z))$ is a Manin matrix and its adjoint matrix can be calculated by standard formulas. Let us denote by $M_{i,j}(z)$ the matrix of the coefficients of expansion in left powers of $e^{-\partial_z}$ of the elements of the last column of $(1 - e^{-\partial_z} T(z))^{\vee}$.
I discuss in the case $n = 3$ how to results by Sklyanin can be framed in our picture ( the following arguments hold (=have been checked) for $n = 2, 3, (4)$).

**Fact**

$M_{i,j}(z)$ **is a Manin matrix.**

- Define $B(z) = Det^{column}(M(z))$; then

$$[B(z), B(u)] = 0. \tag{30}$$

- Consider any root $\beta$ of the equation $B(u) = 0$. Then the system of 3 equations for the single variable $\alpha$

$$\begin{pmatrix} M_{1,0}(z)|_{z\to\beta} & \dots & M_{1,2}(z)|_{z\to\beta} \\ \dots & \dots & \dots \\ M_{3,0}(z)|_{z\to\beta} & \dots & M_{3,2}(z)|_{z\to\beta} \end{pmatrix} \begin{pmatrix} \alpha^2 \\ \alpha \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has a unique solution.

Consider all the roots $\beta_i$ of the equations $B(u) = 0$, and the corresponding variables $\alpha_i$. Then:

- The variables $\alpha_i, \beta_i$ satisfy the following commutation relations:

$$[\alpha_i, \beta_j] = -\alpha_i \delta_{i,j}, \tag{31}$$

$$[\alpha_i, \alpha_j] = 0, \qquad [\beta_i, \beta_j] = 0, \tag{32}$$

- $\alpha_i, \beta_i$ satisfy the "quantum characteristic equation":

$$\forall i: \quad det(1 - e^{-\partial_z} T(z))|_{z \to \beta_i;\ e^{-\partial_z} \to \alpha_i} = 0 \tag{33}$$

- If $T(z)$ is generic, then variables $\alpha_i, \beta_i$ are "quantum coordinates" i.e. all elements of the algebra $R$ can be expressed via $\alpha_i, \beta_i$ and the center (Casimirs) of the algebra $R$.