# Discrete evolution operator for $q$-deformed isotropic top

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Based on the paper
A.P. Isaev and P.N. Pyatov, "Spectral extension of the quantum group cotangent bundle", arXiv:0812.2225[math.QA], to appear in Comm. Math. Phys. (2009).

# Plan

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**Braid group $\mathcal{B}_{M+1}$**

is generated by invertible elements $\sigma_i$ ($i = 1, \dots M$) subject relations:

$$\text{Braid}: \quad \sigma_i \, \sigma_{i+1} \, \sigma_i = \sigma_{i+1} \, \sigma_i \, \sigma_{i+1} \; ,$$

$$\text{Locality}: \quad \sigma_i \, \sigma_j = \sigma_j \, \sigma_i \quad (|i - j| > 1) \; ,$$

Finite-dimensional quotient:

Hecke algebra $H_{M+1}(q)$: $(\sigma_i - q)(\sigma_i + q^{-1}) = 0$,

where $q$ is a parameter such that

$$j_q := \frac{q^j - q^{-j}}{q - q^{-1}} \neq 0 \quad (\forall j = 2, 3, \dots)$$

# 2. Algebraic solutions of YBE and (anti)symmetrizers

Let $x, y$ be *spectral* parameters. The Yang-Baxter equation (YBE) is:

$$\boxed{\sigma_n(x)\,\sigma_{n+1}(xy)\,\sigma_n(y) = \sigma_{n+1}(y)\,\sigma_n(xy)\,\sigma_{n+1}(x)} \quad (\forall n).$$

where "baxterized" elements $\sigma_n(x) \in H_{M+1}$ are ( M.Jimbo (1986)):

$$\sigma_n(x,a) = \frac{\sigma_n - x\sigma_n^{-1}}{a - xa^{-1}} \in H_{M+1} \quad (n = 1, \dots, M) \ ,$$

and, for $a = \pm q^{\pm 1}$, satisfy unitary condition: $\sigma_n(x,a)\sigma_n(x^{-1},a) = 1$. Define elements $A^{(k)}(a) \in H_{M+1}$ by recursive relations

( M.Jimbo (1986)):

$$A^{(1)} = 1 \ , \quad A^{(k+1)}(a) = A^{(k)}(a)\,\sigma_k(a^{2k}, -a^{-1})\,A^{(k)}(a) \ ,$$

which are symmetrizers for $a = -q^{-1}$ and antisymmetrizers for $a = q$.

# 2. $R$-matrices

Let $V$ be a finite dimensional **C**-linear space. For any operator $X \in \mathrm{End}(V \otimes V)$ and integers $i > 0$, $j > 0$ we denote

$$X_{i\,i+1} := I^{\otimes(i-1)} \otimes X \otimes I^{\otimes(j-1)} \in \mathrm{End}(V^{\otimes(i+j)}),$$

where $I \in \mathrm{Aut}(V)$ is the identity operator.

**Def 1.** An operator $\hat{R} \in \mathrm{Aut}(V \otimes V)$ is called an $\mathrm{R}$-*matrix* if

$$\boxed{\hat{R}_{12}\,\hat{R}_{23}\,\hat{R}_{12} \;=\; \hat{R}_{23}\,\hat{R}_{12}\,\hat{R}_{23}}\,.$$

Any R-matrix defines the representation $\rho_R$ of $\mathcal{B}_{M+1}$:

$$\rho_R(\sigma_i) = \hat{R}_{i\,i+1}, \quad 1 \leq i \leq M.$$

An R-matrix $\hat{R}$ is called a Hecke type R-matrix if

$$\boxed{(\hat{R} - q\,\mathbf{1})(\hat{R} + q^{-1}\,\mathbf{1}) = 0\,,} \quad (\mathbf{1} = I \otimes I)\,.$$

In this case $\rho_R$ gives a representation of the Hecke algebra $\mathcal{H}_{M+1}(q)$.

# 2. $GL_q(n)$ type R-matrices

Consider R-matrix image $\mathcal{A}^{(k)} := \rho_R(A^{(k)}(q))$ of antisymmetrizer $A^{(k)}(q) \in \mathcal{H}_{M+1}(q)$:

$$\mathcal{A}^{(k+1)} = \frac{[k]_q}{[k+1]_q} \mathcal{A}^{(k)} \left( \frac{q^k}{[k]_q} - \hat{R}_k \right) \mathcal{A}^{(k)} \in \mathrm{End}(V^{\otimes(k+1)}) \,.$$

**Def 2.** A Hecke type R-matrix $\hat{R}$ for $q$ – generic is called $GL_q(n)$ type R-matrix if it satisfies

$$1.)\ \ \mathcal{A}^{(n+1)} = 0 \ \Leftrightarrow\ \mathcal{A}^{(n)}\Big(\frac{q^n}{[n]_q} I - \hat{R}_n\Big)\mathcal{A}^{(n)} = 0 \,, \quad 2.)\ \ \mathrm{rk}(\mathcal{A}^{(n)}) = 1 \,.$$

An example of $GL_q(n)$ type R-matrix is the standard Drinfeld-Jimbo's R-matrix

$$\hat{R}^\circ \ = \sum_{i,j=1}^{n} q^{\delta_{ij}} E_{ij} \otimes E_{ji} \, + \, (q - q^{-1}) \sum_{i<j} E_{ii} \otimes E_{jj} \,,$$

**Def 3.** $\hat{R}$ is called *skew invertible* if $\exists\,\Psi \in End(V^{\otimes 2})$ such that

$$\mathrm{Tr}_{(2)}\hat{R}_{12}\Psi_{23} = \mathrm{Tr}_{(2)}\Psi_{12}\hat{R}_{23} = P_{13}\ ,$$

where $\mathrm{Tr}_{(i)}$ – trace in $i$-th space, and $P(v_1 \otimes v_2) = v_2 \otimes v_1 \quad (\forall\, v_{1,2} \in V)$.

With any skew invertible $\hat{R}$ we associate operators $D, C \in \mathrm{End}(V)$:

$$D_1 = \mathrm{Tr}_{(2)}\Psi_{12}\ , \quad C_2 = \mathrm{Tr}_{(1)}\Psi_{12}\ ,$$

Then, we define a pair of quantum traces ($q$-traces)

$$Y \mapsto \mathrm{Tr}_{D}(Y) := \mathrm{Tr}(D\,Y)\,, \quad \mathrm{Tr}_{C}(Y) := \mathrm{Tr}(C\,Y)\,, \quad Y \in \mathrm{End}_W(V)\,.$$

which possess many remarkable properties, e.g., ($\varepsilon = \pm 1$)

$$\mathrm{Tr}_{D(2)}(\hat{R}^{\varepsilon}_{12}\, Y_1\, \hat{R}^{-\varepsilon}_{12}) = I_1\, \mathrm{Tr}_{D}(Y)\,, \ \ \mathrm{Tr}_{C(1)}(\hat{R}^{\varepsilon}_{12}\, Y_2\, \hat{R}^{-\varepsilon}_{12}) = I_2\, \mathrm{Tr}_{C}(Y)\,,$$

$$\mathrm{Tr}_{C,D}(1,\dots,k)\left(\left[\hat{R}_{i\,i+1},\ Y^{(k)}\right]\right) = 0 \quad (\forall\, 1 < i < k\,,\ \forall\, Y^{(k)} \in \mathrm{End}_W(V^{\otimes k}))\,.$$

# 3. *RTT* and Reflection equation (RE) algebras

Quantized functions over matrix group (RTT algebra) (L.Faddeev,N.Reshetikhin,L.Takhtajan (1989)).

Let $\hat{R}$ be a skew invertible R-matrix. Consider an associative unital algebra generated by matrix components $\| T^i_j \|^{\dim V}_{i,j=1}$ which satisfy

$$\hat{R}_{12}\, T_1\, T_2 = T_1\, T_2\, \hat{R}_{12} \; .$$

The extension of this algebra by a set of components $\|(T^{-1})^i_j\|^{\dim V}_{i,j=1}$:

$$\sum_k T^i_k\,(T^{-1})^k_j \;=\; \sum_k (T^{-1})^i_k\, T^k_j \;=\; \delta^i_j\, 1 \; ,$$

is a Hopf algebra with coproduct, counit and antipode mappings:

$$\Delta(T^i_j) = \sum_k T^i_k \otimes T^k_j\, , \qquad \epsilon(T^i_j) = \delta^i_j\, , \qquad S(T^i_j) \;=\; (T^{-1})^i_j\, .$$

This algebra is called an *RTT algebra* and denoted by $\mathcal{F}[\hat{R}]$.

**Def 4.** Let $\hat{R}$ be a skew invertible R-matrix. An associative unital algebra $\mathcal{L}[\hat{R}]$ with generators $\|L_j^i\|_{i,j=1}^{\dim V}$ satisfying relations

$$L_1\, \hat{R}_{12}\, L_1\, \hat{R}_{12} \;=\; \hat{R}_{12}\, L_1\, \hat{R}_{12}\, L_1\;,$$

is called a *reflection equation (RE) algebra*.

**Example.** (L.Faddeev, N.Reshetikhin, L.Takhtajan (1989))
Let $\mathcal{A}$ be a quasitriangular Hopf algebra, $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$ - universal $R$-matrix and $T\colon \mathcal{A} \to \mathrm{End}(V)$. Consider $\mathcal{A}$-valued matrices

$$L^{(+)}{}^i_j \;=\; \langle id \otimes T^i_j,\, \mathcal{R}\rangle\,, \qquad L^{(-)}{}^i_j \;=\; \langle T^i_j \otimes id\,, \mathcal{R}^{-1}\rangle\,.$$

The components of $L^{(\pm)}$ satisfy (it follows from the YBE for $\mathcal{R}$)

$$\hat{R}_{12}\, L_2^{(\pm)}\, L_1^{(\pm)} = L_2^{(\pm)}\, L_1^{(\pm)}\, \hat{R}_{12}\,, \quad \hat{R}_{12}\, L_2^{(+)}\, L_1^{(-)} = L_2^{(-)}\, L_1^{(+)}\, \hat{R}_{12}\,,$$

where $\hat{R}_{12} = P_{12}\langle T_1 \otimes T_2,\, \mathcal{R}\rangle$. Then, the components of the matrix

$$L^i_j \;=\; \sum_k ((L^{(-)})^{-1})^i_k L^{(+)}{}^k_j \;=\; \langle id \otimes T^i_j\,, \mathcal{R}_{21}\mathcal{R}_{12}\rangle\,,$$

generate a reflection equation algebra.

Consider REA $\mathcal{L}[\hat{R}]$ for Hecke type $\hat{R}$ and introduce elements ($a_0 = 1$)

$$a_i = \mathrm{Tr}_{D(1,\dots,i)}\left(\mathcal{A}^{(i)} L_{\overline{1}} \dots L_{\overline{i}}\right), \quad p_i = \mathrm{Tr}_{D}(L^i) \quad (i \geq 1)$$

where $L_{\overline{1}} := L_1$, $\quad L_{\overline{k+1}} := \hat{R}_k\, L_{\overline{k}}\, \hat{R}_k^{-1}$. Elements $p_i$ and $a_i$ are central and called *power sums* and *elementary symmetric functions*, resp.

**Prop.** Cayley-Hamilton-Newton identities (A.P.I., O.Ogievetsky, P.Pyatov (1998))

$$i_q\, \mathrm{Tr}_{D(2,\dots,i)}(\mathcal{A}^{(i)} L_{\overline{2}} L_{\overline{3}} \dots L_{\overline{i}}) = (-1)^{i+1} \sum_{j=0}^{i-1} (-q)^j\, a_j\, L_1^{i-j-1} \quad \forall\, i \geq 2\,. \quad (1)$$

**Corollary 1.** Newton relations (multiply (1) by $L_1$ from the left and take $q$-trace $\mathrm{Tr}_{D(1)}$)

$$i_q\, a_i + (-1)^i \sum_{j=0}^{i-1} (-q)^j a_j\, p_{i-j} = 0 \quad \forall\, i \geq 1\,.$$

**Coroll. 2.** Cayley-Hamilton identity ($\hat{R} - GL_q(n)$ type R-matrix; take (1) for $i = n$)

$$\sum_{j=0}^{n} (-q)^j\, a_j\, L^{n-j} = 0\,.$$

**Proposition.** *The set of elementary SF* $\{a_j\}$ $(j = 1, \dots, n)$ *generate the whole center in REA* $\mathcal{L}[\hat{R}_{GL_q(n)}]$.

**Def 5.** A spectral extension of REA $\mathcal{L}[\hat{R}]$ for $GL_q(n)$ type $\hat{R}$-matrix is the extension of $\mathcal{L}[\hat{R}]$ by a set of invertible central elements $\mu_\alpha$ $(\alpha = 1, \dots, n)$ such that $[\mu_\alpha, L_j^i] = 0$ and

$$a_i = \sum_{1 \le j_1 < \dots < j_i \le n} \mu_{j_1} \mu_{j_2} \dots \mu_{j_i} \quad \forall\, i = 1, \dots, n.$$

Then, Cayley-Hamilton identity can be written in factorized form

$$\textstyle\sum_{j=0}^{n} (-q)^j a_j L^{n-j} = \prod_{\alpha=1}^{n} (L - q\mu_\alpha I) = 0 \,,$$

and we have a formal resolution the matrix unity

$$P^\alpha := \prod_{\substack{\beta=1 \\ \beta \neq \alpha}}^{n} \frac{(L - q\mu_\beta I)}{q(\mu_\alpha - \mu_\beta)} \; : \quad P^\alpha P^\beta = \delta_{\alpha\beta} P^\alpha \,, \quad \sum_{\alpha=1}^{n} P^\alpha = I \,,$$

so that: $L\, P^\alpha = P^\alpha L = q\mu_\alpha P^\alpha$.

**Def 6.** A *Heisenberg double (HD) algebra* of the *RTT* and RE algebras is an associative unital algebra $(\mathcal{F} \sharp \mathcal{L})[\hat{R}]$ generated by $T^i_j \in \mathcal{F}[\hat{R}]$ and $L^i_j \in \mathcal{L}[\hat{R}]$ subject to cross commutation relation

$$\boxed{\gamma^2\, T_1\, L_2 \;=\; \hat{R}_{12}\, L_1\, \hat{R}_{12}\, T_1\,,} \qquad (\gamma \in \{\mathbf{C}\backslash 0\}).$$

**Proposition.** *The elementary SF:* $a_i \in \mathcal{L}[\hat{R}] \subset (\mathcal{F}\sharp\mathcal{L})[\hat{R}]$ *commute with generators* $T^m_k$ *as*

$$\gamma^{2i}\, T^m_k\, a_i \;=\; a_i\, T^m_k \;-\; (q^2-1) \textstyle\sum_{j=1}^{i} (-q)^{-j} a_{i-j}\, (L^j\, T)^m_k \;\; \forall\, i \ge 0\,. \qquad (2)$$

Spectral extension of HD $(\mathcal{F} \sharp \mathcal{L})[\hat{R}]$ in case of $GL_q(n)$ type R-matrix.

For spectral extension of $(\mathcal{F} \sharp \mathcal{L})[\hat{R}_{GL_q(n)}]$ we have additional commutators

$$(P^\beta\, T)\, \mu_\alpha \;=\; \gamma^{-2}\, q^{2\delta_{\alpha\beta}}\, \mu_\alpha\, (P^\beta\, T) \qquad \forall\, \alpha, \beta = 1, \dots, n\,,$$

which are compatible with (2).

HD $(\mathcal{F} \sharp \mathcal{L})[\hat{R}]$ is interpreted as quantum group cotangent bundle,

# 5. Discrete time evolution on quantum group cotangent bundle

Consider sequence of automorphisms on the HD $(\mathcal{F} \sharp \mathcal{L})[\hat{R}]$

$$\{T,\ L\} \xrightarrow{\theta^k} \{T(k),\ L(k)\}\,, \quad \forall\, k = 0, 1, 2, \dots\,,$$

$$\begin{aligned} \hat{R}_{12}\, T_1(k)\, T_2(k) \;&=\; T_1(k)\, T_2(k) \hat{R}_{12} \\ \hat{R}_{12}\, L_1(k) \hat{R}_{12}\, L_1(k) \;&=\; L_1(k)\, \hat{R}_{12}\, L_1(k) \hat{R}_{12}\,, \\ \gamma^2\, T_1(k)\, L_2(k) \;&=\; \hat{R}_{12}\, L_1(k)\, \hat{R}_{12}\, T_1(k)\,. \end{aligned}$$

Here $k$ is a discrete time. For any $\hat{R}$-matrix these automorphisms can be realized as (Faddeev–Alekseev discrete time evolution for the quantum top)

$$T(k) = L^k\, T\,, \quad L(k) = L\,.$$

# 5. Discrete time evolution for $SL_q(n)$ case

Consider the case when *RTT* algebra is $SL_q(n)$ quantum group. In this case we require

$$\det_q(T) = \mathrm{Tr}_{(1,\dots,n)} \left( \mathcal{A}^{(n)} T_1 \, T_2 \cdots T_n \right) = 1 \; .$$

Discrete time evolution must conserve this relation, i.e., we have $\det_q(L^k\, T) = 1$ $(\forall k > 0)$. This leads to the conditions

$$a_n = \mathrm{Tr}_{D(1,\dots,n)} \left( \mathcal{A}^{(n)} L_{\overline{1}} \, L_{\overline{2}} \cdots L_{\overline{n}} \right) = q^{-1} \; , \quad \gamma^n = q \; .$$

We will investigate the discrete evolution for HD of $SL_q(N)$ type. The key point is that $\exists$ the special evolution operator $\Theta$:

$$T(k+1) = L\, T(k) = \Theta\, T(k)\, \Theta^{-1} \; , \quad L(k+1) = \Theta\, L(k)\, \Theta^{-1} = L \; .$$

For the case of "ribbon Hopf algebra" the Faddeev-Alekseev evolution is given by $\Theta =$ ribbon element.

# 5. Evolution operator $\Theta$ for $SL_q(n)$ case.

Thus, we have for $k = 1$:

$$L\,T = \Theta\,T\,\Theta^{-1}\;, \quad L = \Theta\,L\,\Theta^{-1},$$

and we assume $\Theta = \Theta(\mu_1, \dots, \mu_n)$, where $a_n = \prod_{\alpha=1}^{n} \mu_\alpha = q^{-1}$.

**Proposition.** *For the HD $(\mathcal{F} \sharp \mathcal{L})[\hat{R}_{SL_q(n)}]$ the evolution operator $\Theta(\mu_\alpha)$ is a solution of equations*

$$\Theta\big(\nabla^\alpha(\mu_\beta)\big) = q^{-1}\mu_\alpha^{-1}\,\Theta(\mu_\beta) \quad \forall\,\alpha = 1, \dots, n\,, \tag{3}$$

*where $\nabla^\alpha$ are finite shift operators $\nabla^\alpha(\mu_\beta) := q^{2X_{\alpha\beta}}\,\mu_\beta$ and the matrix $X$ is a Gram matrix*

$$X_{\alpha\beta} = \langle \vec{e}_\alpha^{\,*}, \vec{e}_\beta^{\,*} \rangle = \delta_{\alpha\beta} - \frac{1}{n} \quad (\alpha, \beta = 1, \dots, n)\,,$$

*for the set of vectors:* $\vec{e}_\alpha^{\,*} = \frac{1}{n}\,(\underbrace{-1, \dots, -1}_{(\alpha-1)\ \textit{times}}, n-1, -1, \dots, -1\,)$.

**Proof.** Applying from the left the projector $P^\alpha$ to both sides of $LT = \Theta T \Theta^{-1}$ we obtain

$$q\mu_\alpha (P^\alpha T) = \Theta(P^\alpha T)\Theta^{-1}, \quad \forall\, \alpha = 1, \ldots, n.$$

Multiplying this equality by $\Theta$ from the right and permuting $\Theta$ with $P^\alpha T$ in the left hand side we finally obtain the functional equation for $\Theta$:

$$\Theta\big(\nabla^\alpha(\mu_\beta)\big) = q^{-1}\mu_\alpha^{-1}\,\Theta(\mu_\beta) \quad (\alpha = 1, \ldots, n).$$

We look for a solution of this eq. as a series in $\mu_\alpha$:

$$\Theta(\mu_\alpha) = \sum_{\vec{k}\in\mathbb{Z}^{n-1}} c(\vec{k})\, \mu_1^{k_1}\, \mu_2^{k_2} \ldots \mu_{n-1}^{k_{n-1}}\,.$$

where $\mathbb{Z}^{n-1} = (k_1, \ldots, k_{n-1})$, $c(\vec{k})$ are coefficient functions and, taking into account condition $\prod_{\alpha=1}^{n} \mu_\alpha = q^{-1}$, we exclude one variable $\mu_n$ from the expansion.

As a result we obtain (partial solution):

**Proposition.** *In case $|q| < 1$ a solution is expressed via multidimensional theta-function*

$$\Theta^{(1)}(\mu_\alpha) = \theta(\vec{z}, \Omega) = \sum_{\vec{k}\in \mathbf{Z}^{n-1}} \exp\left\{\pi\mathrm{i}\,(\vec{k},\, \Omega\,\vec{k}) + 2\pi\mathrm{i}\,(\vec{k}\,,\, \vec{z})\right\},$$

*where $\tau$ is a modular parameter, $\Omega$ is $(n-1)\times(n-1)$ matrix of periods*

$$q = \exp(2\pi\mathrm{i}\,\tau), \quad q^{1/n}\mu_\alpha = \exp(2\pi\mathrm{i}\,z_\alpha) : \ \textstyle\sum_{\alpha=1}^{n} z_\alpha = 0,$$

$$\Omega_{\alpha\beta} = \tfrac{2\tau}{n} A^*_{\alpha\beta} = 2\tau\,(\delta_{\alpha\beta} - \tfrac{1}{n}),$$

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Expression $\Theta^{(1)}(\mu_\alpha)$ converges either if $|q| < 1$, or if $q$ is a rational root of unity, when the series is truncated.

The $(n-1)\times(n-1)$ matrix $A^*_{\alpha\beta}$ is a Gram matrix of a lattice $A^*_{n-1}$ dual to the root lattice $A_{n-1}$, since we have ${A^*}^{-1}_{\alpha\beta} = A_{\alpha\beta} = (\delta_{\alpha\beta} + 1)$ and $A_{\alpha\beta} = (e\alpha, e_\beta)$, where $e_\alpha$ are simple roots of $A_{n-1}$.

**Remark.** The quadratic form $(\vec{k},\, A^*\,\vec{k})$ is often referred to as Voronoi's principal form of the first type.

# 5. ”Noncompact” solution for the evolution operator Θ

**Proposition.** *In case* $|q| \geq 1$ *one can find another solution the evolution equations:*

$$\Theta^{(2)}(z_\alpha) \; := \; \exp\Bigl(-\frac{\pi \mathrm{i}}{2\tau} \textstyle\sum_{\beta=1}^{n} z_\beta^2\Bigr) \, .$$

Written in the independent variables $\vec{z} = \{z_1, \ldots, z_{n-1}\}$ it reads

$$\Theta^{(2)}(\vec{z}) \; = \; \exp\Bigl(-\frac{\pi \mathrm{i}}{\tau} \sum_{1 \leq \alpha \leq \beta \leq n-1} z_\alpha z_\beta\Bigr) \; = \; \exp\Bigl\{-\pi \mathrm{i} \, (\vec{z}, \, \Omega^{-1}\vec{z})\Bigr\} \, ,$$

where the inverse matrix of periods is

$$\Omega^{-1}_{\alpha\beta} = \frac{1}{2\tau} \, (\delta_{\alpha\beta} + 1) = \frac{1}{2\tau} \, A_{\alpha\beta} \; ,$$

and $A_{\alpha\beta} = \langle e_\alpha, e_\beta \rangle$ is the Gram matrix for the root lattice $A_{n-1}$. Note that the logarithmic change of variables: $\log(\mu_\alpha)/(2\pi i) = z_\alpha - \tau/n$ which was rather superficial in case of $\Theta^{(1)}$, is inevitable for the derivation of $\Theta^{(2)}$.

Finally, we comment on relation between the two evolution operators $\Theta^{(1)} = \theta(\vec{z}, \Omega)$ and $\Theta^{(2)}$. The relation is based on the identity for multidimensional theta functions

$$\theta(\Omega^{-1}\vec{z}, -\Omega^{-1}) = \Big(\det(\Omega/\mathrm{i})\Big)^{\frac{1}{2}} \exp\Big\{\pi\mathrm{i}(\vec{z}, \Omega^{-1}\vec{z})\Big\}\, \theta(\vec{z}, \Omega) .$$

With our particular matrix of periods $\Omega$ we find

$$\Theta^{(2)}(\vec{z}) = \frac{1}{\sqrt{n}}\Big(\frac{2\tau}{\mathrm{i}}\Big)^{\frac{n-1}{2}} \frac{\theta(\vec{z}, \Omega)}{\theta(\Omega^{-1}\vec{z}, -\Omega^{-1})} .$$

Note that theta function $\theta(\Omega^{-1}\vec{z}, -\Omega^{-1})$ (in the denominator) commutes with the elements of HD $\mathcal{F} \sharp \mathcal{L}[\hat{R}_{SL_q(n)}]$ and can be thought as an evolution operator on a ‘modular dual’ quantum cotangent bundle $\mathcal{F} \sharp \mathcal{L}[\hat{R}_{SL_{\tilde{q}}(n)}]$.

# 5. Example

In the $SL_q(2)$ case the evolution operator $\Theta^{(1)}$ becomes the Jacobi theta function:

$$\Theta^{(1)}(\mu_1) \;=\; \sum_{k\in\mathbb{Z}} q^{\frac{1}{2}k(k+1)}\mu_1^k \;=\; \sum_{k\in\mathbb{Z}} \exp(\pi\mathrm{i}\, k^2\tau + 2\pi\mathrm{i}\, kz_1) \;=\; \theta_3(z_1;\, q)\,,$$

where $q = \exp(2\pi\mathrm{i}\,\tau)$, $\mu_1 = \exp(2\pi\mathrm{i}\, z_1)q^{-1/2}$. A multiplicative form for $\Theta$ is

$$\frac{1}{\eta(q)}\Theta^{(1)}(\mu_1) = \prod_{n=1}^{\infty}(1+q^n\mu_1)(1+q^{n-1}/\mu_1) = \prod_{n=1}^{\infty}(1+q^n\sigma_1+q^{2n-1})\,,$$

where $\eta(q) = \prod_{n=1}^{\infty}(1-q^n)$. For dual evolution operator we have

$$\widetilde{\Theta}^{(1)}(\mu_1) \;= \sum_{k\in\mathbb{Z}} \exp(-\frac{\pi\mathrm{i}}{\tau}k^2 + \frac{2\pi\mathrm{i}}{\tau}kz_1) = \sum_{k\in\mathbb{Z}} \tilde{q}^{\frac{1}{2}k(k+1)}\tilde{\mu}_1^k\,,$$

where $\tilde{q} = \exp(-\frac{2\pi\mathrm{i}}{\tau})$, $\tilde{\mu}_1 = \exp(\frac{2\pi\mathrm{i}}{\tau}z_1)\tilde{q}^{-1/2}$.

# Summary

- What is a dual HD for the standard HD of $SL_q(n)$ type (which are centralizers for each other)?
- Explicit expressions for evolution operator $\Theta$ in the case of $B, C, D$ quantum groups. In these cases it will be interesting to find Gram matrices $A^*$ and their dual $A = (A^*)^{-1}$.
- It seems that, in $SL_q(n)$ case for $n > 2$, a multiplicative form for multidimensional theta-functions (which gives evolution operators) will explicitly depend only on ESF $\{a_k\}$.
- 3D analogue of RE (tetrahedron RE) were proposed in A.P.Isaev and P.P.Kulish, Mod. Phys. Lett. **A12** (1997) 427 (hep-th/9702013). The analog of 3D *RTT* algebra is also known: $R_{123} T_1 T_2 T_3 = T_3 T_2 T_1 R_{123}$. What kind of cross-commutation relations are needed to describe discrete evolution in 3D case?