MORE ON DISCRETE SPACETIME

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ISAAC NEWTON INSTITUTE FOR MATHEMATICAL SCIENCES
Cambridge (UK)
Quantum Integrable Discrete Systems (DISW01)
23-27 March 2009

1. AN ONTOLOGICAL INTERPRETATION OF THE STRUCTURE SPACETIME

Epistemological pressupositions: we can consider three levels of human knowledge in the comprehension of physic world

- Level 1: Physical magnitudes, such as distance, time interval, mass, event, force and so on, that are given by our senstions and perceptions.
- Level 2: Theoretical models, that are the generalitation of metrical properties given by measurements and numerical relations among them.
- Level 3: Fundamental concepts, representing the ontological properties of physical world given by our consciousness in an attemp to know the reality.

There must be some connections betwen the three levels. In Quantum Mechanics the theoretical models of microphysics in level 2 are related to observable magnitudes in level 1 by correspondence laws.

If we acceept level 3 should be connected to level 2 an inmediate question is to ask the justification of the rules governing the construction of theoretical systems.

For instancee, the unification of Quantum Mechanics and the theory of general Relativity should be made in level 2 where they belong to, but the underlying ontological background should be taken from level 3.

1. AN ONTOLOGICAL INTERPRETATION OF THE STRUCTURE SPACETIME (cont.)

The ontological interpretation Is it possible to make some Ansatz about the nature of these fundamental objects in level 3? If we take the extension as the first property of matter, as Descartes has claimed, space and time should be considered necessary at the beginning of a fundamental theory. We prefer the point of view that the most essential property of material objects is the capacity of producing effects in other objects, which was identified by Leibniz with the concept of force.

This interpretation of matter has been confirmed by modern philosophers who tried to explain cosmic beings in terms of two methaphysical principles.

> *"The essence of human being consists on realizing him-self in conscious actions, namely, on realizing being-in-himself, and being-for-himself... The essence of material being consists on realizing-it-self externally, therefore it is necessary a different principle of being-in-himself. If the principle of being-in-himself is called principle of conscience the principle of acting out there and being-in-others is called principle of matter"*
>
> (E. CORETH, *Methaphysik*)

> *"Matter is the constituting system of material beings in their two aspects: to be real by it-self and to be a system of potentialities. Potentiality is the capacity of producing actions and, some times, is the capacity of determining the structure of reality".*
>
> (J. ZUBIRI, *Space. Time. Matter*)

2. A RELATIONAL MODEL FOR THE DISCRETE SPACETIME

We take a network of material beings acting among themselves as the ontological background of the theoretical model in level 2 for the relational character of spacetime. This ontological background consists on a finite number of material beings (beings-in-others) endowed with the principle of causality.

A very suitable language to describe the network of causal effects among the material beings (hylions) in the concept of graphs.

Graph: $\Gamma = \{V, E\}$ V vertices, E edges. Every edge is incident with two vertices.

Geometrical objects in graphs (by combinatorial rules): paths circuit, length, distance, triangle, plane.

Embedding of a graph Γ into a manifold S. A drawing of Γ on S is a map of vertices and edges into points and curves of S which preserves the relations between edges and vertices.

We call an embedding of Γ into S if all the regions of the drawing are homeomorphic to plane E^2.

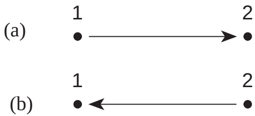
Important remark! The embedding is an artifice to use our intuition for a better understanding of the geometrical properties of the graph. The real networks of material beings and their interactions can be abstracted from this embedding. They are nowhere, as Penrose says.

From the properties of material beings infinite number of possible networks can be constructed, which are considered the arena where the elementary particle are moving. For this arena we present three particular models described by simple graphs:

2.1. THE CAUSAL LATTICE

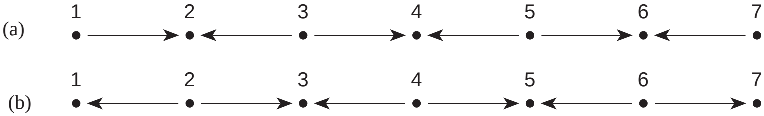
In the ontological level 3 the lattice is made out of material beings (hylions) acting among themselves

- Two interacting hylions



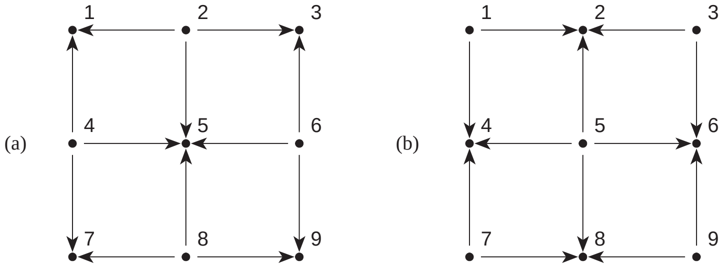
Action of 1 in 2 is a necessary condition for the action of 2 in 1.
Action of 2 in 1 is a necessary condition for the action of 1 in 2.

- Chain of mutual interacting hylions in the relation 1 to 2



We postulate that the actions of (a) are necessary condition for the actions of (b) and actions of (b) are necessary for the action of (a).

- Network of mutual interacting hylions in the relation 1 to 4



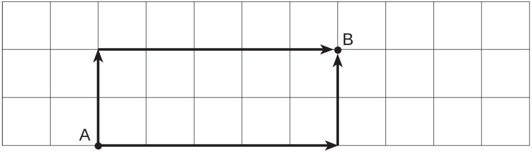
The actions of (a) are necessary conditions for the actions of (b)

The actions of (b) are necessary for new actions of (a)

This ontological model in level 3 can be considered an interpretation of the relational theory of spacetime in level 2

Consequences: Interpretation of Space

Because relational theories use only objects and relations among them we use graphs that are composed of vertices and edges. Take a graph corresponding to the set of relations in the causal lattice:



The logical structure is given in level 2 but it corresponds to some geometrical properties of physical space in level 1.

We take, for simplicity, a set of interacting points in relation 1 to 4.

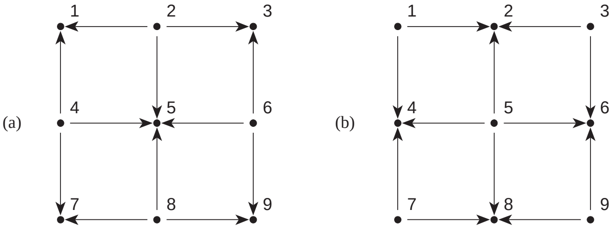
We can define:

Path, length, principal straight line, orthogonal, parallel lines, Cartesian coordinates, euclidean space.

This structure can be easily generalized to n-dim euclidean space by some network where each point is connected with no more and no less than 2n

Consequences: Interpretation of Time

In this case the graph corresponding to causal lattices is oriented: one vertex (the cause) precedes logically the other vertex (the effect)



In the graph (a) the vertices 2,4,6,8 are prior than vertices 1,3,5,7,9

In the graph (b) the vertices 1,3,5,7,9 are prior than vertices 2,4,6,8

We have a series of actions in level 2 that correspond logically to successive instances of time.

Tessellation: a covering of 2-dim surface repeating the same figure without overlapping or holes

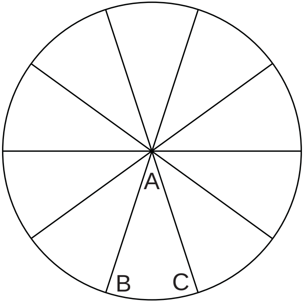
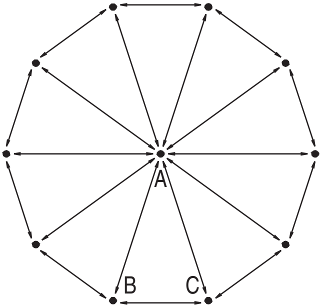


Fig. 1. Tessellation of S^2 (in stereographic projection) by reflecting in the sides of the spherical triangle $T(2, 2, 5)$



Graph obtained from spherical tessellation of Fig. 1.

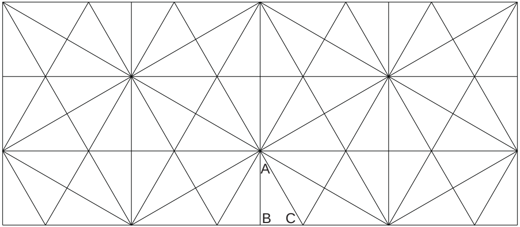
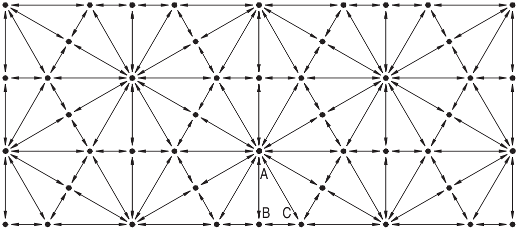


Fig. 2. Tessellation of E^2 generated by reflecting in the sides of the euclidean triangle $T(2,3,6)$



Graph obtained from euclidean tessellation of Fig. 2.

2.2. THE HYPERBOLIC LATTICE (cont.)

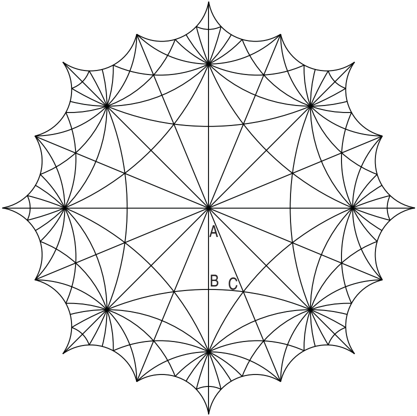
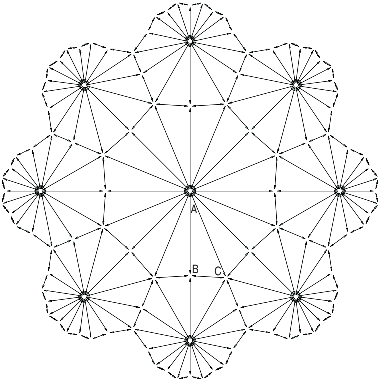


Fig. 3. Tessellation of H^2 generated by reflecting in the sides of the hyperbolic triangle $T(2, 3, 8)$



Graph obtained from hyperbolic tessellation of Fig. 3.

An hyperbolic causal lattice can be obtained from hyperbolic tessellations (which are generated by triangle reflection group) by removing the embedding surface and keeping the vertices and edges of triangles.

In a regular tessellation of spherical, euclidean or hyperbolic type the triangles are geodesic (the adges are geodesic lines) and the angles are submultiples of π

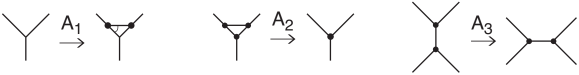
$$\alpha = \frac{\pi}{l},\ \beta = \frac{\pi}{m},\ \gamma = \frac{\pi}{n}$$

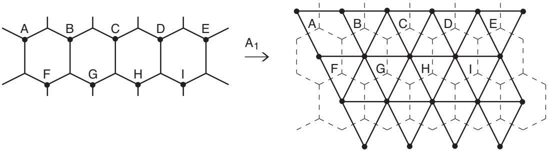
Tessellation.	Corresponding graph
Geodesic triangle	Combinatorial triad
Excess	Excess
$\epsilon = \alpha + \beta + \gamma - \pi$	$\delta = \frac{1}{l} + \frac{1}{m} + \frac{1}{n} - 1$
Area of triangle	Area of triad
$A = \lvert\alpha + \beta + \gamma - \pi\rvert$	$\sigma = \lvert\frac{1}{l} + \frac{1}{m} + \frac{1}{n} - 1\rvert$
Curvature	"Curvature"
Gauss-Bonet theorem	in a planar graph
$\epsilon = \int_A K d\sigma,\ K$ const.	
$K = \frac{\epsilon}{A} = \begin{cases} +1, & \text{spherical} \\ 0, & \text{euclidean} \\ -1, & \text{hyperbolic} \end{cases}$	$K = \frac{\delta}{\sigma} = \begin{cases} +1, & \text{spherical} \\ 0, & \text{euclidean} \\ -1, & \text{hyperbolic} \end{cases}$
	curvature in a planar graph!

2.3. THE EVOLUTION OF SIMPLICIAL LATTICE

Let $\mathcal{S}$ be a 2-dim network of N nodes with three edges arranged in the form of an hexagonal tessellation.

We associate a vector space $\mathcal{H}_n(n = 1, 2 \cdots N)$ to each element of $\mathcal{S}$. The state space $\mathcal{H}$ is the direct product of all the constituents. Local dynamics on $\mathcal{H}$ are created by the following generators that replace some pieces of $\mathcal{S}$ with the same borders to a new ones.





The evolution of the system $\mathcal{S}$ to $\mathcal{S}'$ is realized by Pachner moves with the use of combinatorial rules

$$\mathcal{S} \to \mathcal{S}' \to \mathcal{S}'' = \mathcal{S} \to \mathcal{S}'$$

Our model is not contained (embedded) in a continuous manifold, therefore the measure of length and time intervals is reduced to the problem of counting

3. R. PENROSE'S SPIN NETWORKS

Discrete magnitudes in QM give rise to continuous spacetime

Total angular momentum $\vec{J} = \vec{L} + \vec{S}$

$\vec{n}$-units large spin component

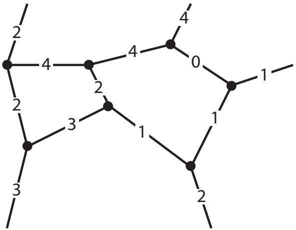
$\vec{n}$-units with large orbital component

Two $\vec{n}$-units interactions

$\longrightarrow$ angle

$\longrightarrow$ displacements

Emergent spacetime from directions and positions



Analogies with Our Model

- "An object is thus located either directionatelly or positionatelly in terms of its relations with other objects. One does not really need a space to begin with"
- The concept of euclidean geometry emerges from the interactions of units among themselves.

Differences with Our Model

- For the construction of discrete magnitudes one needs a background space out of which the geometrical space emerges as a network of relations among $\vec{n}$-unis.
- The elements of the spin networks are physical entities (elementary particles) with discrete magnitudes not evolving in time.

4. R. SORKIN'S CAUSAL SETS

- Following the philosophical ideas of Sakata and Taketani causal sets can be considered "substances" of the real world, different from the pure observations which an operationalist philosopher could only admit.
- Underlying continuous spacetime there is a discrete causal set (a locally finite ordered set of events connected by the principle of causality)
- The structure of spacetime is nothing more than the set of relations coming from interactions among physical events.

Analogies with Our Model

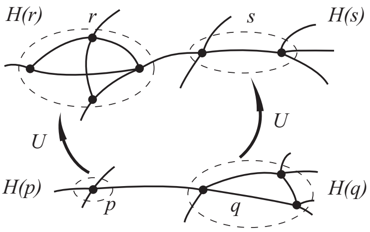
- Causal sets is based in some ontological reality (substance) consisting in cause, effect and its production by the cause.
- Three levels of knowledge: phenomena, physical theory, substance.

Differences with Our Model

- Use of embedding space, necessary to calculate the metric.
- Omission of Hilbert space in order to introduce laws of quantum probability in causal interactions

5. F. MARKOPOULOU'S QUANTUM CAUSAL HISTORIES

- Causal set = partially ordered set locally finite with preceding relation
- Quantum causal set = attach an Hilbert space to each event of a causal set.
- Quantum causal histories the evolution of a quantum causal set is implemented by unitary operators between Hilbert spaces.
- The set of all causal relations among fundamental objects can be taken as the ontological background for the relational theories of spacetime.



Analogies with Our Model

- Causal sets belong to ontological level
- Quantum operators introduce laws of probability
- Evolution of spin network given by combinatorial rules.

Differences with Our Model

- Lack of substantive character of fundamental objects.
- The last model is not embedded in a continuous manifold therefore the measure of distance and time intervals is reduced to the process of counting.

6. C. ROVELLI AND L. SMOLIN'S SPIN FOAM

- A spin foam is a labelled 2.complex whose faces are labelled by representations of some group G, the edges by the intertwiners in the group, and the vertices carry the evolution amplitudes.
- A spin network is a graph whose edges are labelled by representations of the group G and its nodes are labelled by the intertwiners. We regard spin networks as the "spacelike slices" of a spin foam: If we take a spacelike cut through a spin foam, we obtain a graph; its edges are cuts through the spin foam faces, and so we label them with the same representations. Its nodes are cuts of the spin foam edges and so we label them with intertwiners.

Analogies with Our Model

- Abstract spin foam non embedded in some preexisting continuous manifold.
- Transitions from an initial spin network S, to another spin network by a set of combinatorial rules.
- The spin network diagonalize the quantum area and volume operators whose spectrum was discovered to be discrete.

Differences with Our Model

- There is no ontological background for spin foams.

7. THE QUANTUM HARMONIC OSCILLATOR OF DISCRETE VARIABLE

We start from the orthogonal polynomials of a discrete variable, the Kravchuk polynomials $K_n^{(p)}(x)$ and the corresponding normalized Kravchuk functions

$$K_n^{(p)}(x) = d_n^{-1}\sqrt{\rho(x)}\, k_n^{(p)}(x),$$

where $d_n^2 = \frac{N!}{n!(N-n)!}(pq)^n$ is a normalization constant, $\rho(x) = \frac{N! p^x q^{N-x}}{x!(N-x)!}(pq)^n$ is the weight function, with $p > 0, \quad q > 0, \quad p + q = 1, \quad x = 0, 1, \dots N + 1.$

The Kravchuk functions satisfy the orthonormality condition

$$\boxed{\sum_{x=0}^{N} K_n^{(p)}(x) K_{n'}^{(p)}(x) = \delta_{nn'},}$$

and the following difference and recurrence equations:

$$\sqrt{pq(N-x)(x+1)}K_n^{(p)}(x+1) + \\ + \sqrt{pq(N-x+1)x}K_n^{(p)}(x-1) + [x(p-q) - Np + n]\, K_n^{(p)}(x) = 0,$$

$$\sqrt{pq(N-n)(x+1)}K_{n+1}^{(p)}(x) + \\ \sqrt{pq(N-n+1)n}K_{n-1}^{(p)}(x) + [n(q-p) + Np - x]\, K_n^{(p)}(x) = 0,$$

7. THE QUANTUM HARMONIC OSCILLATOR OF DISCRETE VARIABLE (cont.)

From the properties of the Kravchuk polynomials we can construct raising and lowering operators for the Kravchuk functions

$$L^{+}(x,n)K_n^{(p)}(x) = \sqrt{pq(N-n)(n+1)}K_{n+1}^{(p)}(x),$$

$$L^{-}(x,n)K_n^{(p)}(x) = \sqrt{pq(N-n+1)n}K_{n-1}^{(p)}(x).$$

In order to give a physical interpretation of the difference equation and raising and lowering operators for the Kravchuk functions we take the limit when N goes to infinity and the discrete variable x becomes continuous s .

First of all, we take the limit of Kravchuk functions. We write

$$K_n^{(p)}(x) \xrightarrow[n\to\infty]{} \left\{\frac{1}{2^n n!}\right\}^{1/2} \left\{\frac{1}{\sqrt{2\pi Npq}}e^{-s^2}\right\}^{1/2} H_n(s) = \psi_n(s),$$

where the last braket becomes the weight for the Hermite functions and the functions $\psi_n(s)$ are the solution of the continuous harmonic oscillator

$$\frac{1}{\sqrt{Npq}}\, L^{+}(x,n)K_n^{(p)}(x) \xrightarrow[h\to 0]{} \frac{1}{\sqrt{2}}\left\{s - \frac{d}{ds}\right\}\psi_n(s) = \sqrt{n+1}\,\psi_n(s),$$

$$\frac{1}{\sqrt{Npq}}\, L^{-}(x,n)\,K_n^{(p)}(x) \xrightarrow[\substack{N\to\infty \\ h\to 0}]{} \frac{1}{\sqrt{2}}\left\{s + \frac{d}{ds}\right\}\psi_n(s) = \sqrt{n}\,\psi_n(s).$$

Therefore the raising and lowering operators for the Kravchuk functions become, in the limit, creation and annihilation operators for the normalized Hermite functions.

we obtain

$$A^{+}Y_{jm} = \sqrt{\frac{(j+m)(j-m+1)}{2j}}\, Y_{j,m-1} = \frac{1}{\sqrt{2j}} J_{-} Y_{jm}$$

$$A^{-}Y_{jm} = \sqrt{\frac{(j-m)(j+m+1)}{2j}}\, Y_{j,m+1} = \frac{1}{\sqrt{2j}} J_{+} Y_{jm}$$

where J_+, J_- are the generators of SO(3) algebra.

For the commutation relations of these operators we have

$$(AA^{+} - A^{+}A)Y_{jm} = \frac{m}{j}\, Y_{jm} = \frac{1}{2j} 2J_z Y_{jm} = \left(1 - \frac{n}{j}\right) Y_{jm},$$

and then take the limit:

$$\boxed{\left[A, A^{+}\right] Y_{jm} = \left(1 - \frac{n}{j}\right) Y_{jm} \xrightarrow[j\to\infty]{} \left[a, a^{+}\right] \;\psi_n(s).}$$

For the anticommutation relation we have:

$$\left(AA^{+} + A^{+}A\right) Y_{jm} = \frac{1}{j}\left(j(j+1) - m^2\right) Y_{jm} = \frac{1}{j}\left(\vec{J}^2 - J_z^2\right) Y_{jm}.$$

and taking the limit

$$
\begin{aligned}
&(AA^{+} + A^{+}A)Y_{jm} \\
&\quad = \left\{(2n+1) - \frac{n^2}{j}\right\} Y_{jm} \xrightarrow[j\to\infty]{} \left(aa^{+} + a^{+}a\right)\psi_n(s) = (2n+1)\psi_n(s)
\end{aligned}
$$

This correspondence suggests that the operator algebra for the quantum harmonic oscillator on the lattice is expanded by the generators of the SO(3) groups.

The eigenvalues of the Hamilton operator on the lattice are connected with the index $m = j - n$ of the eigenvectors Y_{jm} These eigenvalues are equally separated by $\hbar\omega$ but finite $(m = -j, \ldots + j)$. The eigenvalues of the position operator on the lattice are connected with the index $m' = j - x$ of $Y_{jm'}$. These eigenvalues are equally separated by $\sqrt{\frac{\hbar}{M\omega}}$ but finite $(m' = -j, \ldots + j)$. Therefore the Planck constant $\hbar$ plays a role with respect to the discrete space coordinate similar to the discrete energy eigenvalues.

8. WAVE EQUATION FOR THE HIDROGEN ATOM WITH DISCRETE VARIABLES

Our model is based on the properties of generalized Laguerre polynomials as continuous limit of the Meixner polynomials of discrete variable.

We start from the generalized Laguerre functions

$$\psi_n^\alpha(s) = d_n^{-1}\sqrt{\rho_1(s)}\; L_n^\alpha(s)$$

In the discrete case, we defined the normalized Meixner functions

$$M_n^{(\gamma,\mu)}(x) \equiv d_n^{-1}\sqrt{\rho_1(x)}\; m_n^{(\gamma,\mu)}(x)$$

where $m_n^{(\gamma,\mu)}(x)$ are the Meixner polynomials,

$$d_n^2 = \frac{n!\Gamma(n+\gamma)}{\mu^n(1-\mu)^\gamma\Gamma(\gamma)}, \quad \rho_1(x) = \frac{\mu^x\Gamma(x+\gamma+1)}{\Gamma(x+1)\Gamma(\gamma)},$$

and γ, μ are some constants $0 < \mu < 1, \quad \gamma > 0$. The Meixner functions satisfy the orthonormality condition

$$\boxed{\sum_{x=0}^{\infty} M_n^{(\gamma,\mu)}(x)\; M_{n'}^{(\gamma,\mu)}(x)\frac{1}{\mu(x+\gamma)} = \rho_{nn'}}$$

and the following properties:

i) Difference equation

$$\sqrt{\frac{\mu(x+\gamma)(x+1)(x+\gamma)}{x+\gamma+1}}M_n(x+1) + \sqrt{\mu(x+\gamma)x}M_n(x-1) - [\mu(x+\gamma)+x-n(1-\mu)]\,M_n(x) = 0$$

ii) Raising operator

$$L^+(x,n)\;M_n(x) = \sqrt{\mu(n+\gamma)(n+1)}\;M_{n+1}(x)$$

iii) Lowering operator

$$L^-(x,n)\;M_n(x) = -\sqrt{\mu(n+\gamma-1)n}\;M_{n-1}(x)$$

The commutation relations read

$$L^+(x,n-1)\;L^-(x,n) - L^-(x,n+1)\;L^+(x,n) = -\mu(2n+\gamma)$$

In order to make connection between the Meixner functions of discrete variable and Laguerre functions of continuous variable we substitute $\gamma = \alpha+1,\;\; \mu = 1-h,\;\; x = \frac{s}{h}$ and then take the limit $h \to 0$.

In the limit we obtain

$$M_n^{(\gamma,\mu)}(x) \xrightarrow[\substack{h\to 0\\ x\to\infty\\ hx\to s}]{} \sqrt{\frac{n!}{\Gamma(n+\alpha+1)}e^{-s}s^{\alpha+1}}\, L_n^{\alpha}(s) = \psi_n^{\alpha}(s)$$

In order to make application to the hydrogen atom we take the reduced radial equation

$$\frac{d^2u}{d\rho^2} + \left[\frac{\nu}{\rho} - \frac{1}{4} - \frac{l(l+1)}{\rho^2}\right] u(\rho) = 0$$

where

$$\rho \equiv \frac{\sqrt{8M\,|E|}}{\hbar} r \quad , \quad \nu = \frac{Ze^2}{\hbar}\sqrt{\frac{\mu}{2\,|E|}}$$

The energy eigenvalues are given by

$$\boxed{E_{\nu l} = -\frac{1}{2}\,\frac{M\left(Ze^2\right)^2}{\hbar^2}\,\frac{1}{\nu^2} \quad , \quad \nu = 1, 2\ldots}$$

For fixed ν we still have degeneracy for $l = 0, 1, \ldots \nu - 1$

The corresponding eignvectors are given by

$$\psi_{\nu l}(\rho) = \left\{ \frac{(\nu - l - 1)!}{(\nu + l)!} \right\}^{\frac{1}{2}} \rho^{l+1} \, e^{\frac{\rho}{2}} \, L^{2l+1}_{\nu-l-1} \, (\rho)$$

From the connection between the Meixner and Laguerre functions given above, we can make the ansatz of a discrete model for the hydrogen atom where the reduced radial equation is substituted by the difference equation

The anticommutation relations, which is proportional to hamiltonian of the hydrogen atom (for the radial part), is substituted by the anticommutation realtions. The commutation relations for the raising and lowering operators defining the Lie algebra of the SU(1,1) group, are substituted by commutation relation of discrete type.

The expectation value of the discrete variable x with respect to the Meixner functions $M_n^{(\gamma,\mu)}(x)$ is

$$\langle x \rangle_{n\gamma} = \sum M_n^{(\gamma,\mu)}(x) \; x \; M_n^{(\gamma,\mu)}(x),$$

that can be calculated with the help of the recurrence relation.

9. EXACT SOLUTIONS FOR THE DIRAC-EQUATION ON THE LATTICE

Given a scalar function $\Phi\left(n_\mu\right) \equiv \left(n_\mu \varepsilon_\mu\right)$ defined in the grid points of a Minkowski lattice with elementary lengths ε_μ, and the difference operators

$$\delta_\mu^+ \equiv \frac{1}{\varepsilon_\mu} \Delta_\mu \prod_{\nu \neq \mu} \tilde{\Delta}_\nu, \quad \delta_\mu^- \equiv \frac{1}{\varepsilon_\mu} \nabla_\mu \prod_{\nu \neq \mu} \tilde{\nabla}_\nu$$

$$\eta^+ \equiv \prod_\mu \tilde{\Delta}_\mu, \quad \eta^- \equiv \prod_\mu \tilde{\nabla}_\mu, \quad \mu, \nu = 1, 2, 3, 4$$

the Klein-Gordon equation can be read off

$$\boxed{\left(\delta_\mu^+ \delta^{\mu^-} - m_0^2 c^2 \eta^+ \eta^-\right) \Phi\left(n_\mu\right) = 0}$$

Using the method of separation of variables, we obtain the exact solutions of this difference equation

$$f\left(n_\mu\right) = \prod_{\mu = 0}^{3} \left(\frac{1 - \frac{1}{2} i \varepsilon_\mu k_\mu}{1 + \frac{1}{2} i \varepsilon_\mu k_\mu} \right)^{n_\mu}$$

with k_μ, continuous variables satisfiying the dispersion relations $k_\mu k^\mu = m_0^2 c^2$

Starting from the Hamiltonian

$$H = \varepsilon_1 \varepsilon_2 \varepsilon_3 \sum_{n_1 n_2 n_3 = 0}^{N-1} \tilde{\Delta}_1 \tilde{\Delta}_2 \tilde{\Delta}_3 \psi^+(n) \times$$
$$\times \left\{ \gamma_0 \gamma_1 \frac{1}{\varepsilon_1} \Delta_1 \tilde{\Delta}_2 \tilde{\Delta}_3 + \gamma_0 \gamma_2 \tilde{\Delta}_1 \frac{1}{\varepsilon_2} \Delta_2 \tilde{\Delta}_3 + \right.$$
$$+ \left. \gamma_0 \gamma_3 \tilde{\Delta}_1 \tilde{\Delta}_2 \frac{1}{\varepsilon_3} \Delta_3 + \gamma_0 \tilde{\Delta}_1 \tilde{\Delta}_2 \tilde{\Delta}_3 \right\} \psi_{(n)}$$

we obtain from the Hamilton equations of motion, the Dirac equation

$$\boxed{\left(i \gamma_\mu \delta^{\mu +} - m_0 c \eta^+\right) \psi\left(n_\mu\right) = 0}$$

and from this we recover the Klein-Gordon equation.

Our model for the fermion fields satisfies the following conditions:

i) the Hamiltonian is translational invariant
ii) the Hamiltonian is hermitian
iii) for $m_0 = 0$, the wave equation is invariant under global chiral transformations
iv) there is no fermion doubling
v) the Hamiltonian is non-local

Finally, coupling the vector field to the electro-magnetic vector potential we construct a gauge invariant vector current leading to the correct axial anomaly

10. INTEGRAL LORENTZ TRANSFORMATIONS

We start with the integral transformations of the complete Lorentz group generated by the Kac generators

$$S_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$S_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, S_4 = \begin{pmatrix} 2 & 1 & 1 & 1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{pmatrix}$$

Any integral matrix of the complete Lorentz group can be factorized

$$\boxed{L = P_1^{\eta} P_2^{\theta} P_3^{i} S_4 \ldots S_4 P_1^{\delta} P_2^{\varepsilon} P_3^{\zeta} S_4 \left\{ S_1^{\alpha} S_2^{\beta} S_3^{\gamma} \right\}_{\mathrm{perm.}}}$$

where $P_1 = S_1 S_2 S_3 S_2 S_1, \quad P_2 = S_2 S_3 S_2, \, P_3 = S_3, \qquad \alpha, \beta, \ldots = 0, 1$

11. DISCRETE DIFFERENTIAL FORMS

Given a vectorial space V^n over $\mathcal{Z}$ we can define a linear function

$$f(u) \equiv \langle \omega, u \rangle \quad u \in V^n$$

The forms ω constitue a vectorial linear space (dual space) ${}^*V^n$.

The basis e_β of V^n and ω^α of ${}^*V^n$ can be contracted

$$\left\langle \omega^\alpha, e_\beta \right\rangle \delta^\alpha_\beta$$

If we take $\omega^\beta = \Delta x^\beta$ as coordinate basis for the linear forms we can construct discrete differential forms (a discrete version of the continuous differential forms)

A particular example of this discrete form is the total difference operator:

$$\Delta f(x,y) = \left(\frac{\Delta_x \widetilde{\Delta}_y f}{\Delta x} \right) \Delta x + \left(\frac{\widetilde{\Delta}_x \Delta_y f}{\Delta y} \right) \Delta y$$

We can define the exterior product of two form σ and δ: $\rho \wedge \sigma = -\sigma \wedge \rho$

With the help of this exterior product we can construct a second order discrete differential form or 2-form, namely

$$\rho \wedge \sigma = -\rho_\alpha \Delta x^\alpha \wedge \sigma_\alpha \Delta x^\beta = \frac{1}{2} \left(\rho_\alpha \sigma_\beta - \rho_\beta \sigma_\alpha \right) \Delta x^\alpha \wedge \Delta x^\beta \equiv \sigma_{\alpha\rho} \Delta x^\alpha \wedge \Delta x^\beta$$

where $\sigma_{\alpha\rho}$ is an antisymmetric tensor.

We give now some examples:

- Energy-momentum 1-form

$$\mathbf{P} = -E\Delta t + P_x\Delta x + P_y\Delta y + P_z\Delta z$$

where $(P_x, P_y, P_z, iE) \equiv P_n$ is the four-momentum.

- Vector potential 1-form

$$\mathbf{A} = A_\mu \Delta x^\mu = A_x\Delta x + A_y\Delta y + A_z\Delta z + A_t\Delta t$$

where $A_\mu = (A_x, A_y, A_z, A_t)$ is the four-potential.

- Faraday 2-form

$$\begin{aligned}\mathbf{F} &= E_x\Delta x \wedge \Delta t + E_y\Delta y \wedge \Delta t + E_z\Delta z \wedge \Delta t \\ &+ B_x\Delta y \wedge \Delta z + B_y\Delta z \wedge \Delta x + B_z\Delta x \wedge \Delta y = \frac{1}{2}F_{\mu\nu}\Delta x^\mu \wedge \Delta x^\nu\end{aligned}$$

with $(B_x, B_y, B_z) \equiv \vec{B}$ and $(E_x, E_y, E_z) \equiv \vec{E}$ the magnetic and electric field, respectively.

Given a 1-form in a two-dimensional space

$$\omega = a(x,y)\Delta x + b(x,y)\Delta y$$

we can define the exterior difference, in the similar way as the exterior derivative, namely,

$$\Delta\omega \equiv \Delta a \wedge \Delta x + \Delta b \wedge \Delta y = \left(\frac{\Delta_x \widetilde{\Delta}_y b}{\Delta x} - \frac{\widetilde{\Delta}_x \Delta_y a}{\Delta y} \right) \Delta x \wedge \Delta y$$

Given a 2-form in a 3-dimensional space,

$$\omega = a(x,y,z)\Delta y \wedge \Delta z + b(x,y,z)\Delta z \wedge \Delta x + c(x,y,z)\Delta x \wedge \Delta y$$

we can also define the exterior difference as:

$$\begin{aligned} \Delta\omega &= \Delta a \wedge \Delta y \wedge \Delta z + \Delta b \wedge \Delta z \wedge \Delta x + \Delta c \wedge \Delta x \wedge \Delta y \\ &= \left(\frac{\Delta_x \widetilde{\Delta}_y \widetilde{\Delta}_z a}{\Delta x} + \frac{\widetilde{\Delta}_x \Delta_y \widetilde{\Delta}_z b}{\Delta y} + \frac{\widetilde{\Delta}_x \widetilde{\Delta}_y \Delta_z c}{\Delta z} \right) \Delta x \wedge \Delta y \wedge \Delta z \end{aligned}$$

Given a 3-form in a 4-dimensional space

$$\omega = a\Delta y \wedge \Delta z + \Delta t + b\Delta z \wedge \Delta t \wedge \Delta x + c\Delta t \wedge \Delta x \wedge \Delta y + d\Delta x \wedge \Delta y \wedge \Delta z$$

we can define an exterior difference as before:

$$\omega = \left(\frac{\Delta_x \widetilde{\Delta}_y \widetilde{\Delta}_z \widetilde{\Delta}_t a}{\Delta x} - \frac{\widetilde{\Delta}_x \Delta_y \widetilde{\Delta}_z \widetilde{\Delta}_t b}{\Delta y} + \frac{\widetilde{\Delta}_x \widetilde{\Delta}_y \Delta_z \widetilde{\Delta}_t c}{\Delta z} - \frac{\widetilde{\Delta}_x \widetilde{\Delta}_y \widetilde{\Delta}_z \Delta_t d}{\Delta t} \right) \Delta x \wedge \Delta y \wedge \Delta z \wedge \Delta t$$

From the Faraday 2-form we write down one set of Maxwell difference equations

$$\boxed{\Delta \mathbf{F} = \Delta \left(\Delta \mathbf{A} \right) = 0}$$

$$\left(\frac{\Delta_x \widetilde{\Delta}_y \widetilde{\Delta}_z \widetilde{\Delta}_t B_x}{\Delta x} + \frac{\widetilde{\Delta}_x \Delta_y \widetilde{\Delta}_z \widetilde{\Delta}_t B_y}{\Delta y} + \frac{\widetilde{\Delta}_x \widetilde{\Delta}_y \Delta_z \widetilde{\Delta}_t B_z}{\Delta z} \right) \Delta x \wedge \Delta y \wedge \Delta z$$
$$+ \left(\frac{\widetilde{\Delta}_x \widetilde{\Delta}_y \widetilde{\Delta}_z \Delta_t B_x}{\Delta t} + \frac{\widetilde{\Delta}_x \Delta_y \widetilde{\Delta}_z \widetilde{\Delta}_t E_z}{\Delta y} - \frac{\widetilde{\Delta}_x \widetilde{\Delta}_y \Delta_z \widetilde{\Delta}_t E_y}{\Delta z} \right) \Delta t \wedge \Delta y \wedge \Delta z$$
$$+ \left(\frac{\widetilde{\Delta}_x \widetilde{\Delta}_y \widetilde{\Delta}_z \Delta_t B_y}{\Delta t} + \frac{\widetilde{\Delta}_x \widetilde{\Delta}_y \Delta_z \widetilde{\Delta}_t E_x}{\Delta z} - \frac{\Delta_x \widetilde{\Delta}_y \widetilde{\Delta}_z \widetilde{\Delta}_t E_z}{\Delta x} \right) \Delta t \wedge \Delta z \wedge \Delta x$$
$$+ \left(\frac{\widetilde{\Delta}_x \widetilde{\Delta}_y \widetilde{\Delta}_z \Delta_t B_z}{\Delta t} + \frac{\Delta_x \widetilde{\Delta}_y \widetilde{\Delta}_z \widetilde{\Delta}_t E_y}{\Delta x} - \frac{\widetilde{\Delta}_x \Delta_y \widetilde{\Delta}_z \widetilde{\Delta}_t E_x}{\Delta y} \right) \Delta t \wedge \Delta x \wedge \Delta y$$

From the vector potential 1-forms $\mathbf{A} = A_\mu \Delta x^\mu$ we can construct Faraday 2-form $\mathbf{F} = \Delta \mathbf{A}$ from which the Maxwell equations the second set are derived

$$\boxed{\Delta^* \mathbf{F} = \Delta^* \Delta \mathbf{A} = 4\pi^* \mathbf{J}}$$

Taking the dual of this expression we obtain the wave equation for the vector potential

$$\boxed{\Box A_\mu = {}^*\Delta^* \Delta A = 4\pi J}$$