HITTING PROBABILITIES FOR SYSTEMS
OF STOCHASTIC WAVES

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Dedicated to the memory of
Paul Malliavin
(1925-2010)
Introduction

- \{v(x), x \in \mathbb{R}^m\} is a \mathbb{R}^d\text{-valued stochastic process.}
- \mathcal{I} \subset \mathbb{R}^m, compact, positive Lebesgue measure.

Range of \(v\):

\[ v(\mathcal{I}) = \{v(x), x \in \mathcal{I}\}. \]

Question:

What can be said about \(P\{v(\mathcal{I}) \cap A \neq \emptyset\}\) in terms of \(A \in \mathcal{B}(\mathbb{R}^d)\).

- Upper and lower bounds described by the capacity or the Hausdorff measure of \(A\).
- Characterization of the polar sets \(A\): \(P\{v(\mathcal{I}) \cap A \neq \emptyset\} = 0\).
- Hausdorff dimension of the range of \(v\), questions on level sets, etc.
Introduction

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Bessel-Riesz capacity

Bessel-Riesz kernel. For $\beta, r \in \mathbb{R}$,

$$K_\beta(r) = \begin{cases} r^{-\beta}, & \text{if } \beta > 0, \\ \log_+ \left( \frac{1}{r} \right), & \text{if } \beta = 0, \\ 1, & \text{if } \beta < 0. \end{cases}$$

Energy. $E \in \mathcal{B}(\mathbb{R}^d)$, $\mu$ probability on $E$:

$$I_\beta(\mu) = \int_E \int_E K_\beta(\|x - y\|) \mu(dx) \mu(dy).$$

The capacity of $E$ is:

$$\text{Cap}_\beta(E) = \left[ \inf_{\mu \in \mathcal{P}(E)} I_\beta(\mu) \right]^{-1}.$$
Hausdorff measure

For $\beta \in [0, \infty[, \ E \in \mathcal{B}(\mathbb{R}^d)$:

$$\mathcal{H}_\beta(E) = \liminf_{\varepsilon \to 0^+} \left\{ \sum_{i=1}^{\infty} (2r_i)^\beta : E \subset \bigcup_{i=1}^{\infty} B_{r_i}(x_i), \sup_{i \geq 1} r_i \leq \varepsilon \right\}.$$

For $\beta \in ]-\infty, 0[\ , E \in \mathcal{B}(\mathbb{R}^d)$:

$$\mathcal{H}_\beta(E) = \infty.$$

A useful fact relating capacities and Hausdorff measures

For $\beta_1 > \beta_2 > 0$ and compact $E$,

$$\text{Cap}_{\beta_1}(E) > 0 \implies \mathcal{H}_{\beta_1}(E) > 0 \implies \text{Cap}_{\beta_2}(E) > 0$$

(Frostman’s theorem).
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Example 1: Brownian motion

For a $d$-dim Brownian motion $B$,

$$P(B(\mathbb{R}_+) \cap A \neq \emptyset) > 0 \iff \text{Cap}_{d-2}(A) > 0.$$  

In particular, for $x \neq 0$,

$$P(\exists t: B(t) = x) > 0 \iff d = 1.$$  

Indeed,

$$\text{Cap}_\beta(\{x\}) = \begin{cases} 1, & \beta < 0, \\ 0, & \beta \geq 0. \end{cases}$$

*Kakutani, 1944; Dvorestzky, 1950.*

... And much more

- ...

Some monographs:
Example 2: Brownian sheet

\[ \{ W_{t_1,\ldots,t_m} = (W_{t_1,\ldots,t_m}^1, \ldots, W_{t_1,\ldots,t_m}^d), (t_1, \ldots, t_m) \in \mathbb{R}_+^m \}, \]

Gaussian, independent components, centered,

\[ E (W^i_{t_1,\ldots,t_m} W^i_{s_1,\ldots,s_m}) = (t_1 \wedge s_1) \cdots (t_m \wedge s_m). \]

• Khoshnevisan and Shi, 1999:

\[ c^{-1} \text{Cap}_{d-2m}(A) \leq P\{v(I) \cap A \neq \emptyset \} \leq c \text{Cap}_{d-2m}(A). \]

Remarks

► Extension of Kakutani’s result \((m = 1)\).
► Previous results on multidimensional Markov processes do not apply.
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- Previous results on multidimensional Markov processes do not apply.
Example 3: a system of SPDEs

\[
\begin{aligned}
\frac{\partial^2}{\partial t_1 \partial t_2} u_i(t) &= \sum_{j=1}^{d} \sigma^i_j(u(t)) \frac{\partial^2}{\partial t_1 \partial t_2} W^j_{t_1, t_2} + b^i(u(t)), \\
u_i(t) &= \chi_0,
\end{aligned}
\]

\( i = 1, \ldots, d. \)

If \( \sigma = \text{Id}_d, b = 0, \) then \( u \) is the \textit{Wiener sheet} with \( m = 2. \)

- \textit{Dalang and E. Nualart, 2004:} for \( A \subset \mathbb{R}^d, I \subset \mathbb{R}^2 \) compact sets,

\[
c^{-1} \text{Cap}_{d-4}(A) \leq P\{u(I) \cap A \neq \emptyset\} \leq c \text{Cap}_{d-4}(A).
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\begin{aligned}
&\frac{\partial}{\partial t_1} \frac{\partial}{\partial t_2} u_i(t) = \sum_{j=1}^{d} \sigma_j^i(u(t)) \frac{\partial^2}{\partial t_1 \partial t_2} W_{t_1, t_2}^j + b^i(u(t)), \\
&u_i(t) = x_0, \\
&i = 1, \ldots, d.
\end{aligned}
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If \(\sigma = \text{Id}_d\), \(b = 0\), then \(u\) is the Wiener sheet with \(m = 2\).

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Key ingredients for the extension


In general, we may expect:

- Upper bounds in terms of Hausdorff measure.
- Lower bounds in terms of Bessel-Riesz capacity.
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In general, we may expect:

▶ Upper bounds in terms of Hausdorff measure.

▶ Lower bounds in terms of Bessel-Riesz capacity.
Results on systems of stochastic heat equations

- *Mueller, Tribe*, 2002: Additive, space-time white noise, spatial dimension $k = 1$, $b = 0$.
- *Dalang, Khoshnevisan, E. Nualart* (*in progress*): Multiplicative noise, $k = 1$, space-time white noise.
- *Dalang, Khoshnevisan, E. Nualart* (*in progress*): Multiplicative noise, $k \geq 1$, white noise in $t$, correlated in space.
Criteria for Hitting Probabilities

R.C. Dalang, S.-S., 2009
Summary

- **Lower bounds** for hitting probabilities of $A$ in terms of the Bessel-Riesz capacity of $A$ are obtained assuming:
  1. Joint densities of $(v(x), v(y))$ have *Gaussian type* bounds.
  2. The density of $v(x)$ is bounded away from zero on compact sets.

- **Upper bounds** in terms of the Hausdorff measure are obtained supposing:
  1. The density of $v(x)$ is bounded above on compact sets.
  2. $E \left( \|v(x) - v(y)\|^q \right) \leq C \|x - y\|^{q\delta}.$
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**Theorem 1: Lower bound**

Assume:

1. For any \( x, y \in \mathbb{R}^m \), \( x \neq y \), \((v(x), v(y))\) has a density \( p_{x,y} \), and there exist \( \gamma, \alpha \in ]0, \infty[ \) such that

\[
p_{x,y}(z_1, z_2) \leq C \frac{1}{\|x - y\|^\gamma} \exp \left( -\frac{\|z_1 - z_2\|^2}{\|x - y\|^\alpha} \right),
\]

for any \( z_1, z_2 \in \mathbb{R}^d \).

2. The density \( p_x \) of \( v(x) \) satisfies \( \inf_{w \in K} p_x(w) > 0 \), for any compact \( K \subset \mathbb{R}^d \).

Then, for all \( A \subset [-N, N]^d \), there exists \( c > 0 \) such that

\[
P\{v(I) \cap A \neq \emptyset\} \geq c\text{Cap}_2^{\alpha}(\gamma - m)(A).
\]
Theorem 2: Upper bound

$D \subset \mathbb{R}^n$ is fixed. Assume

1. For any $x \in \mathbb{R}^m$, $v(x)$ has a density $p_x$, and

$$\sup_{x \in I, z \in D} p_x(z) \leq C.$$

2. There exists $\delta \in ]0, 1]$ such that for any $q \in [1, \infty[$, $x, y \in I$,

$$E (\|v(x) - v(y)\|^q) \leq C \|x - y\|^{q\delta}.$$

Then, for any $\theta \in ]0, d[$ and any $A \subset D$,

$$P \{v(I) \cap A \neq \emptyset\} \leq C \mathcal{H}_{\theta - \frac{m}{\delta}}(A).$$

Remark: For some class of Gaussian processes, we can obtain

$$P \{v(I) \cap A \neq \emptyset\} \leq C \mathcal{H}_{d - \frac{m}{\delta}}(A).$$
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Application to Systems of Stochastic Waves
A system of stochastic wave equations

\[
\frac{\partial^2 u_i}{\partial t^2}(t, x) - \frac{\partial^2 u_i}{\partial x^2}(t, x) = \sum_{j=1}^{d} \sigma_{ij}(u(t, x)) \dot{W}^j(t, x) + b_i(u(t, x)),
\]

\[1 \leq i \leq d, \ t \in [0, T], \ x \in \mathbb{R}^k, \ k \geq 1.\]

\[\dot{W} = (\dot{W}^1, \ldots, \dot{W}^d) \text{ Gaussian noise, centered, with covariance}\]

\[E \left( \dot{W}^i(t, x) \dot{W}^j(s, y) \right) = \delta(t - s)\|x - y\|^{-\beta} \delta_{ij},\]

\[\beta \in (0, 2 \land k).\]

\[b, \sigma \text{ Lipschitz continuous.}\]
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  \[\beta \in (0, 2 \wedge k).\]
- \(b, \sigma\) Lipschitz continuous.
References

- Ondreját, 2004 – ⋯, Brzeźniak, ⋯
The Gaussian case

σ = (σ_{ij}) constant, det σ ≠ 0, b_i = 0, i = 1, ..., d:

\[
\frac{\partial^2 u_i}{\partial t^2}(t, x) - \frac{\partial^2 u_i}{\partial x^2}(t, x) = \sum_{j=1}^{d} \sigma_{ij} \dot{W}^j(t, x),
\]

null initial conditions.

Solution:

\[
u_i(t, x) = \sum_{j=1}^{d} \int_{0}^{t} \int_{\mathbb{R}^k} G(t - r, x - y) \sigma_{ij} W^j(dr, dy)
\]

\( t \in [0, T], \ x \in \mathbb{R}^k, \ k \geq 1. \)

\( G(t, \cdot) \) is the fundamental solution to the wave equation:

\[
\mathcal{F} G(t, \cdot)(\xi) = \frac{\sin(t\|\xi\|)}{\|\xi\|}.
\]
Theorem 3: hitting probabilities for Gaussian waves

$I, J$ are compact subsets of $[t_0, T]$ and $\mathbb{R}^k$, respectively, positive Lebesgue measure, $N > 0$, $t_0 > 0$. Then,

There exist positive constants $c_i = c_i(I, J, N, \beta, k, d)$, $i=1,2$, such that, for any Borel set $A \subset [-N, N]^d$,

$$c_1 \text{Cap}_{d- \frac{2(k+1)}{2-\beta}}(A) \leq P\{u(I \times J) \cap A \neq \emptyset\} \leq c_2 \mathcal{H}_{d- \frac{2(k+1)}{2-\beta}}(A).$$
Theorem 3 (cont.): hitting probabilities for sections of
Gaussian waves

1. For any \( t \in I \), there exist positive constants
   \( c_i = c_i(J, N, \beta, k, d) \), \( i = 1, 2 \), such that, for any Borel set
   \( A \subset [-N, N]^d \),
   \[
   c_1 \text{Cap}_{d-\frac{2k}{2-\beta}}(A) \leq P\{u(\{t\} \times J) \cap A \neq \emptyset\} \leq c_2 \mathcal{H}_{d-\frac{2k}{2-\beta}}(A).
   \]

2. For any \( x \in J \), there exist positive constants
   \( c_i = c_i(I, N, \beta, k, d) \), \( i = 1, 2 \), such that, for any Borel set
   \( A \subset [-N, N]^d \),
   \[
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Theorem 3 (cont.): hitting probabilities for sections of Gaussian waves

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2. For any $x \in J$, there exist positive constants $c_i = c_i(I, N, \beta, k, d)$, $i = 1, 2$, such that, for any Borel set $A \subset [-N, N]^d$, $c_1 \text{Cap}_{d-\frac{2}{2-\beta}}(A) \leq P\{u(I \times \{x\}) \cap A \neq \emptyset\} \leq c_2 \mathcal{H}_{d-\frac{2}{2-\beta}}(A)$.

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**Application**

When is a singleton $A = \{a\}$ a polar set for $u$?

- If $A$ is polar, i.e. $P\{u(I \times J) \cap A \neq \emptyset\} = 0$, then
  
  \[ d - \frac{2(k + 1)}{2 - \beta} \geq 0. \]

- If $d - \frac{2(k + 1)}{2 - \beta} > 0$, then $H_{d - \frac{2(k + 1)}{2 - \beta}}(A) = 0$; hence $A$ is polar.

Conjecture: $A$ is polar $\Leftrightarrow d - \frac{2(k + 1)}{2 - \beta} \geq 0$. 
Application

When is a singleton \( A = \{ a \} \) a polar set for \( u \)?

- If \( A \) is polar, i.e. \( P\{ u(I \times J) \cap A \neq \emptyset \} = 0 \), then
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- If \( d - \frac{2(k + 1)}{2 - \beta} > 0 \), then \( \mathcal{H}_{d - \frac{2(k + 1)}{2 - \beta}}(A) = 0 \); hence \( A \) is polar.

Conjecture: \( A \) is polar \( \iff \) \( d - \frac{2(k + 1)}{2 - \beta} \geq 0 \).
Ingredients of the proof of Theorem 3

1. Bounds for the density

The density of $u(t, x)$ is $p_{t,x}(z) = \frac{1}{(2\pi \sigma_{t,x}^2)^{\frac{d}{2}}} \exp \left(-\frac{\|z\|^2}{2\sigma_{t,x}^2}\right)$, and

$$C(t \wedge t^3) \leq \sigma_{t,x}^2 = \int_0^t ds \int_{\mathbb{R}^k} \frac{\sin^2(s\|\xi\|)}{\|\xi\|^2} \mu(d\xi) \leq \tilde{C}(t + t^3),$$

$$\mu(d\xi) = \|\xi\|^\beta - k d\xi.$$  

Hence,

$$\inf_{z \in [-N, N]^d} p_{t,x}(z) \geq C_1,$$

$$\sup_{z \in [-N, N]^d} \sup_{(t, x) \in [t_0, T] \times \mathbb{R}^k} p_{t,x}(z) \leq C_2.$$
2. Estimates of the variance

For any \((t, x), (s, y) \in [t_0, T] \times \mathbb{R}^k\),

\[
C_1 (|t - s| + \|x - y\|)^{2-\beta} \leq E \left( \|u_{t,x} - u_{s,y}\|^2 \right) \\
\leq C_2 (|t - s| + \|x - y\|)^{2-\beta}.
\]

Why this is important?

1. \(L^p\) moments for increments.
2. By-product: regularity of the sample paths (optimal).
3. Improvement of the upper bound.
4. Upper bound for the joint density of \((u(t, x), u(s, y))\).
2. Estimates of the variance

For any \((t, x), (s, y) \in [t_0, T] \times \mathbb{R}^k\),

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4. Upper bound for the joint density of \((u(t, x), u(s, y))\).
3. Gaussian type upper bounds for joint densities

Let $p_{t,x|s,y}$ be the joint density of $(u(t, x), u(s, y))$. We have:

$$p_{t,x|s,y}(z_1, z_2) = p_{t,x|s,y}(z_1, z_2)p_{s,y}(z_2)$$

$$\leq \frac{C}{(|t - s| + \|x - y\|)} \frac{d(2-\beta)}{2} \exp \left( -\frac{\|z_1 - z_2\|^2}{c (|t - s| + \|x - y\|)^{2-\beta}} \right).$$

Hence

$$\gamma = \frac{d(2-\beta)}{2}, \quad \alpha = 2 - \beta,$$

and $\frac{2}{\alpha}(\gamma - m) = d - \frac{2(k+1)}{2-\beta}$ (capacity dimension).
The general case

Assumptions (H)

- $k \in \{1, 2, 3\}$.
- $\sigma_{i,j}, b_i, 1 \leq i, j \leq d$ bounded, infinitely differentiable, bounded partial derivatives of any order.
- $\sigma$ is uniformly elliptic: for any $v \in \mathbb{R}^d$, $\|v\| = 1$,
  \[
  \inf_{x \in \mathbb{R}^d} \|v^t \sigma(x)\| \geq \rho_0 > 0.
  \]
- $f(x) = \|x\|^{-\beta}$ if $x \neq 0$, with $\beta \in ]0, 2 \wedge k[$.
Upper bound for the hitting probabilities of non-linear stochastic waves

**Theorem 4** Assume (H).

Let $I$ and $K$ compact sets of $[0, T]$ and $\mathbb{R}^k$, respectively, positive Lebesgue measure. Fix $\delta \in (0, 1)$.

There exists a positive constant $c = c(I, K, \beta, k, d, \delta)$ such that, for any Borel set $A \subset \mathbb{R}^d$,

$$P\{u(I \times K) \cap A \neq \emptyset\} \leq c \mathcal{H}_{d-\delta-\frac{2(k+1)}{2-\beta}}(A).$$

For sections:

$$P\{u(I \times \{x\}) \cap A \neq \emptyset\} \leq c \mathcal{H}_{d-\delta-\frac{2}{2-\beta}}(A).$$

$$P\{u(\{t\} \times J) \cap A \neq \emptyset\} \leq c \mathcal{H}_{d-\delta-\frac{2k}{2-\beta}}(A).$$
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There exists a positive constant \(c = c(I, K, \beta, k, d, \delta)\) such that, for any Borel set \(A \subset \mathbb{R}^d\),

\[
P\{u(I \times K) \cap A \neq \emptyset\} \leq c \mathcal{H}^{d-\delta - \frac{2(k+1)}{2-\beta}}(A).
\]

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\]

\[
P\{u(\{t\} \times J) \cap A \neq \emptyset\} \leq c \mathcal{H}^{d-\delta - \frac{2k}{2-\beta}}(A).
\]
Ingredients

- Existence of density for the law of $u(t, x)$: $p_{t,x}(z)$.
- $\sup_{(t,x)\in K_1} \sup_{z \in K_2} p_{t,x}(z) \leq C$.
- $E \left( \|u(t, x) - u(s, y)\|^p \right) \leq C \left( |t - s| + \|x - y\| \right)^{\delta p}$, $
\delta \in (0, \frac{2-\beta}{2})$.

Theorem 5: Lower bound for the hitting probabilities of non-linear stochastic waves

Assume (H). Let \( I = [a, b] \subset (0, T) \), \( J \) compact subset of \( \mathbb{R}^k \) of positive Lebesgue measure. Fix \( \delta \in (0, 1) \) and \( N > 0 \).

There exists a positive constant \( c = c(I, J, N, \beta, k, d, \delta) \) such that, for any Borel set \( A \subset [-N, N]^d \),

\[
P \{ u(I \times J) \cap A \neq \emptyset \} \geq c \text{Cap}^d \frac{\delta + 3 - \beta}{2 - \beta} - \frac{2(k + 1)}{2 - \beta}(A).
\]
Theorem 5 (cont.): Lower bound for the hitting probabilities of sections of non-linear stochastic waves

1. Fix $\delta > 0$, $N > 0$ and $x \in J$. There exists a positive constant $c = c(I, N, \beta, k, d, x, \delta)$ such that, for any Borel set $A \subset [-N, N]^d$,

\[ P \{ u(I \times \{x\}) \cap A \neq \emptyset \} \geq c \text{Cap} d^{\frac{\delta + 3 - \beta}{2 - \beta} - \frac{2k}{2 - \beta}}. \]

2. Fix $\delta > 0$, $N > 0$ and $t \in I$. There exists a positive constant $c = c(J, N, \beta, k, d, t)$ such that, for any Borel set $A \subset [-N, N]^d$,

\[ P \{ u(\{t\} \times J) \cap A \neq \emptyset \} \geq c \text{Cap} d^{\frac{\delta + 3 - \beta}{2 - \beta} - \frac{2k}{2 - \beta}}. \]
Remarks

- Upper bound is almost optimal
  - In the Gaussian case: $d - \frac{2(k+1)}{2-\beta}$.
  - In the non-Gaussian case: $d - \delta - \frac{2(k+1)}{2-\beta}$, $\delta$ arbitrarily small.

- Bessel-Riesz capacity dimension is not optimal.
  - In the Gaussian case: $d - \frac{2(k+1)}{2-\beta}$.
  - In the non-Gaussian case: $d \frac{\delta + 3 - \beta}{2-\beta} - \frac{2(k+1)}{2-\beta}$. 
Ingredients

- Strict positivity of the density $p_{t,x}$ (extensions of work in Aida-Kusuoka-Stroock, 1991; Bally-Pardoux, 1998).
- Gaussian type bounds for the densities of $(u(s,y), u(t,x))$
  - For Gaussian waves
    \[
    p_{t,x;s,y}(z_1, z_2) = p_{t,x|s,y}(z_1, z_2)p_{s,y}(z_2) \leq C \frac{d(2-\beta)}{2} \exp \left( - \frac{||z_1 - z_2||^2}{c (|t-s| + ||x-y||)^{2-\beta}} \right).
    \]
  - For non-linear stochastic waves: Watanabe’s formula for the density, based on the integration by parts formula of Malliavin calculus.
Formula for the density of \((u(s, y), u(t, x))\)

\[
p_{t,x;s,y}(z_1, z_2) = \prod_{i=1}^{d} E \left( \mathbb{1}_{\{|u_i(t,x) - u_i(s,y)| > |z_1^i - z_2^i|\}} H_{u(s,y),u(t,x)} \right)
\]

\[
\leq \prod_{i=1}^{d} \left( \mathbb{P}\{|u_i(t,x) - u_i(s,y)| > |z_1^i - z_2^i|\} \right)^{\frac{1}{2d}}
\]

\[
\times \|H_{u(s,y),u(t,x)}\|_{L^2(\Omega)}.
\]

- \(H_{u(s,y),u(t,x)}\) depends on:
  1. the Malliavin derivatives of \((u(s, y), u(t, x))\),
  2. the inverse of the Malliavin covariance matrix of \((u(s, y), u(t, x))\),
  3. \ldots
**Contribution of** $\| H_{u(s,y),u(t,x)} \|_{L^2(\Omega)}$

**Theorem 6** Assume (H), then

$$\| H_{u(s,y),u(t,x)} \|_{L^2(\Omega)} \leq \frac{C}{(|t - s| + \|x - y\|)^\gamma},$$

with $\gamma = \frac{d}{2} (\delta + 3 - \beta)$.

**Remarks**

- In the Gaussian case: $\gamma = \frac{d}{2} (2 - \beta)$.
- Negative powers of $|t - s| + \|x - y\|$ appear in the estimates of the $L^p(\Omega)$-norm of the inverse of the Malliavin covariance matrix of $(u(s,y), u(t,x) - u(s,y))$. 
Contribution of $\| H_{u(s,y),u(t,x)} \|_{L^2(\Omega)}$

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Remarks

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**Contribution of** \( P\{|u_i(t, x) - u_i(s, y)| > |z_1^i - z_2^i|\} \)

\( b = 0 \)

\[
P\{|u_i(t, x) - u_i(s, y)| > |z_1^i - z_2^i|\} \\
\leq C_1 \exp \left\{ -\frac{C_2 |z_1^i - z_2^i|^2}{(|t - s| + \|x - y\|)^{2-\beta}} \right\}.
\]

\( b \neq 0 \)

No suitable Girsanov’s type theorem

\[
P\{|u_i(t, x) - u_i(s, y)| > |z_1^i - z_2^i|\} \\
\leq C \left[ \frac{(|t - s| + \|x - y\|)^\alpha}{|z_1^i - z_2^i|} \right]^{\wedge 1} \leq P,
\]

with \( \alpha \in \left( 0, \frac{2-\beta}{2} \right) \), \( p \geq 1 \).
Contribution of $P\{|u_i(t, x) - u_i(s, y)| > |z_1^i - z_2^i|\}$

$\triangleright b = 0$

$$P\{|u_i(t, x) - u_i(s, y)| > |z_1^i - z_2^i|\} \leq C_1 \exp \left\{-\frac{C_2 |z_1^i - z_2^i|^2}{(|t - s| + \|x - y\|)^{2-\beta}} \right\}.$$  

$\triangleright b \neq 0$

No suitable Girsanov's type theorem

$$P\{|u_i(t, x) - u_i(s, y)| > |z_1^i - z_2^i|\} \leq C \left[\frac{(|t - s| + \|x - y\|)^{a}}{|z_1^i - z_2^i|} \wedge 1\right]^p,$$

with $\alpha \in \left(0, \frac{2-\beta}{2}\right)$, $p \geq 1$. 
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The End