Absence of Absolutely Continuous Spectra for Radial Tree Graphs

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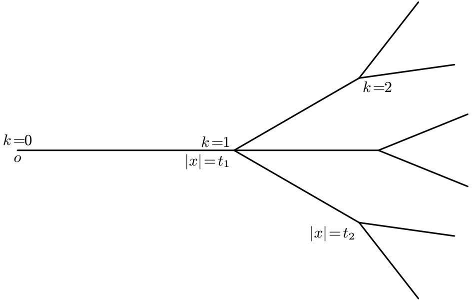
Outline

- radial quantum tree graphs
- unitary equivalence between Hamiltonian on trees and orthogonal sum of Hamiltonians on halflines (Naimark & Solomyak)
- large family of coupling conditions
- absence of absolutely continuous spectra (Breuer & Frank)

Schrödinger Operator on Tree

- rooted metric tree graph
- vertex of the k-th generation
- branching number $b(v) \geq 1$; $b(v) = b(w) = b_k$ for v, w in k-th generation
- radial tree graph: branching numbers for all vertices of the same generation are equal, edges emanating from the vertices of the same generation have equal lengths



Hamiltonian on Tree

- the Hamiltonian acts as $\mathbf{H} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(|x|)$
- the potential depends only on the distance $|x|$ from the root and is real, bounded and measurable
- domain of the Hamiltonian: functions in $\oplus_e H^2(e)$ satisfying the coupling conditions
- Robin coupling condition at the root

$$f_o' + f_o \tan\frac{\theta_0}{2} = 0\,, \quad \theta_0 \in (\pi/2, \pi/2]\,.$$

Coupling Conditions

$$\sum_{j=1}^{b_k} f'_{vj+} - f'_{v-} = \frac{\alpha_{\mathrm{t}k}}{2}\left(\frac{1}{b_k}\sum_{j=1}^{b_k} f_{vj+} + f_{v-}\right) + \frac{\gamma_{\mathrm{t}k}}{2}\left(\sum_{j=1}^{b_k} f'_{vj+} + f'_{v-}\right),$$

$$\frac{1}{b_k}\sum_{j=1}^{b_k} f_{vj+} - f_{v-} = -\frac{\bar{\gamma}_{\mathrm{t}k}}{2}\left(\frac{1}{b_k}\sum_{j=1}^{b_k} f_{vj+} + f_{v-}\right) + \frac{\beta_{\mathrm{t}k}}{2}\left(\sum_{j=1}^{b_k} f'_{vj+} + f'_{v-}\right).$$

$$(U_k - I)V_k\Psi_v + i(U_k + I)V_k\Psi'_v = 0\,,$$

where the index j distinguishes the edges emanating from v (denoted by the subscript $+$), the subscript minus refers to the incoming edge, and

$$\Psi_v := (f_{v1+}, f_{v2+}, \dots, f_{vb_k+})^{\mathrm{T}},$$

$$\Psi'_v := (f'_{v1+}, f'_{v2+}, \dots, f'_{vb_k+})^{\mathrm{T}}$$

- the coefficients $\alpha_{\mathrm{t}k}, \beta_{\mathrm{t}k} \in \mathbb{R}$, and $\gamma_{\mathrm{t}k} \in \mathbb{C}$ are the same for all the vertices belonging to the k-th generation – describe coupling in the subspace of the radial functions
- coupling between vectors Ψ_v and Ψ'_v: described by a $(b_k - 1) \times (b_k - 1)$ unitary matrix U_k, while V_k stands for an arbitrary $b_k \times (b_k - 1)$ matrix with orthonormal rows which all are perpendicular to the vector $(1, 1, \dots, 1)$.
- the subspace of the radial functions and its complement is not coupled
- $(b_k - 1)^2 + 4$ real parameters describing the coupling at each vetex
- only the eigenvalues of U_k influence the spectrum
- suppose that the coupling parameters are the same for all vertices of the same generation

GPI Interaction on the Line

- first description of the coupling on the line

$$
\begin{aligned}
y'_+ - y'_- &= \frac{\alpha_\mathrm{h}}{2}(y_+ + y_-) + \frac{\gamma_\mathrm{h}}{2}(y'_+ + y'_-)\,, \\
y_+ - y_- &= -\frac{\bar{\gamma}_\mathrm{h}}{2}(y_+ + y_-) + \frac{\beta_\mathrm{h}}{2}(y'_+ + y'_-)
\end{aligned}
$$

- second description of the coupling on the line

$$
\begin{pmatrix} y'_+ \\ -y'_- \end{pmatrix} = \begin{pmatrix} a_\mathrm{h} & c_\mathrm{h} \\ \bar{c}_\mathrm{h} & d_\mathrm{h} \end{pmatrix} \begin{pmatrix} y_+ \\ y_- \end{pmatrix}\,,
$$

Mapping of Coupling Conditions in Radial Subspace

- isometry $\Pi : f \to \varphi$, $\varphi(t) = f(x)$ for $t = |x|$ of $L^2_{0,\mathrm{rad}}(\Gamma)$ into the weighted space $L^2(\mathbb{R}_+, g_0)$ with the norm

$$\|\varphi\|^2_{L^2(\mathbb{R}_+,g_0)} = \int_{\mathbb{R}_+} |\varphi(t)|^2 g_0(t)\,\mathrm{d}t\,,$$

$$g_0(t) = b_0 \dots b_k,\, t_k \leq t < t_{k+1}$$

- with the isometry $y(t) := g_0^{1/2}(t)\varphi(t)$ leads to

$$f_{v-} \to y_{k-}\,, \qquad f'_{v-} \to y'_{k-}\,,$$

$$\frac{1}{b_k}\sum_{j=1}^{b_k} f_{vj+} \to b_k^{-1/2} y_{k+}\,, \qquad \sum_{j=1}^{b_k} f'_{jv+} \to b_k^{1/2} y'_{k+}\,.$$

- transformation of the coupling constants:

$$a_{\mathrm{h}} = b^{-1} a_{\mathrm{t}}\,, \quad c_{\mathrm{h}} = b^{-1/2} c_{\mathrm{t}}\,, \quad d_{\mathrm{h}} = d_{\mathrm{t}}$$

- and for the first type of coupling

$$\alpha_{\mathrm{h}} = \frac{16\alpha_{\mathrm{t}}}{4(b^{1/2}+1)^2 + \det\mathcal{A}_{\mathrm{t}}(b^{1/2}-1)^2 + 4(1-b)\,\mathrm{Re}\,\gamma_{\mathrm{t}}}\,,$$

$$\beta_{\mathrm{h}} = \frac{16\,b\,\beta_{\mathrm{t}}}{4(b^{1/2}+1)^2 + \det\mathcal{A}_{\mathrm{t}}(b^{1/2}-1)^2 + 4(1-b)\,\mathrm{Re}\,\gamma_{\mathrm{t}}}\,,$$

$$\gamma_{\mathrm{h}} = 2\,\frac{(1-b)(4+\det\mathcal{A}_{\mathrm{t}}) + 8ib^{1/2}\,\mathrm{Im}\,\gamma_{\mathrm{t}} + 4(b+1)\,\mathrm{Re}\,\gamma_{\mathrm{t}}}{4(b^{1/2}+1)^2 + \det\mathcal{A}_{\mathrm{t}}(b^{1/2}-1)^2 + 4(1-b)\,\mathrm{Re}\,\gamma_{\mathrm{t}}}\,.$$

where $\det\mathcal{A}_{\mathrm{t}} = \alpha_{\mathrm{t}}\beta_{\mathrm{t}} + |\gamma_{\mathrm{t}}|^2$,

Construction of the Unitary Equivalence

- unitary operator R_v

$$R_v : \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_{b_k}(x) \end{pmatrix} \mapsto \begin{pmatrix} \sum_{j=1}^{b_k} (W_k \cdot V_k)_{1j}\, f_j(x) \\ \sum_{j=1}^{b_k} (W_k \cdot V_k)_{2j}\, f_j(x) \\ \vdots \\ \sum_{j=1}^{b_k} (W_k \cdot V_k)_{(b_k-1)j}\, f_j(x) \end{pmatrix},$$

- domains of Hamiltonians

$$\mathrm{dom}\,\mathbf{H}_{vs,\mathrm{rad}} = \{f \in H^2(\Gamma_{\succeq v}) \ominus L^2_{0,\,\mathrm{rad}}(\Gamma_{\succeq v}) \,|$$

$$|\, \mathrm{supp}\,(R_v f) \subset \Gamma_{\succeq v,s},\, (R_v f)'_{vs+} + (R_v f)_{vs+} \tan \frac{\theta_{ks}}{2} = 0,$$

$f \in L^2_{0,\,\mathrm{rad}}(\Gamma_{\succeq w})$ and satisfies the coupling condition for all $w \succeq v\}$.

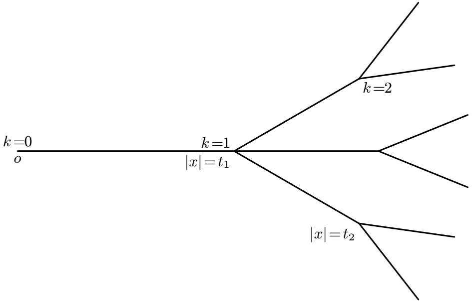
where $\Gamma_{\succeq v,s}$ is the s-th subtree emanating from v.

Theorem

The Hamiltonian $\mathbf{H}$ *on a radial tree graph* Γ *is unitarily equivalent to*

$$\mathbf{H} \cong H_{L_0} \oplus \bigoplus_{n=1}^{\infty} \bigoplus_{s=1}^{b_n - 1} (\oplus\, b_0 \dots b_{n-1}) H_{L_{ns}}\,.$$

where $(\oplus\, m) H_{L_{ns}}$ *is the orthogonal sum of m identical copies of the operator* $H_{L_{ns}}$.



Absence of Absolutely Continuous Spectra

- assumption: there is no potential on the tree ($V(|x|) = 0$)
- the proof goes similarly to the one in Breuer & Frank
- sparse tree: there exist a subsequence of edges which grows to infinity
- it holds for most of the coupling conditions
- there exist nontrivial coupl. conditions on the tree which correspond to free condition on the halfline $\Rightarrow$ in some cases there is absolutely continuous spectrum
- there are even the cases with purely absolutely continuous spectrum

- main idea: prove the absence of a. c. spectrum for halflines with Dirichlet condition at the beginning
- a. c. spectrum is not affected by the change of coupling at one vertex
- m-function defined as

$$m_{\pm}(z,t) := \pm \frac{f'_{\pm}(z,t)}{f_{\pm}(z,t)},$$

- right limit: set $\omega(H(\{t_n\}_{n=1}^{\infty}, \{\mathcal{A}_n\}_{n=1}^{\infty}))$ of Hamiltonians $\hat{H}$ for which there is a strictly increasing sequence $\{s_m\}$, $s_m \to \infty$, such that

$$d(H'(\{t_n + s_m\}_{n=1}^{\infty}, \{\mathcal{A}_n\}_{n=1}^{\infty}), \hat{H}) \to 0$$

- Remling-type theorem: any right limit is reflectionless ($\hat{m}_+(E+i0,t) = -\bar{\hat{m}}_-(E+i0,t)$) on the absolutely continuous spectrum

Important Assumptions

- the tree is sparse $\limsup_{n\to\infty}(t_{n+1} - t_n) = \infty$
- $\varepsilon = \inf_{n,m;n\neq m}|t_n - t_m| > 0$
- a) $|\beta_{\mathrm{h}n}| > \delta > 0$ and $|c_{\mathrm{h}n}| > \delta > 0$
  or
  b) $\beta_{\mathrm{h}n} = 0$, $|\gamma_{\mathrm{h}n}| < K$, and at least one of the following conditions is valid for all $n > N$: $\mathrm{Re}\,\gamma_{\mathrm{h}n} > \delta$ or $\mathrm{Re}\,\gamma_{\mathrm{h}n} < -\delta$ or $\alpha_{\mathrm{h}n} > \delta$ or $\alpha_{\mathrm{h}n} < -\delta$.
- there is at most finitely many coupling conditions which separate the halfline
- then the absolutely continuous spectrum of H is empty

Summary

- generalization of the unitary equivalence between tree and halflines
- $(b_k - 1)^2 + 4$ coupling parameters, only $(b_k - 1) + 4$ influence the spectrum
- absence of continuous spectra is general behaviour for sparse graphs
- there exist nontrivial sparse graphs with absolutely continuous spectrum, even graphs with purely continous spectrum

Thank you for your attention!

Summary

- generalization of the unitary equivalence between tree and halflines
- $(b_k - 1)^2 + 4$ coupling parameters, only $(b_k - 1) + 4$ influence the spectrum
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- there exist nontrivial sparse graphs with absolutely continuous spectrum, even graphs with purely continous spectrum

Thank you for your attention!