Properties of zeta functions of quantum graphs

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AGAW – 27/7/10

Outline

1. Spectral properties of graphs
2. Star graph zeta function
3. General graph zeta function
4. Example: piston graph

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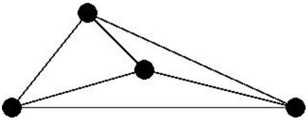
Quantum graph model



V no. of vertices.
B no. of bonds.

Bond b corresponds to interval $[0, L_b]$, $\mathcal{L} = \sum_{b=1}^{B} L_b$.

Hilbert space $\mathcal{H} := \bigoplus_{b=1}^{B} L^2\big([0, L_b]\big)$.

Operator on bond $-\frac{\mathrm{d}^2}{\mathrm{d}x_b^2}$ with vertex matching conditions so op. self-adjoint.

Spectral determinant

Graph spectrum: $0 \leqslant \lambda_0 \leqslant \lambda_1 \leqslant \dots$

Spectral determinant

$$\det{}'(-\triangle) = \prod_{j=1}^{\infty} \lambda_j$$

(99) Pascaud & Montambaux.

(00) Akkermans, Comtet, Desbois, Montambaux & Texier.

(06) Friedlander: Dirichlet to Neumann map, δ-coupling conditions.

(10) Texier: δ-coupling conditions with potential, conjecture for general matching conditions.

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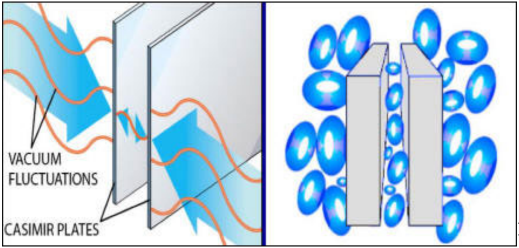
Casimir effect

(48) Casimir predicts quantum mechanical attraction between uncharged parallel plates.

(97) Accurate observations of Casimir effect by Lamoreaux at Los Alamos and Mohideen and Roy at UC Riverside.



[1]

[1] Pictures from Wikipedia

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Vacuum energy

- Casimir effect due to change in vacuum energy E_c.
- Formally vacuum energy $E_c = \frac{1}{2}\sum_{j=0}^{\infty}{}' \sqrt{\lambda_j}$.

Models of vacuum energy:

(01) Leboeuf, Monastra, & Bohigas: Riemannium.

(07) Fulling, Kaplan & Wilson: star graphs, repulsive Casimir effect for $B > 3$.

(09) Berkolaiko, JH & Wilson: general graphs, periodic orbit expansion of E_c.

Heat kernel

Heat kernel

$$K(\tau) = \sum_{j=1}^{\infty} \mathrm{e}^{-\lambda_j \tau}$$

(84) Roth – trace formula

(03/04) Berkolaiko, Keating & Winn – star graphs

(07) Kostrykin, Potthoff & Schrader

(09) Bolte & Endres – general trace formula

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Spectral zeta function

$$\zeta(s) = \sum_{j=1}^{\infty}{}' \lambda_j^{-s} \qquad\qquad \operatorname{Re} s > 1$$

(99) Carlson: eigenvalue problems

(09) Endres and Steiner: zeta fn of Berry-Keating op.

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Spectral zeta function

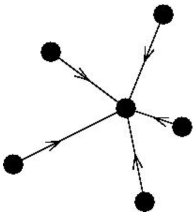
$$\zeta(s) = \sum_{j=1}^{\infty}{}' \lambda_j^{-s} \qquad \mathrm{Re}\, s > 1$$

Spectral properties

- $E_c = \frac{1}{2}\sum_{n=1}^{\infty}\sqrt{\lambda_j} = \frac{1}{2}\zeta(-1/2)$
- $\det'(-\triangle) = \prod_{n=1}^{\infty}\lambda_j = \exp\Big(-\zeta'(0)\Big)$
- $K(\tau) \underset{\tau\to 0}{\sim} \sum_{\ell=0,1/2,1,\dots}^{\infty} \varepsilon_\ell \tau^{\ell-1/2} \qquad \varepsilon_\ell = \mathrm{Res}(\zeta(s)\Gamma(s))|_{s=1/2-\ell}$

Neumann star graph

- Central vertex and B external vertices (*nodes*).
- Neumann bcs at nodes, $\psi_b'(0) = 0$.
- **Center:** fn continuous, $\psi_b(L_b) = \psi$, and $\sum_{b=1}^{B} \psi_b'(L_b) = 0$.



Eigenproblem
$-\triangle\psi = k^2\psi$

Conditions at nodes and continuity at center implies

$$\psi_b(x_b) = \psi \frac{\cos kx_b}{\cos kL_b} \ .$$

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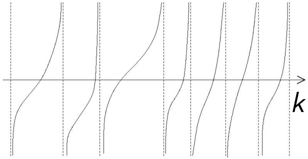
Secular equation

$$\sum_{b=1}^{B} \tan kL_b = 0$$



- Solns k_j of secular eqn correspond to evals k_j^2 of Laplace op.
- Poles $\bigcup_{b\in\mathcal{B}}\{(m+1/2)\pi/L_b\}_{m\in\mathbb{Z}}$.
- $\{L_b\}$ incommensurate: Poles and zeros distinct with one zero between every pair of adjacent poles.

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Equal bond lengths

$L_b = L$ for all b, secular eqn reduces to $\tan kL = 0$.

k-spectrum

$$\left\{\frac{n\pi}{L}\right\}_{n\in\mathbb{Z}} \text{ and } \left\{\frac{(m+1/2)\pi}{L}\right\}_{m\in\mathbb{Z}} \text{ multiplicity } B-1.$$

Zeta fn

$$\begin{aligned}\zeta(s) &= \sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^{-2s} + (B-1)\sum_{m=0}^{\infty}\left(\frac{(m+1/2)\pi}{L}\right)^{-2s} \\ &= \left(\frac{\pi}{L}\right)^{-2s}\left((B-1)2^{2s} - B + 2\right)\zeta_R(2s)\end{aligned}$$

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Zeta fn $L_b = L$

$$\zeta(s) = \left(\frac{\pi}{L}\right)^{-2s} \left((B-1)2^{2s} - B + 2\right) \zeta_R(2s)$$

Spectral properties

- $\det'(-\triangle) = \exp\Big(-\zeta'(0)\Big) = \dfrac{2^B \mathcal{L}}{B}$
- $E_c = \dfrac{1}{2}\zeta(-1/2) = \dfrac{\pi}{4L}(3-B)\zeta_R(-1) = \dfrac{\pi}{48L}(B-3)$
- $K(\tau) \underset{\tau \to 0}{\sim} \dfrac{\mathcal{L}}{\sqrt{4\pi\tau}} - \dfrac{1}{2}$

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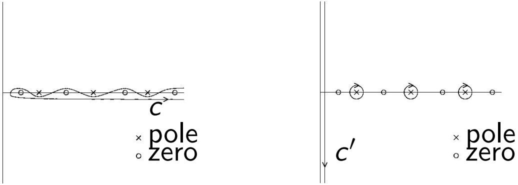
Incommensurate bond lengths

Let $f(z) = \frac{1}{z} \sum_{b=1}^{B} \tan z L_b$, secular eqn $f(k) = 0$.

Argument principle

$$\zeta(s) = \sum_{j=1}^{\infty} k_j^{-2s} = \frac{1}{2\pi \mathrm{i}} \int_c z^{-2s} \frac{f'(z)}{f(z)} \, \mathrm{d}z$$



Transform c to c' $\zeta(s) = \zeta_{\mathrm{Im}}(s) + \zeta_{\mathrm{P}}(s)$.

Pole contribution

At a pole z_0 of f subtract residue z_0^{-2s}.

$$\zeta_{\mathrm{P}}(s) = \sum_{b=1}^{B} \left(\frac{\pi}{L_b}\right)^{-2s} \sum_{m=0}^{\infty} \left(m + \frac{1}{2}\right)^{-2s}$$
$$= (2^{2s} - 1)\zeta_{\mathrm{R}}(2s) \sum_{b=1}^{B} \left(\frac{\pi}{L_b}\right)^{-2s}$$

$$\zeta_{\mathrm{P}}'(0) = 2\log 2\,\zeta_{\mathrm{R}}(0)B$$
$$= -B\log 2$$

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Imaginary axis integral

$z = \mathrm{i}t$ and $f(\mathrm{i}t) = -\hat{f}(t)/t$, where $\hat{f}(t) = \sum_{b=1}^{B} \tanh tL_b$.

$$\zeta_{\mathrm{Im}}(s) = \frac{1}{2\pi \mathrm{i}} \int_{\infty}^{-\infty} (\mathrm{i}t)^{-2s} \frac{\mathrm{d}}{\mathrm{d}t} \log\left(f(\mathrm{i}t)\right) \mathrm{d}t \qquad 0 < \mathrm{Re}\, s < 1$$

$$= \frac{\sin \pi s}{\pi} \left[\int_0^1 t^{-2s} \frac{\mathrm{d}}{\mathrm{d}t} \log\left(\frac{1}{t}\hat{f}(t)\right) \mathrm{d}t + \int_1^{\infty} t^{-2s} \frac{\mathrm{d}}{\mathrm{d}t} \log \hat{f}(t)\, \mathrm{d}t - \frac{1}{2s} \right] \qquad \mathrm{Re}\, s < 1$$

$$\zeta'_{\mathrm{Im}}(0) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} \log\left(\frac{\hat{f}(t)}{t}\right) \mathrm{d}t + \int_1^{\infty} \frac{\mathrm{d}}{\mathrm{d}t} \log \hat{f}(t)\, \mathrm{d}t$$

$$= -\log \mathcal{L} + \log B$$

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Spectral determinant

$$\det{}'(-\triangle) = \exp -\zeta'(0) = \exp(B\log 2 + \log\mathcal{L} - \log B) = \frac{2^B\mathcal{L}}{B}$$

Vacuum energy

$$E_c = \frac{\pi}{48}\sum_{b=1}^{B} L_b^{-1} - \frac{1}{2\pi}\int_0^\infty t\frac{\hat{f}'(t)}{\hat{f}(t)}\,\mathrm{d}t \qquad \hat{f}(t) = \sum_{b=1}^{B}\tanh tL_b$$

Setting $L_b = L$

$$E_c = \frac{\pi B}{48L} - \frac{L}{2\pi}\int_0^\infty t\frac{B\mathrm{sech}^2(tL)}{B\tanh(tL)}\,\mathrm{d}t\ .$$

Integrating again $E_c = \pi(B-3)/48L$.

General star graph

- B nodes with Dirichlet bcs, $\psi_b(0) = 0$.
- Matching at center defined by two $B \times B$ matrices via

$$\mathbb{A}\boldsymbol{\psi} + \mathbb{B}\boldsymbol{\psi}' = \mathbf{0} \ .$$

where $\mathbb{A}, \mathbb{B}$ define a self-adjoint Laplace op. iff rank$(\mathbb{A}, \mathbb{B}) = B$ and $\mathbb{A}\mathbb{B}^\dagger = \mathbb{B}\mathbb{A}^\dagger$ (Kostrykin and Schrader).

e.g. Neumann matching at center

$$\mathbb{A} = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & -1 \\ 0 & \dots & 0 & 0 & 0 \end{pmatrix} , \qquad \mathbb{B} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \\ 1 & 1 & \dots & 1 \end{pmatrix} .$$

Secular equation

$\psi_b(x_b) = c_b \sin kx_b$ let $\mathbf{c} = (c_1, \dots, c_B)^T$ & $\mathbf{L} = \mathrm{diag}\{L_1, \dots, L_B\}$.
The matching condition at the center is then

$$\Big(\mathbb{A}\sin(k\mathbf{L}) - k\mathbb{B}\cos(k\mathbf{L})\Big)\mathbf{c} = \mathbf{0} \ . \qquad (1)$$

k is an eigenvalue iff it is a soln of

$$\det\Big(\mathbb{A}\sin(k\mathbf{L}) - k\mathbb{B}\cos(k\mathbf{L})\Big) = 0 \ . \qquad (2)$$

Secular equation of a general star

$$\det\begin{pmatrix} \mathbb{A} & \mathbb{B} \\ \mathrm{I}_B & \frac{1}{k}\tan(k\mathbf{L}) \end{pmatrix} = 0$$

General graph

- Matching conditions specified by $2B \times 2B$ matrices

$$\mathbb{A}\boldsymbol{\psi} + \mathbb{B}\boldsymbol{\psi}' = \mathbf{0} \ . \qquad (3)$$

- Wavefunction on bond $\psi_b(x_b) = c_b \sin kx_b + \hat{c}_b \cos kx_b$.

$$\left[\mathbb{A} \begin{pmatrix} 0 & \mathrm{I} \\ \sin(k\mathbf{L}) & \cos(k\mathbf{L}) \end{pmatrix} + k\mathbb{B} \begin{pmatrix} \mathrm{I} & 0 \\ -\cos(k\mathbf{L}) & \sin(k\mathbf{L}) \end{pmatrix} \right] \begin{pmatrix} \mathbf{c} \\ \hat{\mathbf{c}} \end{pmatrix} = \mathbf{0}$$

Secular equation

$$\det \left(\mathbb{A} + k\mathbb{B} \begin{pmatrix} -\cot(k\mathbf{L}) & \csc(k\mathbf{L}) \\ \csc(k\mathbf{L}) & -\cot(k\mathbf{L}) \end{pmatrix} \right) = 0$$

Theorem (H, Kirsten)

For a graph with vertex matching conditions defined by $\mathbb{A}, \mathbb{B}$ with $\{L_b\}_{b=1}^{B}$ incommensurate,

$$\zeta(s) = \frac{\sin \pi s}{\pi}\left[\int_0^1 t^{-2s}\frac{\mathrm{d}}{\mathrm{d}t}\log \hat{f}(t)\,\mathrm{d}t \int_1^\infty t^{-2s}\frac{\mathrm{d}}{\mathrm{d}t}\log(t^{-N}\hat{f}(t))\,\mathrm{d}t\right] + \frac{N\sin\pi s}{2\pi s} + \zeta_R(2s)\sum_{b=1}^{B}\left(\frac{\pi}{L_b}\right)^{-2s} \qquad -\frac{1}{2} < \mathrm{Re}\, s < 1$$

$$\hat{f}(t) = \det\left(\mathbb{A} - t\mathbb{B}\begin{pmatrix}\coth(t\mathbf{L}) & -\mathrm{csch}(t\mathbf{L}) \\ -\mathrm{csch}(t\mathbf{L}) & \coth(t\mathbf{L})\end{pmatrix}\right)$$

$$\underset{t\to\infty}{\sim} \det(\mathbb{A} - t\mathbb{B}) = a_N t^N + \cdots + a_1 t + \det\mathbb{A}$$

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Spectral determinant

$$\det{}'(-\triangle) = \exp\Big(- \zeta'(0) \Big)$$

Neumann star

$$\det{}'(-\triangle) = \frac{2^B \mathcal{L}}{B}$$

General graph

$$\det{}'(-\triangle) = \frac{2^B}{a_N \prod_{b=1}^{B} L_b} \det\left(\mathbb{A} - \mathbb{B} \begin{pmatrix} \mathbf{L}^{-1} & -\mathbf{L}^{-1} \\ -\mathbf{L}^{-1} & \mathbf{L}^{-1} \end{pmatrix} \right)$$

Vacuum energy

$$\log \hat{f}(t) \underset{t\to\infty}{\sim} N \log t + \log a_N + \frac{a_{N-j}}{a_N t^j} + O(t^{-(j+1)}) \tag{4}$$

a_N and a_{N-j} first 2 non-vanishing coeffs. in expansion of $\det(\mathbb{A} - t\mathbb{B})$.

Subtracting asymptotic behavior integral convergent for $\mathrm{Re}\, s > -(j+1)/2$.

$$\zeta(s) = \frac{\zeta_R(2s)}{\pi^{2s}} \sum_{b=1}^{B} L_b^{2s} + \frac{\sin \pi s}{\pi} \left[\int_0^1 t^{-2s} \frac{\mathrm{d}}{\mathrm{d}t} \log \hat{f}(t)\, \mathrm{d}t \right.$$
$$\left. + \int_1^\infty t^{-2s} \frac{\mathrm{d}}{\mathrm{d}t} \left(\log \left(t^{-N} \hat{f}(t) \right) - \frac{a_{N-j}}{a_N\, t^j} \right) \mathrm{d}t + \frac{N}{2s} - \frac{a_{N-j}\, j}{a_N\, (2s+j)} \right]$$

$E_c = \frac{1}{2}\zeta(-1/2)$ divergent for $j = 1$ generic case.

But only changes in E_c observable.

Casimir force on bond b_0

$$\begin{aligned} F_{c,b_0} &= \frac{\partial}{\partial L_{b_0}} E_c \\ &= \frac{\pi}{24 L_{b_0}^2} + \frac{1}{\pi} \int_0^\infty \frac{\partial}{\partial L_{b_0}} \log \hat{f}(t) \mathrm{d}t \\ \hat{f}(t) &= \det \left(\mathbb{A} - t\mathbb{B} \begin{pmatrix} \coth(t\mathbf{L}) & -\mathrm{csch}(t\mathbf{L}) \\ -\mathrm{csch}(t\mathbf{L}) & \coth(t\mathbf{L}) \end{pmatrix} \right) \end{aligned}$$

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Heat kernel asymptotics

Restriction $\mathrm{Re}\, s > -1/2$ comes from $t \to \infty$ behavior,

$$\hat{f}(t) \sim \det(\mathbb{A} - t\mathbb{B}) = t^N \sum_{j=0}^{N} \frac{a_{N-j}}{t^j} \tag{5}$$

$$\log \hat{f}(t) \sim N \log t + \log a_N + \sum_{n=1}^{\infty} \frac{b_n}{t^n} \tag{6}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \log \hat{f}(t) \sim \frac{N}{t} - \sum_{n=1}^{\infty} \frac{n b_n}{t^{n+1}} \tag{7}$$

Heat kernel expansion

$$K(\tau) \underset{\tau \to 0}{\sim} \frac{\mathcal{L}}{\sqrt{4\pi\tau}} + \frac{N}{2} - \sum_{k=1,3/2,2,\ldots}^{\infty} \frac{b_{2k-1}}{\Gamma\left(k - \frac{1}{2}\right)} \tau^{k-1/2}$$

Piston graph

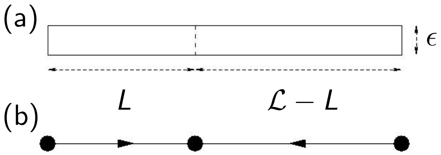

Figure: (a) An ϵ thick piston. (b) The corresponding piston graph.

- $\psi_1(0) = \psi_2(0) = 0$.
- $\psi_1(L) = \psi_2(\mathcal{L} - L) = \psi$ and $\psi_1'(L) + \psi_2'(\mathcal{L} - L) = \gamma\psi$.

$$\mathbb{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -\gamma & 0 \end{pmatrix} \qquad \mathbb{B} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\hat{f}(t) = \det \left(\mathbb{A} - t\mathbb{B} \begin{pmatrix} \coth(t\mathbf{L}) & -\mathrm{csch}(t\mathbf{L}) \\ -\mathrm{csch}(t\mathbf{L}) & \coth(t\mathbf{L}) \end{pmatrix} \right)$$

$$\underset{t\to\infty}{\sim} \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -\gamma & -t \end{pmatrix} = -2t - \gamma$$

$$K(\tau) \underset{\tau\to 0}{\sim} \frac{\mathcal{L}}{\sqrt{4\pi\tau}} - \frac{1}{2} - \sum_{k=1,3/2,2,\ldots}^{\infty} \frac{(-1)^{2k}}{(2k-1)\Gamma\left(k-\frac{1}{2}\right)} \left(\frac{\gamma}{2}\right)^{2k-1} \tau^{k-1/2}$$

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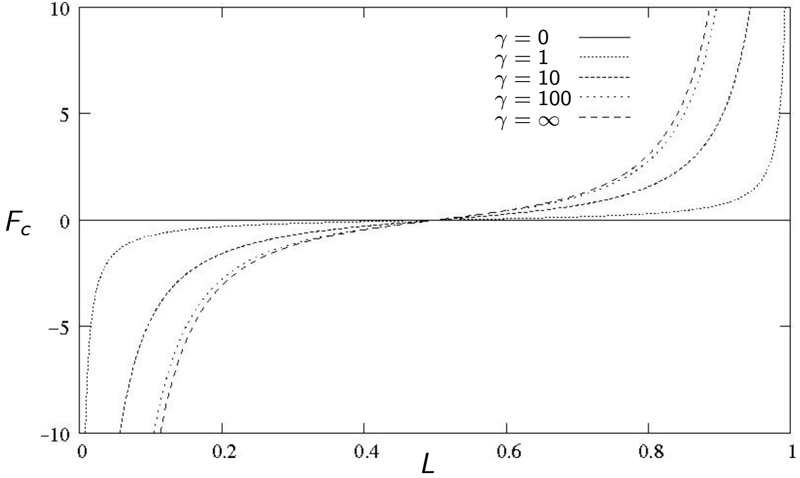

Figure: Casimir force on on the central vertex of the piston graph.

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Conclusions

- Obtained general expressions for zeta fns of quantum graphs.
- Formulation of spectral determinant for general matching conditions.
- Integral formulation of vacuum energy.
- Heat kernel expansion in terms of vertex matching conditions.

Outlook

- Relationship with zeta fns of combinatorial graphs.
- Zeta fns on thin networks.

References

- J. M. Harrison and K. Kirsten, "Zeta functions of quantum graphs," arXiv:0911.2509
- J. M. Harrison and K. Kirsten, "Vacuum energy, spectral determinant and heat kernel asymptotics of graph Laplacians with general vertex matching conditions," arXiv:0912.0036

JMH supported by NSF grant DMS-0604859.
KK supported by NSF grant PHY-0554849.