# Line integrals of 1-forms on the Sierpinski gasket

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# Outline

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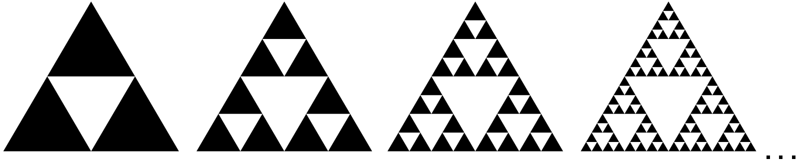
# Definition

• $T \subset \mathbb{R}^2$ equilateral triangle, side length 1, vertices $V_0 := \{v_1, v_2, v_3\}$

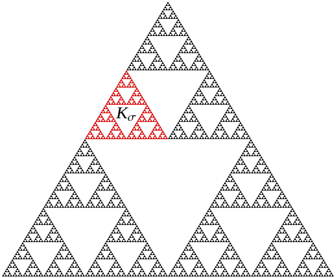
• $w_i : x \in \mathbb{R}^2 \to \frac{1}{2}(x - v_i) + v_i \in \mathbb{R}^2$, $i = 1, 2, 3$

Then there is a unique compact $K \subset \mathbb{R}^2$ s.t. $K = W(K) \equiv \cup_{i=1}^3 w_i(K)$, also obtained as $\lim_{n\to\infty} W^n(T)$.



$K$ is called Sierpinski gasket.

# Definitions




Similarities: $w_\sigma = w_{\sigma_1} \cdot \ldots \cdot w_{\sigma_n}$, $|\sigma| = n$
Cells: $K_\sigma = w_\sigma(K)$
Then $K = \bigcup_{|\sigma|=n} K_\sigma$
Edges: $E = \bigcup_n E_n$, $E_0 = \{e_1, e_2, e_3\}$,
$E_n = \{w_\sigma(e), |\sigma| = n, e \in E_0\}$.
$o(e)$ = origin of $e$, $t(e)$ = terminus of $e$
Lacunas: $\ell_\sigma = w_\sigma(\ell)$

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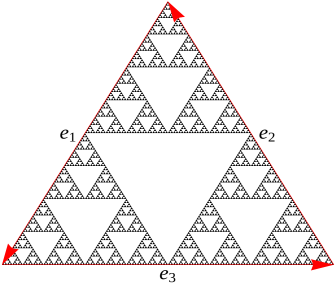


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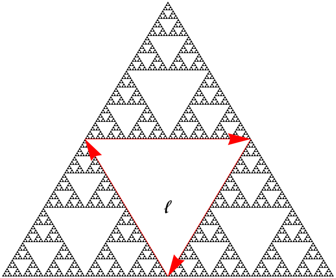


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# Topological and metric properties

$K$ is connected, locally connected, arcwise connected, but it is not semilocally simply connected [ $\Longrightarrow$ no universal cover].

- $\alpha$-dimensional Hausdorff measure of $E \subset \mathbb{R}^n$:

$$H^\alpha(E) := \lim_{\delta \to 0} \inf_{\substack{E \subset \bigcup_i A_i \\ \operatorname{diam} A_i \leq \delta}} \sum_{i=1}^{\infty} (\operatorname{diam} A_i)^\alpha$$

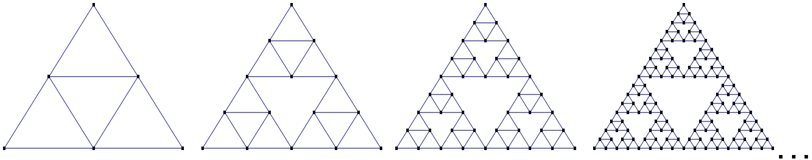
- Hausdorff dimension of $E$:

$$\begin{aligned} d_H(E) &= \inf\{\alpha > 0 : H^\alpha(E) = 0\} \\ &= \sup\{\alpha > 0 : H^\alpha(E) = +\infty\} \end{aligned}$$

Then $d_H(K) = \frac{\log 3}{\log 2} =: d$, $\quad H^d(K) \in (0, \infty)$, and $H^d(K) = \frac{1}{3} \sum_{i=1}^{3} H^d(w_i^{-1}(K))$.

# Dirichlet form

Different way of approximating $K$: sequence of graphs $(V_n, E_n)$



• Dirichlet (or energy) form on $(V_n, E_n)$:
$\mathcal{E}_n[f] := \sum_{x \sim y}(f(x) - f(y))^2$. Then $\left(\frac{5}{3}\right)^n \mathcal{E}_n[f] \nearrow \mathcal{E}[f]$.
• Set $\mathcal{F} := \{f : \mathcal{E}[f] < \infty\}$. Then $\mathcal{F} \subset C(K)$.
• Choose Borel regular prob. measure $\mu$ on $K$. Then $(\mathcal{E}, \mathcal{F})$ is local regular Dirichlet form on $L^2(K, \mu)$. Therefore $\mathcal{E}(f, g) = (f, \Delta_\mu g)$, $\Delta_\mu \geq 0$, compact resolvent.

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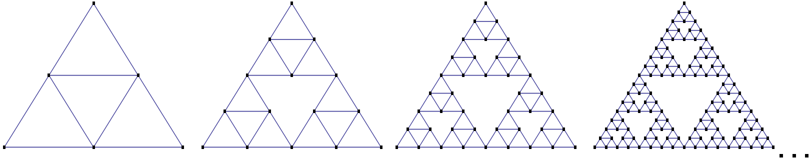

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# Laplacian for Bernoulli measure

Different construction of $\Delta_\mu$ for $\mu$ Bernoulli measure.

- for $f : V_n \setminus V_0 \to \mathbb{R}$ set $\Delta_n f(x) := \sum_{y\sim x}(f(x) - f(y))$
- for $f \in C(K)$ set $\Delta f(x) := \frac{3}{2} \lim_{n\to\infty} 5^n \Delta_n f(x)$, if limit $\exists$
- $\mathrm{dom}(\Delta) := \{f \in C(K) : \Delta f \in C(K)\} \subset \mathcal{F}$

Obs. $f \in \mathrm{dom}(\Delta) \implies f^2 \notin \mathrm{dom}(\Delta)$.

- for $f : V_n \to \mathbb{R}$ set $\left(\frac{\partial f}{\partial \nu}\right)_n(x) := \sum_{y\sim x}(f(x) - f(y)), x \in V_0$
- for $f \in C(K)$ set $\frac{\partial f}{\partial \nu}(x) := \lim_{n\to\infty} \left(\frac{5}{3}\right)^n \left(\frac{\partial f}{\partial \nu}\right)_n(x)$, if limit $\exists$

**Theorem (Gauss-Green)**

*$\mu$ Bernoulli measure, $f \in \mathrm{dom}(\Delta)$, $g \in \mathcal{F}$. Then*
$\mathcal{E}(f, g) = \int_K g\Delta f \, d\mu + \sum_{p\in V_0} g(p)\frac{\partial f}{\partial \nu}(p)$.

# Weyl-type asymptotics

Classical Weyl asymptotic: $\Omega \subset \mathbb{R}^n$ bdd connected open set.
Define eigenvalue counting function:
$N(x) := \sum_{\lambda \leq x} \dim\{f \in \mathrm{dom}(\Delta) : \Delta f = \lambda f\}$.
Then $N(x) = cx^{n/2}(1 + o(1))$, $x \to \infty$.
As for the gasket, $\exists G$, a nonconstant $\frac{1}{2}\log 5$-periodic function,
s.t. $N(x) = \{G(\log \frac{x}{2}) + o(1)\}x^{d_S/2}$, $x \to \infty$,
where $d_S := \frac{\log 9}{\log 5}$, is the spectral exponent.

# Harmonic functions

- for $m \in \mathbb{N} \cup \{0\}$, $u : V_m \to \mathbb{R}$, $\exists! f \in \mathcal{F}$ s.t. $f|_{V_m} = u$, $\mathcal{E}[f] = \min\{\mathcal{E}[g] : g \in \mathcal{F}, g|_{V_m} = u\}$. $f$ is said $m$-harmonic.

**Theorem (weak maximum principle)**

*$f$ m-harmonic, $\sigma$ multiindex, $|\sigma| \geq m$, $x \in w_\sigma(K)$. Then $\min_{w_\sigma(V_0)} f \leq f(x) \leq \max_{w_\sigma(V_0)} f$.*

- Obs. Harmonic functions are dense in $C(K)$.

# Outline

1. The Sierpinski gasket
2. 1-forms on the Sierpinski gasket
3. Coverings and potentials

# Universal 1-forms

Want to construct differential 1-forms on $K$.

• $d : g \in \mathcal{F} \to 1 \otimes g - g \otimes 1 \in \mathcal{F} \otimes \mathcal{F}$

• $\Omega^1(\mathcal{F})$ the $\mathcal{F}$-bimodule generated by $\{fdg : f, g \in \mathcal{F}\}$. It's called bimodule of universal 1-forms. Actions of $\mathcal{F}$:

$$\begin{cases} h.(fdg) = (hf)dg, & h \in \mathcal{F} \\ (fdg).h = fd(gh) - (fg)dh & fdg \in \Omega^1(\mathcal{F}). \end{cases}$$

• $\partial g(e) := g(t(e)) - g(o(e))$, $g \in \mathcal{F}$, $e \in E$

# 1-forms

Define

- $(fdg, fdg) = \lim_n \left(\frac{5}{3}\right)^n \sum_{e \in E_n} |f(o(e))|^2 |\partial g(e)|^2$, extended by bilinearity,
- $\int_e fdg = \lim_n \sum_{e_1 \in E_n, e_1 \subseteq e} f(o(e_1)) \partial g(e_1)$, extended by linearity.

Then

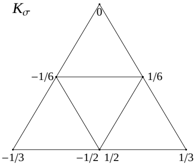
- the limits above are well defined and finite for any $\omega \in \Omega^1(\mathcal{F})$,
- if $\int_e \omega = 0$ for any $e \in E$, then $\|\omega\| = 0$.

**Definition (1-forms)**

Set $\Omega = \Omega^1(\mathcal{F})/\sim$, where $\omega \sim 0$ if $\int_e \omega = 0$ for any $e$.

# $n$-exact 1-forms

There exists a map $\{\{f_\sigma\}_{|\sigma|=n}, \mathcal{E}_{K_\sigma}[f_\sigma] < \infty\} \longrightarrow \omega \in \Omega$ with $\omega|_{K_\sigma} = df_\sigma$. The form $\omega$ will be called *n*-exact.




$\forall \sigma \in \Sigma$, let $dz_\sigma$ attain the $\min\{\|\omega\| : \omega \in \Omega, \text{is } n+1\text{-exact}, \int_{\ell_\sigma} \omega = 1\}$. The form $dz_\sigma$ is zero on $K_\sigma^c$, and is given by the (harmonic) function $z_\sigma$ on $K_\sigma$.

Then

- the set $\{dz_\sigma\}_{\sigma\in\Sigma}$ is an orthogonal system, with $\|dz_\sigma\|^2 = 5/6(5/3)^{|\sigma|}$,
- the $dz_\sigma$'s are co-closed: $(df, dz_\sigma) = 0$, $\forall f \in \mathcal{F}$.

Obs. Since $K$ is topologically 1-dimensional, any 1-form is closed, hence we say that $dz_\sigma$ is a harmonic 1-form.

# Hodge decomposition

$\forall \omega \in \Omega\ \exists! \{k_\sigma\}_{\sigma \in \Sigma}$ s.t., setting $\omega_0 = \sum_\sigma k_\sigma dz_\sigma$, we have

- $N(k_\sigma) = \sup_n (5/3)^n \sum_{|\sigma|=n} |k_\sigma| < \infty$.
- $\omega_0 \in \Omega$,
- $\|\omega_0\|^2 = 5/6 \sum_\sigma (5/3)^{|\sigma|} |k_\sigma|^2 < \infty$,
- $\omega - \omega_0$ is exact, i.e. $\exists U_1 \in \mathcal{F}$ s.t. $\omega = dU_1 + \omega_0$.

Therefore

- $\|\omega\| = 0 \implies \omega \sim 0$, i.e. $\Omega$ is a pre-Hilbert space,
- Hodge decomposition: any 1-form in $\Omega$ can be uniquely decomposed into an exact and a harmonic part.

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# Coverings with finitely generated homotopy

Let $T$ = convex hull$(K)$, then $i_n : K \hookrightarrow T_n := \cup_{|\sigma|=n} w_\sigma(T)$,
$i_{n*} : \pi_1(K) \to \pi_1(T_n)$ = free group with $\#\{|\sigma| < n\}$ generators.
Let $\widetilde{T}_n$ be the universal covering of $T_n$. Then
there exists a covering $\widetilde{K}_n$ of $K$ such that the diagram

$$\begin{array}{ccc} \widetilde{K}_n & \hookrightarrow & \widetilde{T}_n \\ p_n \downarrow & \circlearrowleft & \downarrow \\ K & \hookrightarrow & T_n \end{array}$$

commutes, and deck$(\widetilde{K}_n)$ = deck$(\widetilde{T}_n) = \pi_1(T_n)$.

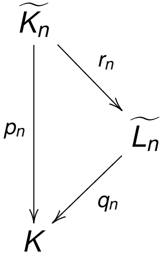
# Coverings with finitely generated homotopy

The family $\{(\widetilde{K}_n, p_n)\}$ is projective.
Take projective limit $(\widetilde{K}, p) := \varprojlim (\widetilde{K}_n, p_n)$.

Then, any path $\gamma$ in $K$ has a lifting $\tilde{\gamma}$ in $\widetilde{K}$, unique up to the starting point.
Obs. $\mathrm{deck}(\widetilde{K}) = \varprojlim \mathrm{deck}(T_n) = \check{\pi}_1(K)$, the first Čech homotopy group of $K$, and $\check{\pi}_1(K) \supset \pi_1(K)$.

# Abelian coverings

A smaller covering. Set $\widetilde{L}_n := \widetilde{K}_n/[\text{deck}(\widetilde{K}_n), \text{deck}(\widetilde{K}_n)]$.




Then

- $\text{deck}(\widetilde{L}_n) = \text{deck}(\widetilde{K}_n)/[\text{deck}(\widetilde{K}_n), \text{deck}(\widetilde{K}_n)]$ is a free abelian group with $\#\{|\sigma| < n\}$ generators,
- $\{(\widetilde{L}_n, q_n)\}$ is a projective family.

Set $(\widetilde{L}, q) = \lim\limits_{\leftarrow} (\widetilde{L}_n, q_n)$. Then

- $\text{deck}(\widetilde{L}) = \lim\limits_{\leftarrow} \text{deck}(\widetilde{L}_n)$,
- any path $\gamma$ in $K$ has a lifting $\tilde{\gamma}$ in $\widetilde{L}$, unique up to the starting point.

# Affine functions. Potentials of $n$-exact 1-forms

Say $f \in \mathcal{C}(\widetilde{K}_n)$ is deck($\widetilde{K}_n$)-affine if $\exists \varphi \in \hom(\mathrm{deck}(\widetilde{K}_n), (\mathbb{C}, +))$ s.t. $f(gx) - f(x) = \varphi(g), \quad \forall x \in \widetilde{K}_n, g \in \mathrm{deck}(\widetilde{K}_n)$.

Then

Any $n$-exact 1-form $\omega$ has a unique (up to an additive constant) deck($\widetilde{K}_n$)-affine potential $U$, for which

- $\int_e \omega = \partial U(e) := U(t(\tilde{e})) - U(o(\tilde{e}))$, $\tilde{e}$ a lifting of $e$ to $\widetilde{K}_n$
- $\|\omega\|^2 = \mathcal{E}[U] = \lim_n \left(\frac{5}{3}\right)^n \sum_{e \in E_n} |\partial U(e)|^2$.

We denote by $z_\sigma$ the potential of $dz_\sigma$.

# Potentials of $n$-exact 1-forms live on abelian coverings

Let $\omega, U, \varphi$ be as above. Since $(\mathbb{C}, +)$ is abelian, $\varphi$ vanishes on commutators. Therefore,

• the potential $U$ is a deck$(\widetilde{L}_n)$-affine function on $\widetilde{L}_n$.

Obs.

• the projective limit topology on $\widetilde{L}$ is generated by $\{z_\sigma : \sigma \in \Sigma\}$,

• any deck$(\widetilde{L})$-affine function on $\widetilde{L}$ is the lifting of a deck$(\widetilde{L}_n)$-affine function on $\widetilde{L}_n$, for some $n$.

# Restricting the covering, pseudometrics

Set $N'(a_\sigma) = \sum_n \left(\frac{3}{5}\right)^n \sup_{|\sigma|=n} |a_\sigma|$, so $\left|\sum_\sigma a_\sigma k_\sigma\right| \leq N'(a_\sigma)N(k_\sigma)$,
and define $d(x,y) = N'\big(z_\sigma(y) - z_\sigma(x)\big)$, $x, y \in \widetilde{L}$.

Then

• $d$ is a pseudometric on $\widetilde{L}$, namely a metric which is allowed to be infinite. The $d$-topology is finer than the projective limit topology.

• For any $g \in \text{deck}(\widetilde{L})$, $\ell(g) = d(x, gx)$ does not depend on $x$, and $\Gamma = \{g \in \text{deck}(\widetilde{L}) : \ell(g) < \infty\}$ is a subgroup of deck$(\widetilde{L})$.

• $\Gamma$ acts on any $d$-component of $\widetilde{L}$, namely on any subset of $\widetilde{L}$ consisting of points having finite mutual distance.

# Integration on paths

- Any 1-form $\omega \in \Omega$ has a $\Gamma$-affine potential $U$ on any $d$-component of $\widetilde{L}$.
- If $\omega = dU_1 + \sum_\sigma k_\sigma dz_\sigma$, then $U = U_0 + U_1$, with $U_0 = \sum_\sigma k_\sigma z_\sigma$. Such a sum converges uniformly on compact sets to a $\Gamma$-affine, $d$-continuous function on any $d$-component of $\widetilde{L}$. Indeed, $|U_0(x) - U_0(y)| \leq N(k_\sigma)d(x,y)$.
- For any path $\gamma$ in $K$, set $\ell(\gamma) = d(\tilde{\gamma}(1), \tilde{\gamma}(0))$. Then, if $\ell(\gamma) < \infty$, and $U$ is the potential of $\omega \in \Omega$,

$$\int_\gamma \omega = U(\tilde{\gamma}(1)) - U(\tilde{\gamma}(0)).$$