On the Splitting of Primes in Coverings

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Outline

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In this talk we will deal with finite connected galois graph covers. We will allow multiple edges and loops. Directed graphs will not be considered.

Definition

An undirected finite graph Y is said to **Cover** an undirected finite graph X if after an arbitrary assignment of directions to the edges of X there exists an assignment of directions of the edges of Y and a covering map $\pi : V(Y) \to V(X)$ which is bijective on neighborhoods and preserves directions.

The spectator may be convinced he has spotted a more simplistic definition for covering, the complications in the preceding definition arise from allowing finite graphs to have multiple edges or loops.

Definition

A closed path in X is said to **Split Completely** if when lifted to Y it remains a closed path.

Let $N_{e,n}$ be the number of closed non-backtracking tailless paths of length n in X, which when lifted to Y return to form a closed path or 'split completely.' Here, e is viewed as the identity element of the galois group. If we were to substitute other elements of the galois group into the first component, we denote the number of closed paths in X, which when lifted to sheet e, end at the specified elements sheet.
We will let $\pi_{n,e}$ denote the number of prime paths (not classes) in X which split completely.
In this talk we will attempt to calculate these quantities.

Prime Paths are More Complicated then Closed Paths.

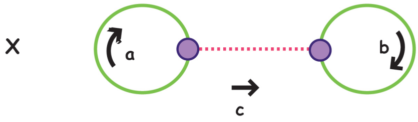

Figure: A picture of the Dumbbell Graph, X

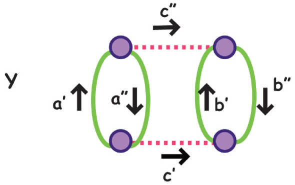

Figure: A picture of a 2-Sheeted Cover of the Dumbbell Graph, Y

Splitting Paths in Y/X

One calculates that

n	1	2	3	4	5	6
$N_{e,n}$	0	4	0	20	0	76
n	7	8	9	10	11	12
$N_{e,n}$	0	228	0	1084	0	4052

Table: List of values of $N_{e,n}$ for the cover Y/X

Surprisingly, the four paths which split completely of minimal length 2 are $a^{\pm 2}, b^{\pm 2}$. All are non-prime paths.

One can establish that

Theorem

$$N_{e,n} = \frac{1}{|\mathrm{Gal}(Y/X)|} \sum_{\rho \in \widehat{\mathrm{Gal}(Y/X)}} d_\rho \, \mathrm{tr}(W_{1,\rho}^n) = \frac{N_{n,Y}}{|\mathrm{Gal}(Y/X)|}$$

This gives a relatively simple method of calculating paths which 'split completely' using data from the base graph or data from the cover. But how to calculate the number of prime paths with these type of properties?

Notice,

Theorem

$$N_{e,n} = \frac{N_{n,Y}}{|\operatorname{Gal}(Y/X)|} = \sum_{d|n} \sum_{\sigma^{\frac{n}{d}}=e} \pi_{\sigma,n}$$

Hence,

$$\frac{1}{|G|} N_{n,Y} = \sum_{d|n} \sum_{\sigma^{\frac{n}{d}}=e} \pi_{\sigma,d}$$

Notice, if $(|\operatorname{Gal}(Y/X)|, n) = 1$, then

$$\frac{1}{|G|} N_{n,Y} = \sum_{d|n} \pi_{e,d}$$

$$\pi_{e,n} = \sum_{d|n} \frac{1}{|G|} \mu(d) N_{\frac{n}{d},Y}$$

Hence, applying the prime path theorem for graphs one obtains that if $(|\operatorname{Gal}(Y/X)|, n) = 1$, then

$$\pi_{e,n} \sim \frac{\Delta Y}{|G|}\lambda_{max}^{N}.$$

In general,

$$\pi_{e,n} \leq \sum_{d|n} \frac{1}{|G|} N_{\frac{n}{d},Y}$$

Thus, one can obtain the growth control $O(\frac{\Delta Y}{|G|}\lambda_{max}^{N})$. Here, λ_{max} is the value of the largest eigenvalue in absolute value of the nonbacktacking path math matrix, W_1.

Furhter Questions

- An assymtotic in the non-coprime case?
- What if we want to know about paths which end at a sheet indexed by specific element of the galois group?