Heat kernel estimates on metric trees

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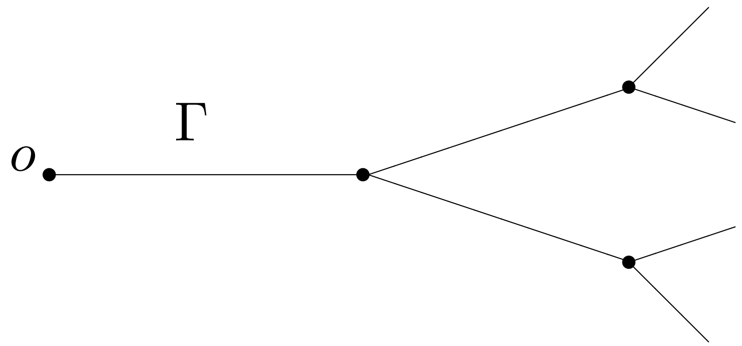
Metric trees



- $\mathcal{V}$ ... set of vertices, $\mathcal{E}$ ... set of edges, no loops.
- $|x| = \mathsf{dist}(o, x)$.
- $\sup_{x\in\Gamma} |x| = \infty$.
- We assume that Γ is **symmetric**, i.e. all the vertices of the same generation produce the same number of edges, which have equal length.

Laplace operator on Γ

We denote by $-\Delta$ a differential operator which acts on each edge as

$$-\Delta\, u = -\frac{d^2 u}{dx^2}\,,$$

where

$$u \in H^2(e) \quad \forall\, e \in \mathcal{E}, \quad \sum_{e \in \mathcal{E}} \|u\|^2_{H^2(e)} < \infty$$

and at each vertex $z \in \mathcal{V}$ u is continuous and satisfies

$$\sum_{e \in \mathcal{E}_z} \frac{du}{dx}(z) = 0 \qquad \text{(Kirchhoff b.c.)}$$

At the root we have

$$\frac{du}{dx}(o) = 0 \qquad \text{(Neumann b.c.)}$$

The heat kernel

The operator $-\Delta$ is associated with the quadratic form

$$\int_\Gamma |u'|^2\, dx \qquad u \in H^1(\Gamma)$$

and generates the semigroup $e^{t\Delta}$, $t > 0$ on $L^2(\Gamma)$ with the integral kernel

$$k(t,x,y) := e^{t\Delta}(x,y), \qquad x, y \in \Gamma$$

The heat kernel: a uniform estimate

Theorem 1 Let Γ be a symmetric tree with infinite volume. Then

$$\sup_{x\in\Gamma} k(t,x,x) \leq (\pi\, t)^{-1/2} \qquad t>0.$$

- The example $\Gamma = \mathbb{R}_+$ shows that the right hand side is sharp.

The proof is based on the symmetrization of a given function $u \in \Gamma$, vanishing at infinity. We denote by u^* the unique non-increasing function on $[0,\infty)$ with the same distribution function as $|u|$. Similarly as **[Friedlander, 2005]** we get

$$\int_\Gamma |u'|^2\, dx \geq \int_0^\infty |(u^*)'|^2\, dr\,.$$

Hence the Euclidean logarithmic Sobolev inequality (applied to symmetric functions) implies that

$$\frac{a^2}{\pi}\int_\Gamma |u'|^2\,dx \geq \int_\Gamma |u|^2 \ln\left(\frac{|u|^2}{\|u\|^2}\right)\,dx + (1+\ln(a/2))\int_\Gamma |u|^2\,dx\,.$$

for any $u \in H^1(\Gamma)$ and any $a > 0$. The assertion of Thm. 1 then follows by standard methods, **[E.B. Davies, 1989]**.

The heat kernel: pointwise estimates

Let Γ be a **symmetric** tree. Following **[Naimark-Solomyak, 2000]** we describe its branching by the non-decreasing function

$$\mathbf{g_0(r) := \#\{x \in \Gamma \, : \, |x| = r\}}$$

It is known, **[Ekholm-Frank-K., 2007]**, that if

$$\int_0^\infty \frac{dr}{g_0(r)} < \infty \, ,$$

then $-\Delta$ satisfies a Hardy type inequality

$$\int_\Gamma \psi(|x|)|u(x)|^2 \, dx \, \leq \, C(\Gamma, \psi) \int_\Gamma |u'(x)|^2 \, dx, \quad u \in H^1(\Gamma), \ \psi > 0.$$

The heat kernel: pointwise estimates

This means that $-\Delta$ is a subcritical operator on $L^2(\Gamma)$ and hence

$$\int_0^\infty e^{t\Delta}(x,y)\,dt = \int_0^\infty k(t,x,y)\,dt < \infty \qquad \forall\, x \neq y$$

So if the branching of Γ is fast enough, then $k(t,x,y)$ must decay to zero **faster** than $t^{-1/2}$.

Question:

What is the decay rate of $k(t,x,y)$ as $t \to \infty$ and how does it depend on the geometry of the tree Γ ?

- From now on we consider only $k(t,x,x)$.

The heat kernel: pointwise estimates

Theorem 2 Let Γ be a symmetric tree and suppose that

$$c^{-1}\, r^{d-1} \,\le\, g_0(r) \,\le\, c\, r^{d-1} \qquad \forall\, r \ge 1$$

for some $c > 0$ and $d \ge 1$. Then there exists $C(d)$ such that

$$k(t, x, x) \,\le\, C(d)\, g_0(|x|)\, t^{-d/2} \qquad \forall\, x \in \Gamma,\; t > 0.$$

- We call d a **global dimension** of Γ (not necessarily an integer).

Sketch of the proof

We use an orthogonal decomposition of $L^2(\Gamma)$ and $H^1(\Gamma)$ due to **[Carlson, 2000]** and **[Naimark-Solomyak, 2000]**.

Let b_k be the branching number of the vertices of the k^{th} generation, number of out-going edges, and let r_k be their distance to the root. Define

$$g_k(r) := \begin{cases} 0, & r < r_k\,, \\ 1, & r_k \leq r < r_{k+1}\,, \\ b_{k+1}b_{k+2}\cdots b_n, & r_n \leq r < r_{n+1}, \quad k < n\,. \end{cases}$$

If $v \in \mathcal{V}$ is a vertex of the k^{th} generation, we denote by $\Gamma_{v,m}$, $m = 1, ..., b_k$ the mutually disjoint subtrees rooted at v. Let $1 \leq \sigma \leq b_k - 1$ and put

$$Y_{k,v,\sigma}(x) := \frac{1}{\sqrt{b_k \; g_k(|x|)}} \sum_{m=1}^{b_k} \omega_k^{m\sigma} \chi_{v,m}(x), \qquad \omega_k := \exp\left(\frac{2\pi i}{b_k}\right).$$

Sketch of the proof

If f is a function on Γ we put

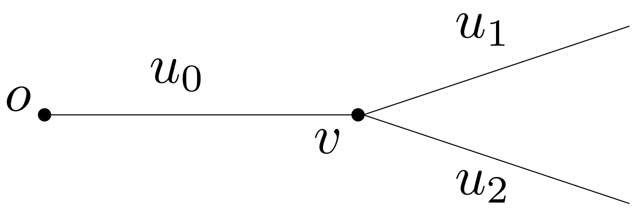
$$f_{k,v,\sigma}(r) := \frac{1}{\sqrt{g_k(r)}} \sum_{|x|=r} \overline{Y_{k,v,\sigma}(x)}\, f(x)\,.$$

Lemma (partial wave decomposition) Let $u \in H^1(\Gamma)$. Denote by $\sum_{k,v,\sigma}$ summation over all $k \in \mathbb{N}_0$, all v with gen $v = k$ and all σ with $1 \leq \sigma \leq b_k - 1$. Then for any $j \in \{0,1\}$ we have

$$u^{(j)}(x) = \sum_{k,v,\sigma} u^{(j)}_{k,v,\sigma}(|x|)\, \sqrt{g_k(|x|)}\, Y_{k,v,\sigma}(x) \qquad \forall\, x \in \Gamma,$$

$$\sum_{|x|=r} |u^{(j)}(x)|^2 = \sum_{k,v,\sigma} |u^{(j)}_{k,v,\sigma}(r)|^2\, g_k(r) \qquad r > 0.$$

Example



$$u_0(v) = u_1(v) = u_2(v)$$

$$\begin{aligned}
k = 0: \quad u_o(r) &= u_0(x) && |x| = r < |v| \\
u_o(r) &= \frac{1}{2}(u_1(x) + u_2(x)) && |x| = r \geq |v| \\
k = 1: \quad u_v(r) &= 0 && |x| = r < |v| \\
u_v(r) &= \frac{1}{2}(u_1(x) - u_2(x)) && |x| = r \geq |v|
\end{aligned}$$

Let A_k, $k = 0, 1, 2 \ldots$ be the non-negative operators in $L^2((r_k, \infty),\, g_k dr)$ generated by the quadratic forms

$$\int_{r_k}^{\infty} |u'|^2\, g_k\, dr, \qquad u \in H_0^1((r_k, \infty), g_k\, dr) \qquad k = 1, 2 \ldots$$

$$\int_0^{\infty} |u'|^2\, g_0\, dr, \qquad u \in H^1((0, \infty), g_0\, dr) \qquad k = 0.$$

Since

$$\|u\|^2_{L^2(\Gamma)} = \sum_{k,v,\sigma} \int_{r_k}^{\infty} |u_{k,v,\sigma}|^2\, g_k\, dr, \qquad \|u'\|^2_{L^2(\Gamma)} = \sum_{k,v,\sigma} \int_{r_k}^{\infty} |u'_{k,v,\sigma}|^2\, g_k\, dr$$

we have

$$(-\Delta u)_{k,v,\sigma} = A_k\, u_{k,v,\sigma}\, .$$

Hence by the spectral theorem

$$(e^{t\Delta}\, u)_{k,v,\sigma} \;=\; e^{-tA_k}\, u_{k,v,\sigma}$$

This implies

$$k(t,x,y) = \sum_{k,v,\sigma} \sqrt{g_k(|x|)\; g_k(|y|)}\; Y_{k,v,\sigma}(x)\, \overline{Y_{k,v,\sigma}(y)}\; p_k(t,|x|,|y|)\,.$$

where

$$p_k(t,|x|,|y|) = e^{-tA_k}(|x|,|y|) \qquad \text{in } \; L^2((r_k,\infty), g_k\, dr)$$

are extended by zero to all negative values of $|x| - r_k$ and $|y| - r_k$.

For $r_L < |x| \leq r_{L+1}$ we thus get

$$k(t,x,x) = p_0(t,|x|,|x|) + \sum_{k=1}^{L} \frac{b_k - 1}{b_k}\, p_k(t,|x|,|x|)\,.$$

Lemma For all $k \geq 1$ and all $t > 0$ one has

$$p_k(t,r,s) \leq b_0 \cdots b_k\, p_0(t,r,s) \qquad \forall\, r,s \geq r_k.$$

$$k(t,x,x) \leq p_0(t,|x|,|x|)\Big(1 + \sum_{k=1}^{L} b_0 \cdots b_{k-1}(b_k - 1)\Big) = p_0(t,|x|,|x|)\, g_0(|x|)$$

To estimate $p_0(t, |x|, |x|)$ we need

Lemma(Nash inequality) Let $d \geq 1$ be a global dimension of Γ. Then there exists a constant c_d such that

$$\|f'\|_2 \, \|f\|_1^{2/d} \, \geq \, c_d \, \|f\|_2^{(2+d)/d}$$

holds for all $f \in H^1(\mathbb{R}_+, g_0) \cap L^1(\mathbb{R}_+, g_0)$.

Corollary For any $t > 0$ it holds

$$\sup_{r>0} p_0(t, r, r) \, \leq \, C_d \, t^{-\frac{d}{2}}.$$

Corollary Let Γ be a symmetric tree with global dimension d. Then

$$k(t,x,x) \quad \leq \quad C(d)\, \min\left\{t^{-\frac{1}{2}},\ g_0(|x|)\, t^{-\frac{d}{2}}\right\}$$

$$k(t,x,x) \quad \leq \quad C(a,d)\ g_0(|x|)^{\frac{a}{d-1}}\ t^{-\frac{1+a}{2}} \qquad\qquad \forall\, a \in [0, d-1]$$

hold for all $t > 0$ and all $x \in \Gamma$.

- Can the weight $g_0(|x|)$ on the right hand side of

$$k(t,x,x) \;\leq\; C(d)\, g_0(|x|)\; t^{-d/2} \qquad\qquad \forall\, x\in\Gamma,\;\; t>0.$$

be improved?

Schrödinger operators

Let $V : \Gamma \to [0, \infty)$ be a real valued measurable function and consider the Schrödinger operator

$$-\Delta - V \qquad \text{in } L^2(\Gamma),$$

We assume that $V \to 0$ as $|x| \to \infty$ and consider the number $N(-\Delta - V, 0)$ of negative eigenvalues of $-\Delta - V$.

By Lieb's inequality, **[Lieb, 1976], [Rozenblum-Solomyak, 1997]**

$$N(-\Delta - V, 0) \leq C \int_\Gamma \int_0^\infty k(t,x,x)\, t^{-1} \, (t\, V(x) - 1)_+ \, dt \, dx$$

Inserting the upper bound

$$k(t,x,x) \leq C \, g_0(|x|) \, t^{-\frac{d}{2}} \qquad \forall \, x \in \Gamma$$

we get

Schrödinger operators

Theorem 3 Let Γ be a symmetric tree with global dimension $d > 2$. Then there exists L_d such that

$$N(-\Delta - V, 0) \, \leq \, L_d \int_\Gamma V(x)^{\frac{d}{2}} \, g_0(|x|) \, dx$$

- For symmetric potentials this result was proved by **[Ekholm-Frank-K., 2007]**. Extensions for general potentials on trees are due to **[Solomyak, 2008]**.

- Related results for Γ^d graphs, i.e. $\mathcal{V} = \mathbb{Z}^d$, were obtained in **[Molchanov-Vainberg, 2008]**.

- More general graphs are treated in **[Rosenblum-Solomyak, 2010]**.

Weighted Sobolev inequalities

Theorem 4 (Sobolev inequality) Let Γ be a symmetric tree with global dimension $d > 2$. Then the inequality

$$\int_\Gamma |u'|^2 \, dx \, \geq \, C_s \left(\int_\Gamma |u|^{\frac{2d}{d-2}} \, g_0(|x|)^{-\frac{\alpha}{d-2}} \, dx \right)^{\frac{d-2}{d}}$$

holds for some C_s and all $u \in H^1(\Gamma)$ if and only if $\alpha \geq 2$.

Since $g_0 \geq 1$, it suffices to prove the inequality for $\alpha = 2$. This is a direct a consequence of Theorem 3: put

$$V(x) = \frac{\beta \, |u(x)|^{\frac{4}{d-2}} \, g_0(|x|)^{-\frac{2}{d-2}}}{\left(L_d \int_\Gamma |u|^{\frac{2d}{d-2}} \, g_0(|x|)^{-\frac{2}{d-2}} \, dx \right)^{2/d}}, \qquad \beta < 1,$$

Weighted Sobolev inequalities

Then

$$N(-\Delta - V, 0) \leq L_d \int_\Gamma V^{d/2} g_0 \, dx = \beta^{d/2} < 1,$$

which implies

$$\int_\Gamma |u'|^2 \, dx \geq \int_\Gamma V \, |u|^2 \, dx = C \left(\int_\Gamma |u|^{\frac{2d}{d-2}} \, g_0(|x|)^{-\frac{2}{d-2}} \, dx \right)^{\frac{d-2}{d}}$$

Suppose now that $\alpha < 2$. Since $g_0(r) \simeq r^{d-1}$, necessarily we must have

$$\sup_{e \in \mathcal{E}} |e| = \infty.$$

Let e_n be an edge of the n^{th} generation of length $|e_n| = a_n$ connecting the vertices r_{n-1} and r_n. Consider the sequence of functions $u_n(x)$ with the support on e_n and defined by

$$\begin{aligned} u_n(x) &= \frac{|x| - |r_{n-1}|}{2a_n} \quad x \in e_n, \quad |x| \leq |r_{n-1}| + a_n/2 \\ u_n(x) &= \frac{|r_n| - |x|}{2a_n} \quad x \in e_n, \quad |x| > |r_{n-1}| + a_n/2 \end{aligned}$$

Since $a_n \to \infty$ as $n \to \infty$, we get

$$\int_\Gamma |u_n'|^2 \, dx \simeq a_n^{-1}$$

$$\left(\int_\Gamma |u_n|^{\frac{2d}{d-2}} g_0(|x|)^{-\frac{\alpha}{d-2}} \, dx \right)^{\frac{d-2}{d}} \simeq a_n^{-1+\frac{(d-1)(2-\alpha)}{d}}$$

and the Sobolev inequality leads to a contradiction if $\alpha < 2$.

Weighted Sobolev inequalities

In particular, this shows that the estimate

$$k(t,x,x) \,\leq\, C\, g_0(|x|)^\gamma\; t^{-d/2} \qquad\qquad \forall\, x\in\Gamma,\;\; t>0.$$

fails if $\gamma < 1$.

Homogeneous trees

Following **[Sobolev-Solomyak, 2002]** we call a rooted tree Γ_b **homogeneous** if all the edges have the same length τ and if the branching number $b_k = b > 1$ is independent of k. By scaling we may assume that $\tau = 1$. The branching function then satisfies

$$g_0(r) \simeq b^r,$$

and by **[Sobolev-Solomyak, 2002]** we have

$$\inf \sigma(-\Delta_b) = \inf \sigma_{es}(-\Delta_b) = \lambda_b = \left(\arccos \frac{1}{R_b}\right)^2, \quad R_b = \frac{b^{\frac{1}{2}} + b^{-\frac{1}{2}}}{2}$$

Theorem 5 Let Γ_b be a homogeneous tree with branching number $b > 1$. Then

$$e^{t\Delta_b}(x,x) \le C_b\,(1+|x|)^2\, t^{-3/2}\, e^{-t\lambda_b} \qquad \forall\, x \in \Gamma, \quad \forall\, t > 0$$

Open questions

- Symmetric trees without global dimension.
- Non symmetric trees.
- More general b.c. at the vertices.