A Solution to an Ambarzumyan Problem on Trees and Related Topics

Chun-Kong Law (National Sun Yat-sen Univ., Kaohsiung, Taiwan, R.O.C.)
&
Eiji Yanagida (Tokyo Institute of Technology, Tokyo, Japan)
with
Vyacheslav Pivovarchik (South-Ukrainian State Pedagogical University

AGAW06, Isaac Newton Institute, Cambridge

Outline

Introduction

Ambarzumyan Theorem (1929)

For the following Neumann Sturm-Liouville problem on $(0,1)$

$$\begin{aligned} -y'' + qy &= \lambda y \\ y'(0) = y'(1) &= 0 \end{aligned}$$

If the spectrum $\sigma = \{(n\pi)^2 : n \geq 0\}$, then $q = 0$.

Literature

- 1946 Borg

- 1987 Poschel and Trubowitz showed that for Dirichlet boundary conditions, there exists infinitely many L^2 potentials q such that $\sigma(q) = \{(n\pi)^2\}$

- 1995 Chern and Shen proved the Ambarzumyan Theorem for vectorial Sturm-Liouville system

$$\begin{aligned} -Y'' + P(x)Y &= \lambda Y \\ Y'(0) = Y'(1) &= \mathbf{0} \end{aligned}$$

where P is a real symmetric continuous $I \times I$ matrix function of x. If $(n\pi)^2 \in \sigma(P)$ with multiplicity I, then $P = \mathbf{0}$.

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- 2001 Chern-L-Wang showed that :
  For Dirichlet b.c., if $\rho_n = n\pi$ for all n and $\int_0^1 q(x)\cos(2x)\,dx = 0$, then $q = 0$.

- 2009 C.F. Yang, Huang and Yang showed that :
  For periodic b.c., if $\rho_n = 2n\pi$ with multiplicity 2 for all n, then $q = 0$.

- So we start to call **Ambarzumyan problems**, for different b.c.

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Spectral problems on trees

Let Γ be a graph. Then $\Gamma = (V, E)$, set of vertices and edges. Also let

$$I = |E|, a_i = |\gamma_i| \text{ for } \gamma_i \in E$$

Sturm-Liouville problem on graphs :

$$\begin{aligned} -y_i'' + q_i y_i &= \lambda y_i \quad (0 < x < a_i) \\ y_i'(0) &= 0 \quad \text{for all } \gamma_i \in \partial\Gamma. \end{aligned}$$

At each $\mathbf{v}$ in interior of Γ, we require

1. continuity conditions : $y_i(\mathbf{v}) = y_j(\mathbf{v})$ when $\gamma_i, \gamma_j \in E(\mathbf{v})$.
2. Kirchhoff conditions : $\sum_{\gamma_i \in E(\mathbf{v})} y_i'(\mathbf{v}) = 0$.

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Ambarzumyan problems on trees

- 2004 Pivovarchik solved the Neumann Amb. P. for star graphs with edges of constant lengths.
- 2007 Carlson-Pivovarchik solved Neumann Amb.P. for trees with edges of constant lengths.
  $\Rightarrow$ solved Neumann Amb.P. for trees with edges of rational lengths, i.e. lengths of rational ratios.
- **Question** : How about those edges with irrational ratios ?

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Main Theorem

Theorem (L,Yanagida)

For a Neumann Ambarzumyan problem on trees with arbitrary lengths, if $\sigma(Q) = \sigma(0)$, *then* $Q = 0$.

Three Key Lemmas

1. Characteristics functions. For a 3-star graph, with $q = 0$, the system of equations

$$\begin{cases} y_i = A_i \cos(\rho x) \\ A_1 \cos(\rho a_1) = A_2 \cos(\rho a_2) = A_3 \cos(\rho a_3) \\ \sum \rho A_i \sin(\rho a_i) = 0 \end{cases}$$

has a solution in A_1, A_2, A_3

$$\Leftrightarrow \Phi(\rho) = \begin{bmatrix} \cos(\rho a_1) & -\cos(\rho a_2) & 0 \\ \cos(\rho a_1) & 0 & -\cos(\rho a_3) \\ \rho\sin(\rho a_1) & \rho\sin(\rho a_2) & \rho\sin(\rho a_3) \end{bmatrix} \quad \text{is invertible}$$

$$\begin{aligned} \Leftrightarrow \varphi(\rho) &= \det \Phi(\rho) \\ &= \rho(\sin(\rho a_1)\cos(\rho a_2)\cos(\rho a_3) + \sin(\rho a_2)\cos(\rho a_1)\cos(\rho a_3) \\ &\quad + \sin(\rho a_3)\cos(\rho a_1)\cos(\rho a_2)) \\ &= 0 \end{aligned}$$

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For arbitrary q, the characteristic function

$$\varphi(\rho) = \rho(C_1' C_2 C_3 + C_1 C_2' C_3 + C_1 C_2 C_3')$$

where by the theory of transformation operators,

$$\begin{aligned} C_i &= \cos(\rho a_i) + \int_0^{a_i} K(a,t)\cos(\rho t)\,dt \\ S_i &= \frac{\sin(\rho a_i)}{\rho} + \int_0^{a_i} K(a,t)\sin(\rho t)\,dt \end{aligned}$$

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Let φ_{NK} denote the characteristic fcn w.r.t. q with Neumann boundary conditions and Kirchhoff conditions. Let φ_D be the characteristic fcn w.r.t. q with Dirichlet b.c. at the first vertex but Neumann b.c. at all other boundary vertices. When there is only one edge, we let $\varphi_{NK}(\rho) = C_1'$ and $\varphi_D(\rho) = S_1'$.

Lemma

(a) $\varphi_{NK}(\rho) = C_1 \tilde{\varphi}_{NK}(\rho) + C_1' \tilde{\varphi}_D(\rho)$.

(b) $\varphi_D(\rho) = S_1 \tilde{\varphi}_{NK}(\rho) + S_1' \tilde{\varphi}_D(\rho)$.

Hence for a caterpillar graph, the characteristic fcn

$$\begin{aligned} \varphi &= C_4'C_5\tilde{\varphi}_D + C_4(C_5'\tilde{\varphi}_D + C_5\tilde{\varphi}_{NK}) \\ &= (C_4'C_5 + C_4C_5')(S_3'C_1C_2 + C_1'C_2S_3 + C_1C_2'S_3) \\ &\quad + C_4C_5(C_3'C_1C_2 + C_1'C_2C_3 + C_1C_2'C_3)\} \end{aligned}$$

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Hence for a caterpillar graph, the characteristic fcn

$$\begin{aligned}\varphi &= C_4'C_5\tilde{\varphi}_D + C_4(C_5'\tilde{\varphi}_D + C_5\tilde{\varphi}_{NK}) \\ &= (C_4'C_5 + C_4C_5')(S_3'C_1C_2 + C_1'C_2S_3 + C_1C_2'S_3) \\ &\quad +C_4C_5(C_3'C_1C_2 + C_1'C_2C_3 + C_1C_2'C_3)\}\end{aligned}$$

Three Key Lemmas

2. Let φ_{NK} and φ_D denote the characteristic fcns w.r.t. q ;
ψ_{NK} and ψ_D denote the characteristic fcns w.r.t. $q_0 = 0$.

For a tree with only one edge with length a, let

$$\varphi_D(\rho) = C_i(a), \qquad \varphi_{NK}(\rho) = C_i'(a)$$
$$\psi_D(\rho) = \cos(\rho a), \qquad \psi_{NK}(\rho) = -\sin(\rho a)$$

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For a tree with only one edge with length a, let

$$\begin{aligned} \varphi_D(\rho) &= C_i(a), & \varphi_{NK}(\rho) &= C_i'(a) \\ \psi_D(\rho) &= \cos(\rho a), & \psi_{NK}(\rho) &= -\sin(\rho a) \end{aligned}$$

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Hence

$$\begin{aligned}\varphi_D(\rho) &= \cos(\rho a) + \int_0^a K(a,t)\cos(\rho t)\,dt \\ &= \cos(\rho a) + K(a,a)\frac{\sin(\rho a)}{\rho} \\ &\quad + \int_0^a K_t(a,t)\frac{\sin(\rho t)}{\rho}\,dt \\ &= \cos(\rho a) K\frac{\sin(\rho a)}{\rho} + o(\frac{1}{\rho}) \\ &= \psi_D(\rho) - \frac{1}{\rho}K\psi_{NK}(\rho) + o(\frac{1}{\rho}),\end{aligned}$$

where $K := K(a,a) = \frac{1}{2}\int_0^a q$.

Three Key Lemmas

$$\begin{aligned}\varphi_{NK}(\rho) &= -\rho\sin(\rho a) + K\cos(\rho a) \\ &\quad + \int_0^a K_x(a,t)\cos(\rho t)\,dt \\ &= \rho\psi_{NK}(\rho) + \frac{1}{\rho}K\psi_D(\rho) + o(1)\end{aligned}$$

By induction, we have

Lemma

As $\rho \to \infty$,

$$\begin{aligned}\varphi_{NK}(\rho) &= \rho\psi_{NK}(\rho) + (\sum K_i)\psi_D(\rho) + o(1) \\ \varphi_D(\rho) &= \psi_D(\rho) - \frac{1}{\rho}(\sum K_i)\psi_{NK}(\rho) + o(\frac{1}{\rho})\end{aligned}$$

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3.

Lemma

(a) *There exist infinite sequences of natural numbers* $\{m_n\}$ *and* $\{k_{i,n}\}$ $(i = 1, \ldots, I)$ *such that* $m_n \to \infty$ *as* $n \to \infty$, *and for each* n,

$$\sum_{i=1}^{I} k_{i,n} = m_n \quad \text{and} \quad \left|\frac{a_i}{L} - \frac{k_{i,n}}{m_n}\right| < \frac{1}{m_n^{1+1/I}}$$

(b) *let* $\mu_n := 2m_n\pi/L$. *Then for any* $\rho = \mu_n + O(m_n^{-(1+1/I)})$,

(i) $\psi_{NK}(\rho) = O(m_n^{-1/I})$ *and* $\frac{d\psi_{NK}}{d\rho}(\rho) = -L + O(m_n^{-1/I})$.

(ii) $\psi_D(\mu_n) = 1 + O(m_n^{-1/I})$ *and* $\frac{d\psi_D}{d\rho}(\mu_n) = O(m_n^{-1/I})$.

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Hence for sufficiently large n's, there exist $\rho_n = \mu_n + O(\frac{1}{m_n^{1+1/I}})$ such that $\varphi_{NK}(\rho) = 0$.

Proof of Main Theorem

So by Lemma

$$\varphi_{NK}(\rho) = \rho\psi_{NK}(\rho) + (\sum K_i)\psi_D(\rho) + o(1)$$

Take $\rho = \rho_n$ such that $\varphi_{NK}(\rho_n) = 0 = \psi_{NK}(\rho)$, since $\sigma(Q) = \sigma(\mathbf{0})$, implying

$$\sum K_i = \frac{1}{2}\sum\int_0^{a_i} q_i = 0$$

Also, 0 is the first eigenvalue. So

$$0 = \min \frac{\sum_{i=1}^{I}\int_0^{a_i}(y_i'^2 + q_i y_i^2)\,dx}{\sum_{i=1}^{I} y_i^2\,dx}.$$

Let $y_i = 1$ for all $i = 1, \cdots, I$. The minimum value is achieved. So $(1, \cdots, 1)$ is the first eigenfunction. Therefore $q_i = 0$ for all i.

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Second theorem

Let $\mathbf{v}$ be a root of a metric tree Γ with complementary subtrees $\Gamma^{(1)}$ and $\Gamma^{(2)}$. For the Schrodinger operator defined on Γ, define ϕ_{NK} and and ϕ_D to be the characteristic functions with Neumann-Kirchhoff condition and Dirichlet condition at $\mathbf{v}$ resp. Then

Theorem (L,Pivovarchik)

(a) $\phi_{NK} = \phi_{NK}^{(1)}\phi_D^{(2)} + \phi_D^{(1)}\phi_{NK}^{(2)}$.

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Corollary

(a) $\varphi_{NK}(\rho) = C_1\tilde{\varphi}_{NK}(\rho) + C_1'\tilde{\varphi}_D(\rho)$.

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Illustrative proof

For Neumann Schrodinger operator defined on a caterpillar metric graph, let the solution at each edge γ_i be $A_iC_i(x,\rho)+B_iS_i(x,\rho)$, where $C_i(0)=1=S_i'(0)$, $C_i'(0)=0=S_i(0)$, $A_i, B_i \in \mathbf{R}$.

Then $B_1=B_2=B_4=B_5=0$, and

$$\begin{aligned}
A_1C_1 &= A_2C_2\\
A_2C_2 &= A_3C_3+B_3S_3\\
A_3 &= A_4C_4\\
A_4C_4 &= A_5C_5\\
A_1C_1'+A_2C_2'+A_3C_3'+B_3S_3' &= 0\\
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$$\phi(\rho) = \det \begin{bmatrix} C_1 & -C_2 & 0 & 0 & | & 0 & 0 \\ 0 & C_2 & -C_3 & -S_3 & | & 0 & 0 \\ C_1' & C_2' & C_3' & S_3' & | & 0 & 0 \\ 0 & 0 & 0 & 1 & | & C_4' & C_5' \\ -- & -- & -- & -- & -- & -- & -- \\ 0 & 0 & 1 & 0 & | & -C_4 & 0 \\ 0 & 0 & 0 & 0 & | & C_4 & -C_5 \end{bmatrix}$$

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Illustrative proof

Hence

$$\begin{aligned}
\phi(\rho) &= \phi_{NK}^{(1)}\phi_{D}^{(2)} + \phi_{D}^{(1)}\phi_{NK}^{(2)} \\
&= C_4C_5(C_1C_2C_3' + C_1C_2'C_3 + C_1'C_2C_3) + \\
&\quad (C_1C_2S_3' + C_1'C_2S_3 + C_1C_2'S_3)(C_4C_5' + C_5C_4')
\end{aligned}$$

Remarks

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- Formula works for arbitrary boundary conditions.
- Complementary subgraphs may have loops.
- Formula also works for any δ-type vertex conditions.
- Similar result appeared in Texier (2008) - spectral determinant.

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Conclusions

(1) So we 'completely' solve the Neumann Ambarzumyan Problem.

(2) The proof does not seem to work for eigenvalues with multiple multiplicity. Hence it does not work for other b.p.

(3) We develop a reduction formula for the characteristic function of a graph :

$$\phi_{NK} = \phi_{NK}^{(1)} \phi_D^{(2)} + \phi_D^{(1)} \phi_{NK}^{(2)}.$$

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Thank you for your attention

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