Limits of self-similar graphs and criticality of the Abelian Sandpile Model

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The space of rooted graphs

$\mathcal{X}_*$ denotes the compact space of all rooted graphs (up to isomorphism) with uniformly bounded degrees endowed with pointed Hausdorff-Gromov topology:

$(G_n, v_n) \stackrel{n\to\infty}{\longrightarrow} (G, v)$ iff for any $r > 0$ there exists N such that for all $n \geq N$ the ball $B_{G_n}(v_n, r)$ is isomorphic to the ball $B_G(v, r)$.

example: sequence of graph coverings rooted at vertices belonging to the same fibre.

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Measures on rooted graphs

If we are given a sequence $\{G_n\}_{n\geq 1}$ of graphs (with bounded degrees) without any rooting, we can choose a root in each G_n uniformly at random. This defines a sequence of probability measures on $\mathcal{X}_*$; its weak limit ρ is the random weak limit of $\{G_n\}_{n\geq 1}$ (Benjamini, Schramm).

Thus, ρ is a probability distribution on the limits of $\{(G_n, v_n)\}_{n\geq 1}$ in $\mathcal{X}_*$ for all possible choices of roots v_n in G_n.

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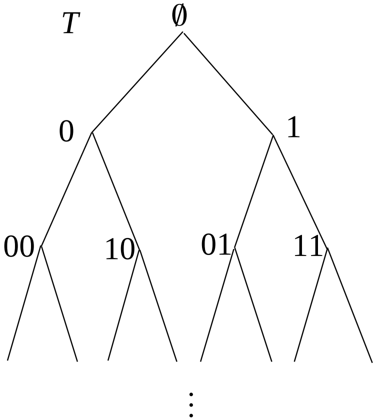
Actions by automorphisms of rooted trees provide many sequences of graphs converging to interesting limits.



Let $\Gamma \leq Aut(T)$ be finitely generated by $S \cup S^{-1}$. For each $n \geq 1$, the Schreier graph $G_n \equiv G(\Gamma, S, \{0,1\}^n)$ is given by

- $V(G_n) = \{0,1\}^n$;
- $v, w \in V(G_n)$, $v \sim w$ iff $\exists s \in S \cup S^{-1} | s(v) = w$.

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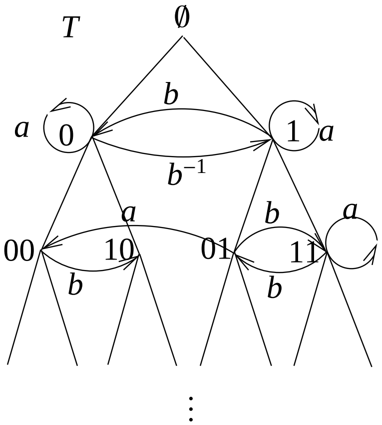

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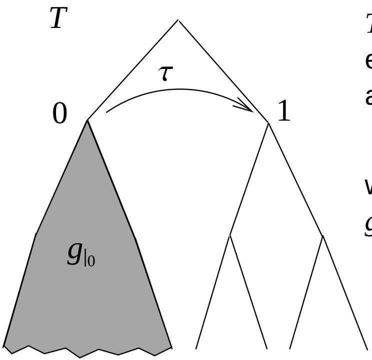


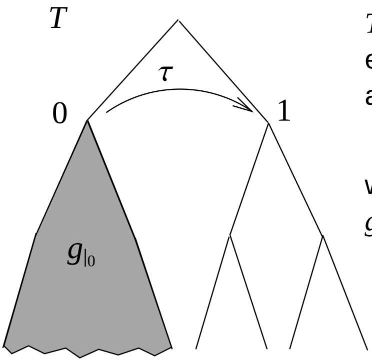
T has a self-similar structure: each $g \in Aut(T)$ can be written as

$$g = \tau(g|_0, g|_1)$$

where $\tau \in Sym(2)$ and $g|_0, g|_1 \in Aut(T)$.

Definition (Grigorchuk, Nekrashevych)

A group Γ acting by automorphisms on T is self-similar if, for all $g \in \Gamma$ and $v \in V(T)$, $g|_v \in \Gamma$.



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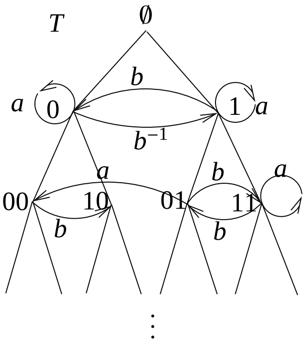
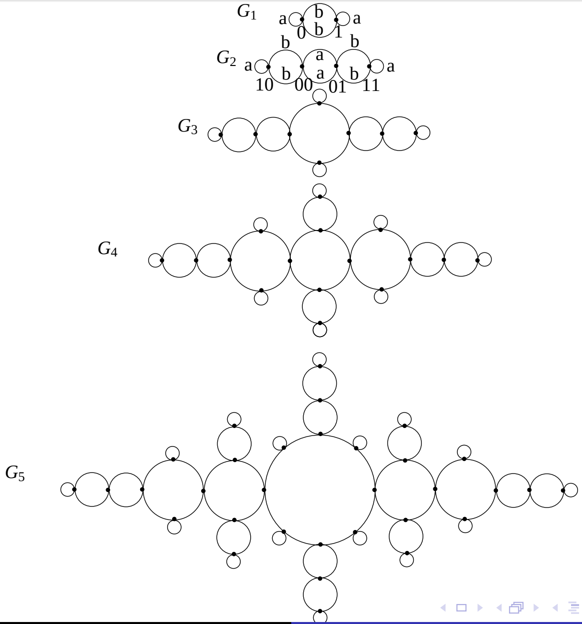
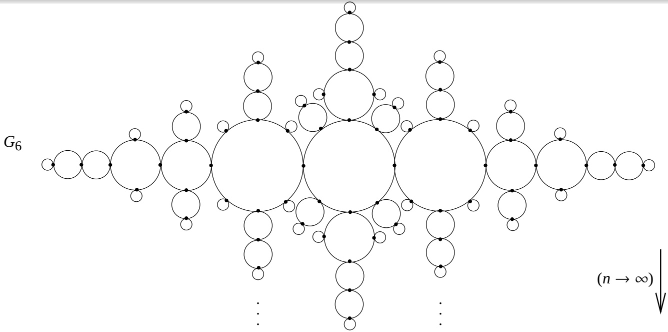
An interesting example of a self-similar group is the Basilica group $\mathcal{B}$ introduced by Grigorchuk and Żuk '02. $\mathcal{B}$ is generated by two elements having the following self-similar structure:

$$a = e(b, id) \qquad \text{and} \qquad b = \tau(a, id).$$




G1
a
b
b
a
0
1
b
b
G2
a
a
a
a
b
b
a
10
00
01
11
G3
G4
G5



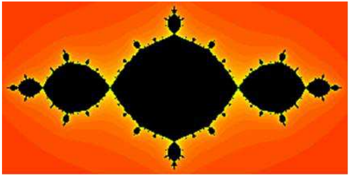
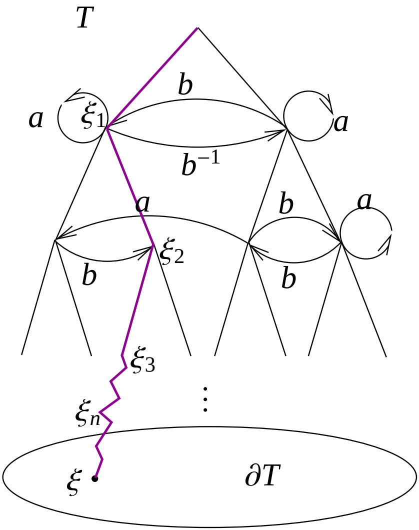


Figure: The Julia set $J(z^2 - 1)$



If $\Gamma \leq Aut(T)$ acts spherically transitively, then

$$(G_n, \xi_n) \longrightarrow (G_\xi, \xi)$$

for pointed Hausdorff-Gromov convergence, where G_ξ is the infinite orbital Schreier graph of the action of Γ on ∂T.

Applications

- Schreier graphs approximate Julia sets: spectra of Laplacians on Julia sets [Rogers, Teplyaev for Basilica];
- Spectra on infinite graphs [Grigorchuk, Šunić];
- New examples of random weak limit;
- Statistical physics models (Ising model, Potts models, dimer model, Abelian sandpile model,...):
  - Usually defined on sequences of finite graphs exhausting an infinite lattice (direct limit).
  - What about considering them on covering sequences of Schreier graphs of self-similar groups (projective limit)?

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The Basilica group

Depending of $\xi \in \partial T$, $\{(G_n, \xi_n)\}_{n \geq 1}$ converges to various interesting infinite limits.

Theorem (D'Angeli, Donno, Matter, Nagnibeda, '10)

For $i = 1, 2, 4$, let $E_i = \{\xi \in \partial T | (G_\xi, \xi) \text{ has } i \text{ ends}\}$. Then,

- *$\partial T = E_1 \sqcup E_2 \sqcup E_4$;*
- *let λ be the uniform probability distribution on ∂T; then $\lambda(E_1) = 1$;*
- *there are uncountably many isomorphism classes of one-ended graphs, each of them but one being uncountable;*
- *there is one isomorphism class of 4-ended graphs (containing one orbit).*

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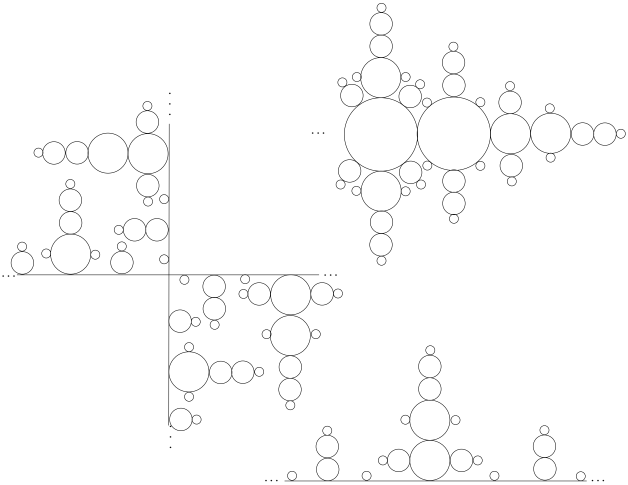

A statistical-physics model

The Abelian Sandpile Model (ASM):

- Self-organized criticality (earthquakes, forest fires) [Bak, Tang, Wiesenfeld, '88];
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Local rules

Let $G = (V, E)$ be a finite, connected (multi)graph. The model consists in

- configurations $\eta : V \longrightarrow \mathbb{N}$, encoding the repartition of an amount of grains of sand (or chips) on the vertices of G;
- local transformation rules between successive configurations: if $\eta(v) \geq \deg(v)$ for some $v \in V$, then v is fired:

$$T_v(\eta)(w) = \begin{cases} \eta(w) - \deg(w) & \text{if } v = w, \\ \eta(w) + m(v, w) & \text{if } v \neq w. \end{cases}$$

  where $m(v, w)$ denotes the number of edges between v and w.

Introduce one (or more) dissipative vertex $p \in V$; chips reaching p leave the game, ensuring that the firing process eventually stops.
Abelian Property: the stable configuration reached through a sequence of firings is independent of the order in which unstable vertices are fired.

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Global dynamics

Look at

{stable configurations} $\supset \mathcal{R} :=$ {recurrent configurations}

Recurrent configurations are characterized by a Markov chain: given a stable configuration η,

- drop an extra-chip on a randomly chosen vertex v;
- if the configuration $\eta + \delta_v$ is unstable, let it stabilize into a new stable configuration η'.

The sequence of firings transforming $\eta + \delta_v$ into η' is an avalanche. By irreducibility, there is a unique stationary distribution μ supported by $\mathcal{R}$; μ is the uniform distribution.

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Avalanches and criticality

Let $G_1, G_2 \ldots G_n \ldots$ be a sequence of finite graphs converging (in some sense) to an infinite graph G.

Consider the random variable $Mav_{G_n} : (\mathcal{R}_n, \mu_n) \longrightarrow \mathbb{N}$ encoding the mass of the avalanche triggered by adding an extra chip on a fixed vertex v_n.

Definition

The ASM on $\{G_n\}_{n\geq 1}$ is critical if

$$\lim_{n\to\infty} \mathbb{P}_{\mu_n}(Mav_{G_n}(\cdot, v_n) = M) \sim M^{-\delta}$$

for some exponent $\delta > 0$ (called the critical exponent).

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Examples

Many numerical simulations for exhibiting criticality and determining the exponent δ, but few rigorous results.

- regular infinite tree: $\delta = 3/2$ [Dhar, Majumdar, '90];
- $\mathbb{Z}^d$:
  - $d = 1$: not critical;
  - $d = 2$: conjectured that $\delta = 5/4$ [Priezzhev et al., '96];
  - $d > 4$: conjectured that $\delta = 3/2$ [Priezzhev, '00];
- Sierpiński gasket: experiments yield $\delta \simeq 1.46$ [Kutnjak-Urbanc et al. '96]

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ASM on Schreier graphs of the Basilica group

Theorem (Matter, Nagnibeda, '10)

Given $\xi \in \partial T$, let $\{(G_n, \xi_n)\}_{n\geq 1}$ be the sequence of Schreier graphs of the action of the Basilica group $\mathcal{B}$ on T. Then, for almost every ξ, the ASM on $\{(G_n, \xi_n)\}_{n\geq 1}$ is critical with critical exponent $\delta = 1$.

Thus, we have exhibited an uncountable family of graphs on which the ASM is critical. Moreover, these graphs are 1-ended, 4-regular and of quadratic growth (properties that they are sharing with $\mathbb{Z}^2$).

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There are uncountably many isomorphism classes of two-ended graphs (each of them containing two orbits).

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