Necklace Graphs and Slowing down of Light

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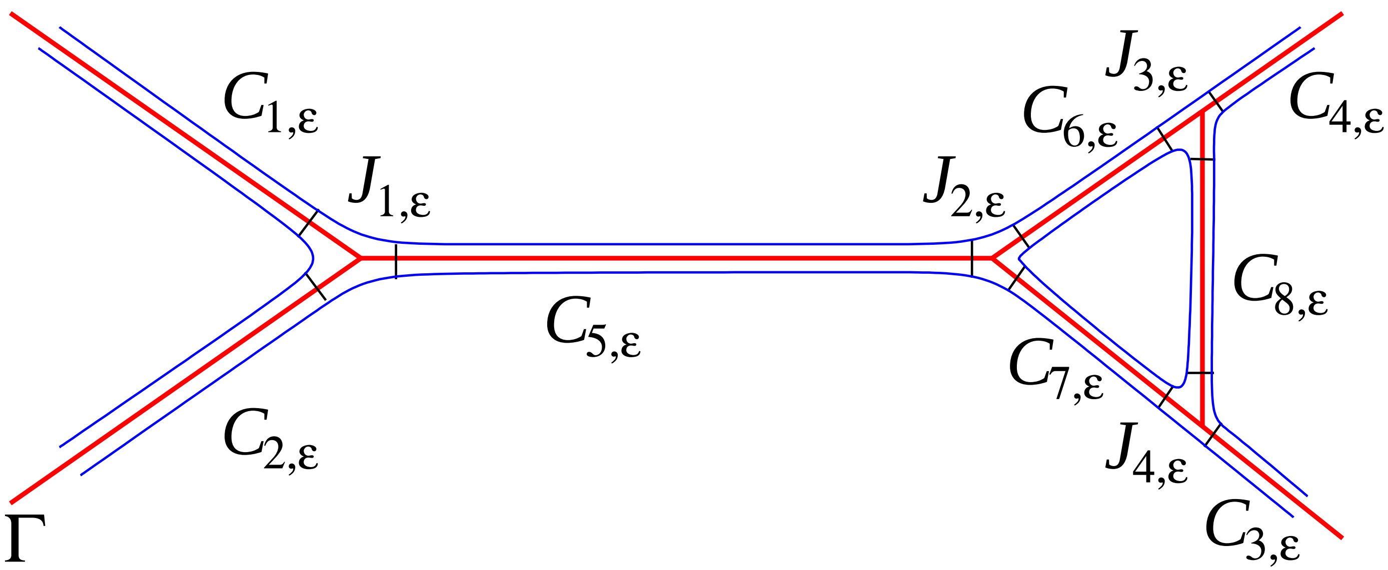
PART I.Transition from networks to graphs

Consider the stationary wave (Helmholtz) equation

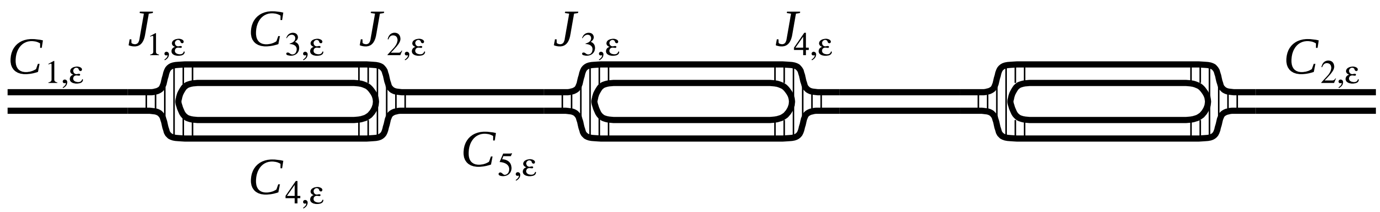
$$H_\varepsilon u = -\Delta u = \omega^2 u + f, \quad x \in \Omega_\varepsilon, \quad Bu = 0 \quad \text{on } \partial\Omega_\varepsilon,$$

in a domain $\Omega_\varepsilon \subset R^d$, $d \geq 2$, with infinitely smooth boundary (for simplicity), which has the following structure:

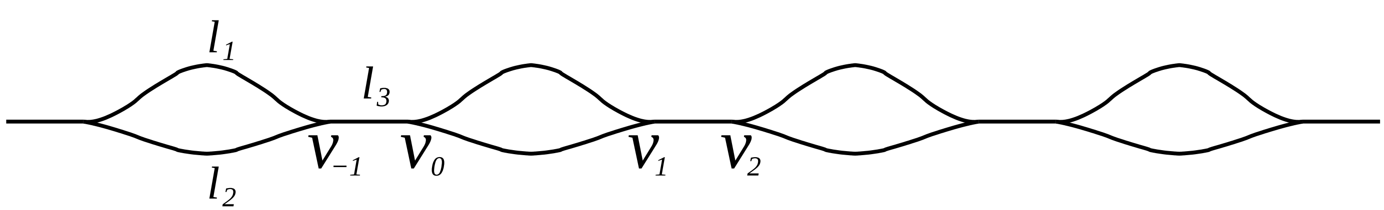


$B = 1$ (the Dirichlet BC) or $B = \frac{\partial}{\partial n}$ (the Neumann BC) or $B = \varepsilon\frac{\partial}{\partial n} + \alpha(x)$.

Network of the necklace type:



The domain Ω_ε shrinks to a one-dimensional metric graph Γ as $\varepsilon \to 0$. The axes of the channels form edges Γ_j, $1 \leq j \leq N$, of Γ, and the distances between points of Γ_j are defined by the distances between the corresponding points of the channels. The junctions shrink to vertices of the graph Γ. We denote the set of vertices v_j by V.



For the sake of simplicity of this talk, we assume that all the channels $C_{j,\varepsilon}$ have the same cross-section U_ε. We also assume self-similarity of junctions:

$$J_{v,\varepsilon} = \{(\widehat{x} + \varepsilon x) : x \in J_v\}.$$

From the latter it follows that $U_\varepsilon = \varepsilon U$, $U \subset R_y^{d-1}$.

Let

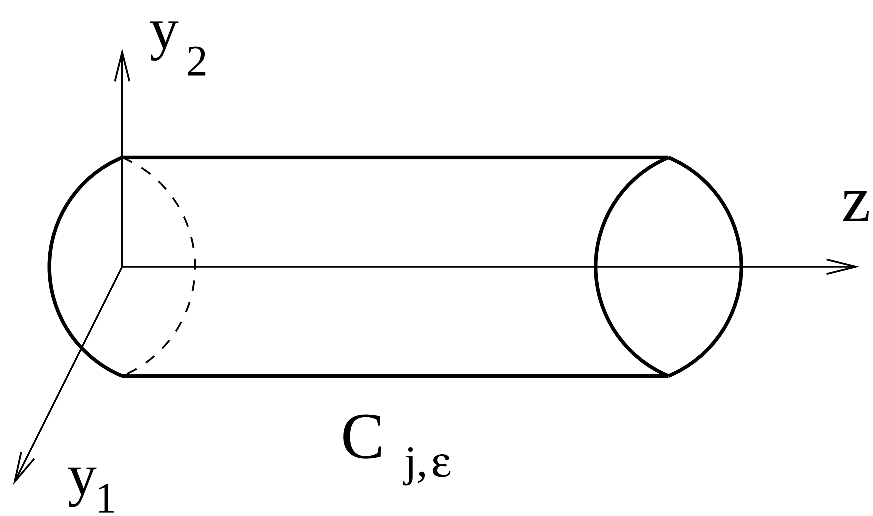
$$-\Delta_{d-1}\varphi_n(y) = \lambda_n \varphi_n(y), \quad y \in U \in R^{d-1}, \quad B\varphi_n(y) = 0 \text{ on } \partial U.$$

Then

$$-\Delta_{d-1}\varphi_n(y/\varepsilon) = \varepsilon^{-2}\lambda_n \varphi_n(y/\varepsilon) \text{ in } U_\varepsilon, \quad B\varphi_n(y/\varepsilon) = 0 \text{ on } \partial U_\varepsilon.$$

The absolutely continuous spectrum of the problem in Ω_ε is $[\varepsilon^{-2}\lambda_0, \infty)$, and propagation of waves is possible when $(\varepsilon\omega)^2 > \lambda_0$.

We introduce Euclidean coordinates (z, y) in channels $C_{j,\varepsilon}$ chosen in such a way that the z-axis is parallel to the axis of the channel.



Important: Waves governed by the operator $-\Delta-\omega^2$ do not propagate through the channels if $(\varepsilon\omega)^2<\lambda_0$, there exists only one propagating mode if $(\varepsilon\omega)^2\in(\lambda_0,\lambda_1)$:

$$e^{\pm i\sigma z}\varphi_0(y/\varepsilon), \quad \sigma=\sqrt{\omega^2-\varepsilon^{-2}\lambda_0},$$

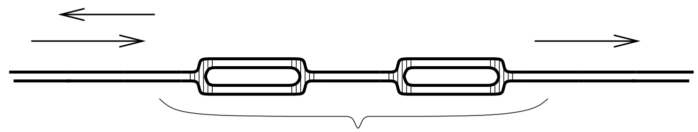
and there are many similar modes

$$e^{\pm i\sigma_j z}\varphi_j(y/\varepsilon), \quad \sigma_j=\sqrt{\omega^2-\varepsilon^{-2}\lambda_j}, \quad \text{if } (\varepsilon\omega)^2>>\lambda_1.$$

Many particular cases of the problem with $(\varepsilon\omega)^2=\lambda_0+O(\varepsilon^2)$ or $(\varepsilon\omega)^2<\lambda_0$ (waves do not propagate) were considered:
Fredlin, Wentzel, Exner, Kuchment, Rubinstein, Post, Grieser, Dell'Antonio, Tenuta, Kostrykin.....

Molchanov-Vainberg (2006-2010): $(\varepsilon\omega)^2\geq\lambda_0$.
The main result: the scattering solutions and the resolvent of the operator H_ε can be approximated by the corresponding solutions of the one-dimensional problem on the limiting graph Γ with the GC expressed in terms of the scattering matrices of the individual extended junctions.

Scattering solutions in Ω_ε



Let $\lambda_0 < (\varepsilon\omega)^2 < \lambda_1$. A function $\Psi = \Psi_{p,\varepsilon}$, $1 \le p \le m$, (m is the number of infinite channels) is called a solution of the scattering problem in Ω_ε if

$$(-\Delta - \omega^2)\Psi = 0, \quad x \in \Omega_\varepsilon; \qquad B\Psi = 0 \text{ on } \partial\Omega_\varepsilon, \quad \text{and}$$

$$\Psi_{p,\varepsilon} = [\delta_{p,j} e^{-i\sigma z} + t_{p,j} e^{i\sigma z}]\varphi_0(y/\varepsilon) + O(e^{-\frac{\alpha z}{\varepsilon}}), \quad z \to \infty, \quad \alpha > 0.$$

Here $\sigma = \sqrt{\omega^2 - \varepsilon^{-2}\lambda_0}$, $\delta_{p,j} = 1$ if $p = j$, $\delta_{p,j} = 0$ if $p \ne j$. (The incident wave comes through the channel $C_{p,\varepsilon}$ and exits through channels $C_{j,\varepsilon}$, $j \le m$.)

The matrix

$$T = [t_{p,j}], \quad p, j \le m,$$

is called *the scattering matrix*. It is unitary and symmetric.

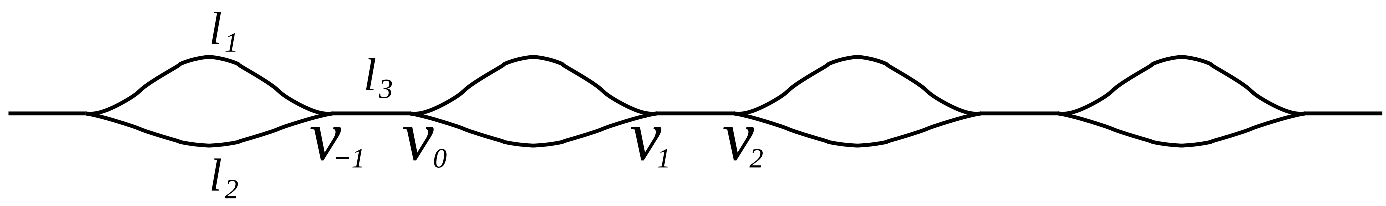
Scattering solutions on the graph.

It happens that the scattering solutions $\Psi_{p,\varepsilon}$ can be approximated with an exponential in ε accuracy using the scattering solutions $\psi = \psi_{p,\varepsilon}$ of a one-dimensional problem on the limiting graph Γ which are solutions of the equation

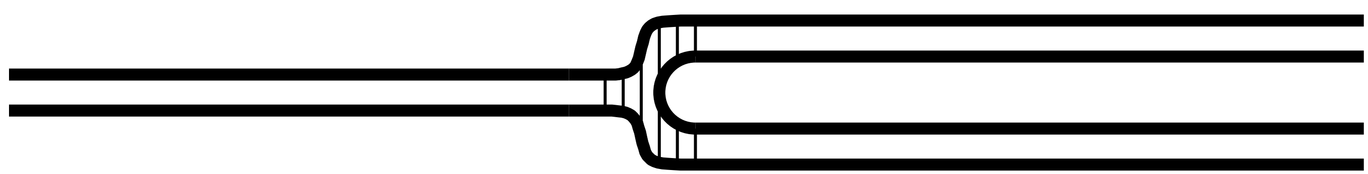
$$(\frac{d^2}{dz^2} + \sigma^2)\psi = 0 \text{ on } \Gamma, \quad \sigma^2 = \omega^2 - \varepsilon^{-2}\lambda_0, \tag{1}$$

with appropriate conditions at infinity and satisfy the following GC (gluing conditions) at vertices v of Γ:

$$i[I_v + T_v(\varepsilon\omega)]\frac{d}{dz}\psi^{(v)}(z) - \sigma[I_v - T_v(\varepsilon\omega)]\psi^{(v)}(z) = 0, \quad z = 0, \quad v \in V.$$



Here $\psi^{(v)} = \begin{pmatrix} \psi_1^{(v)}(z) \\ \psi_2^{(v)}(z) \\ \psi_3^{(v)}(z) \end{pmatrix}$, where $\psi_j^{(v)}(z)$ are restrictions of ψ on edges Γ_j of the graph adjacent to the vertex v,



$T_v(\varepsilon\omega)$ are the scattering matrices for the extended junctions of the network:

If det$[I_v + T_v(\varepsilon\omega)] \neq 0$, the GC (gluing conditions)

$$i[I_v + T_v(\varepsilon\omega)]\frac{d}{dz}\psi^{(v)}(z) - \sigma[I_v - T_v(\varepsilon\omega)]\psi^{(v)}(z) = 0, \qquad z = 0, \qquad v \in V.$$

can be written in the form

$$\frac{d}{dz}\psi^{(v)}(z) - \sigma A_v \psi^{(v)}(z) = 0, \qquad z = 0, \qquad v \in V_2,$$

where matrix

$$A_v = -i[I_v + T_v(\varepsilon\omega)]^{-1}[I_v - T_v(\varepsilon\omega)]$$

is real valued and symmetric $(A_v = A_v')$.

If $(\varepsilon\omega) = \lambda_0$, then T_v is orthogonal with eigenvalues ± 1 and the GC is a generalized Kirchhoff condition.

Theorem 1. *For any interval* $[\lambda_0, \lambda']$, $\lambda' < \lambda_1$, *there exists* ρ *such that scattering solutions* $\Psi_{p,\varepsilon}(x)$ *of the problem in* Ω_ε *have the following asymptotic behavior on the channels of* Ω_ε *as* $\varepsilon \to 0$

$$\Psi_{p,\varepsilon}(x) = \psi_{p,\varepsilon}(\gamma)\varphi_0(\frac{y}{\varepsilon}) + r_p^{(\varepsilon)}(x), \quad \gamma \in \Gamma,$$

where $\psi_{p,\varepsilon}(\gamma)$ *are the scattering solutions of the problem on the graph* Γ *and*

$$|r_p^{(\varepsilon)}(x)| \leq Ce^{\frac{-\rho d(\gamma)}{\varepsilon}}, \qquad (\varepsilon\omega)^2 \in [\lambda_0, \lambda'] \backslash F.$$

Here $\gamma = \gamma(x)$ *is the point on* Γ *which is defined by the cross-section of the channel through the point* x, $d(\gamma)$ *is the distance between* γ *and the closest vertex of the graph, and* F *is an exponentially in* ε *small neighborhood of some finite set of points.*

Similar statement is valid for the resolvent, i.e. the GC on the graph for any limiting ($\varepsilon \to 0$) solution and for the resolvent (not only for scattering solutions) are defined by the scattering matrices

PART II. Slowing down of light and transparency.

Introduction. The problem of creating compact and efficient optical delay devices has been very actively discussed in literature for the last several years. These devices can find many applications, for example in synchronizing the work of very fast optical elements and much slower electronics. My next goal is to present a simple mathematical model for optical delay devices (a necklace waveguide) where the slowing down is accompanied by the transparency.

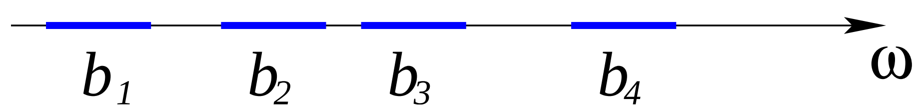
Consider

$$u_{tt} = u_{xx} - q(x)u, \quad x \in R^1, \quad t > 0; \qquad\qquad q(x+L) = q(x).$$

Let $u(t,x) = \psi(x,\omega)e^{-i\omega t}$. Then

$$H\psi := -\psi_{xx} + q(x)\psi = \omega^2\psi \tag{2}$$

The spectrum of this periodic Schrodinger operator has a band-gap structure, i.e. the spectrum is absolutely continuous and consists of segments b_n, called bands, separated by gaps.



The equation (2) has the Bloch solutions of the form

$$\psi(x,\omega) = \varphi(x,\omega)e^{\pm ik(\omega)x/L}, \quad \varphi(x+L,\omega) = \varphi(x,\omega),$$

and Im$k(\omega) = 0$ when $k \in \cup b_n$. Plane (Bloch) waves:

$$u(t,x) = \psi(x,\omega)e^{-i\omega t} = \varphi(x,\omega)e^{\pm ik(\omega)x/L - i\omega t}, \;\; V = V_{phase} = L\omega/k(\omega) >>> 1$$

Consider a wave packet

$$u = \int_{|\omega - \omega_0| < \varepsilon} \psi(x, \omega) \alpha(\omega - \omega_0) e^{-i\omega t} d\omega. \tag{3}$$

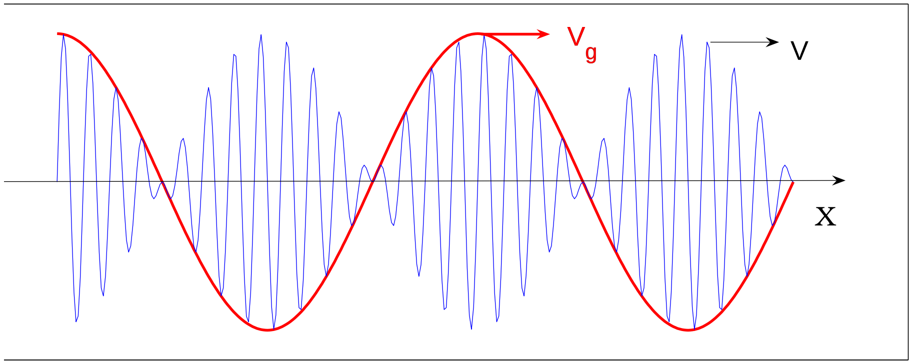
angular frequencies from a small interval around ω_0. If ε and $\varepsilon^2 |x|$ are small, then

$$u \sim \gamma(x) \widetilde{\alpha}(\frac{x}{V_g} - t) e^{i(k(\omega_0)\frac{x}{L} - \omega_0 t)}, \tag{4}$$

where $\widetilde{\alpha}$ is the Fourier transform of $\alpha(\omega)$, function γ is periodic, and

$$V_g = L/k'(\omega_0)$$

is the group velocity. The wave packet u has the form of a plane wave which propagates with the phase velocity $V(\omega_0)$ and which is modulated by another wave $\widetilde{\alpha}$ propagating with the speed V_g.

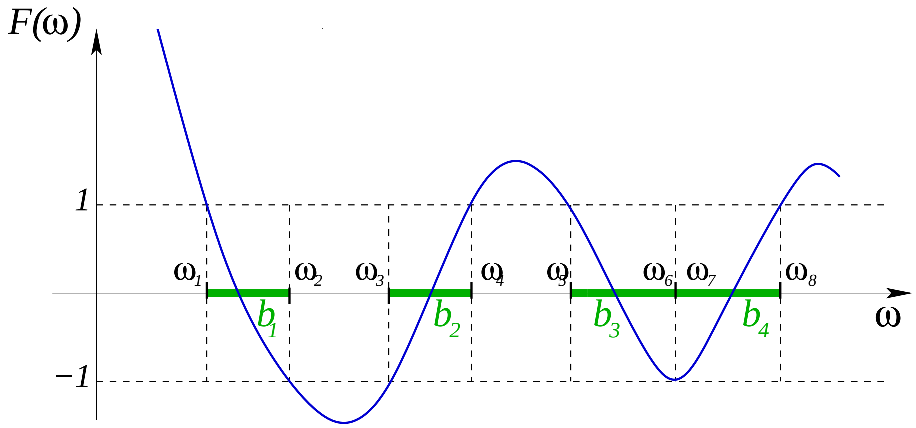


Let M_ω be the (Prüffer) monodromy operator:

$$M_\omega = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \begin{pmatrix} \psi(0) \\ \omega^{-1}\psi'(0) \end{pmatrix} \to \begin{pmatrix} \psi(L) \\ \omega^{-1}\psi'(L) \end{pmatrix}$$

The Hill discriminant:

$$F(\omega) = \frac{1}{2} Tr M_\omega = (\psi_1 + \psi_2')(\omega, L) = (\alpha + \delta)(\omega)\,.$$



Functions $e^{\pm ik(\omega)}$ are eigenvalues of M_ω, i.e.

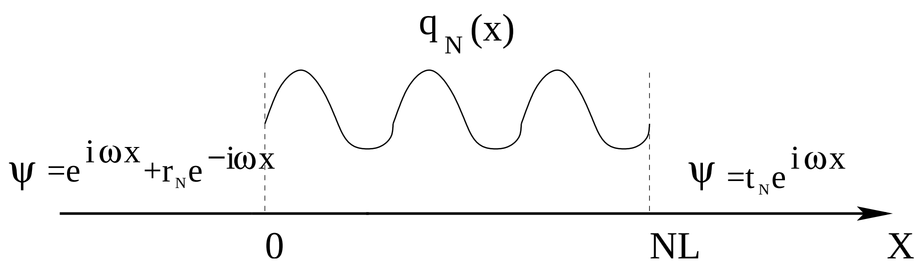
$$\cos k(\omega) = F\left(\omega\right) = \frac{1}{2} Tr M_\omega .$$

The function $k(\omega)$ changes by π on a band, i.e. if a band is narrow then $|k'(\omega)| >> 1$ and $|V_g| = |L/k'(\omega)| << 1$

Reflection

$$\psi_{tt} = \psi_{xx} + q_N(x)\psi.$$

Solution ψ of the scattering problem for the truncated potential:



The goal is to find a potential for which both $|V_g| << 1$ and $|r_N| << 1$. The reflection coefficient $r = r_N$ can be estimated as follows

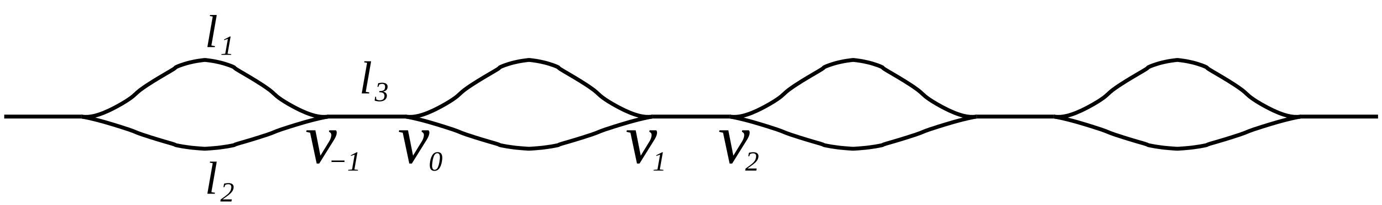
$$|r_N| = |\frac{sinNk(\sigma)}{sink(\sigma)}|(||M_\sigma||^2 - 2)^{1/2},$$

where $||M_\sigma||$ is the Gilbert-Shmidt norm of the monodromy matrix, i.e.

$$||M_\sigma||^2 = ||(m_{i,j})||^2 = \sum_{i,j \le 2} m_{i,j}^2.$$

Part III Necklace waveguide

In this section we consider a truncated periodic metric graph Γ of the necklace type. One can manufacture the corresponding device using a chain of modified Mach-Zehnder interferometers.



One cell of periodicity consists of two arches of lengths l_1 and $l_2 \leq l_1$ connected at end points and of a segment of length l_3 starting at one of these points.

We will use local coordinates $z = \pm s$ on the edges of the graph, where s is the arc length of edges measured from their left ends and z is the arc length, measured from a vertex for all the edges adjacent to this vertex (z is convenient for GC)

Our goal in this section is to study the propagation of waves on Γ governed by the equation

$$H\psi = -\frac{d^2}{dz^2}\psi = \sigma^2\psi, \quad \gamma \in \Gamma,$$

and GC

$$\frac{d}{dz}\psi^{(v)}(z) - \sigma A_v(\varepsilon\omega)\psi^{(v)}(z) = 0, \qquad z = 0, \qquad v \in V, \;\; A_v = -i[I_v + T_v(\varepsilon\omega)]^{-1}[I_v - T \tag{5}$$

at the vertices of Γ. Here and below we use

$$\sigma = \sqrt{\omega^2 - \varepsilon^{-2}\lambda_0}.$$

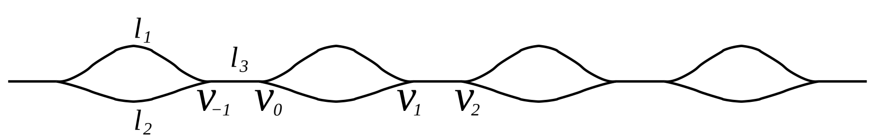
Note that both the equation and GC depend on the frequency ω and ε.

We assume that all junctions of the network Ω_ε have the same geometry and that they are symmetric with respect to the axis of the graph. This implies the following structure of the matrices A_v(they are symmetric): they do not depend on v and they are preserved under simultaneous permutations of the first two rows and of the first two columns, i.e.

$$A = A_v = \begin{pmatrix} a & b & d \\ b & a & d \\ d & d & c \end{pmatrix} = \begin{pmatrix} B & \delta \\ \delta^* & c \end{pmatrix}, \quad \delta = \begin{pmatrix} d \\ d \end{pmatrix}, \quad B = \begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

Our first step will be to define and evaluate the (Prüffer) monodromy operator M_σ (transfer operator over the period). Let $\begin{pmatrix} \psi \\ \sigma^{-1}\psi_s \end{pmatrix}(\alpha)$ be the Cauchy data (it always will have the factor σ^{-1} in the second component) of the solution ψ of the equation $H\psi = \sigma^2\psi$ evaluated at a point α of a straight segment of Γ. When $\alpha = v_n$ we understand this vector as the limit of the corresponding vectors evaluated at α as α approaches v_n moving along the segment (not along one of the arches). We denote by M_σ the monodromy operator:

$$M_\sigma : \begin{pmatrix} \psi \\ \sigma^{-1}\psi_s \end{pmatrix}(v_0) \to \begin{pmatrix} \psi \\ \sigma^{-1}\psi_s \end{pmatrix}(v_2).$$



and we denote by T_σ the (Prüffer) transfer operator over the loop:

$$T_\sigma : \begin{pmatrix} \psi \\ \sigma^{-1}\psi_s \end{pmatrix}(v_0) \to \begin{pmatrix} \psi \\ \sigma^{-1}\psi_s \end{pmatrix}(v_1).$$

We will need below the following matrices

$$S = \begin{pmatrix} \sin \sigma l_1 & 0 \\ 0 & \sin \sigma l_2 \end{pmatrix}, \quad C = \begin{pmatrix} \cos \sigma l_1 & 0 \\ 0 & \cos \sigma l_2 \end{pmatrix}, \quad P = C + SB,$$

$$M = (I - P^2)^{-1} PS, \quad N = (I - P^2)^{-1} S.$$

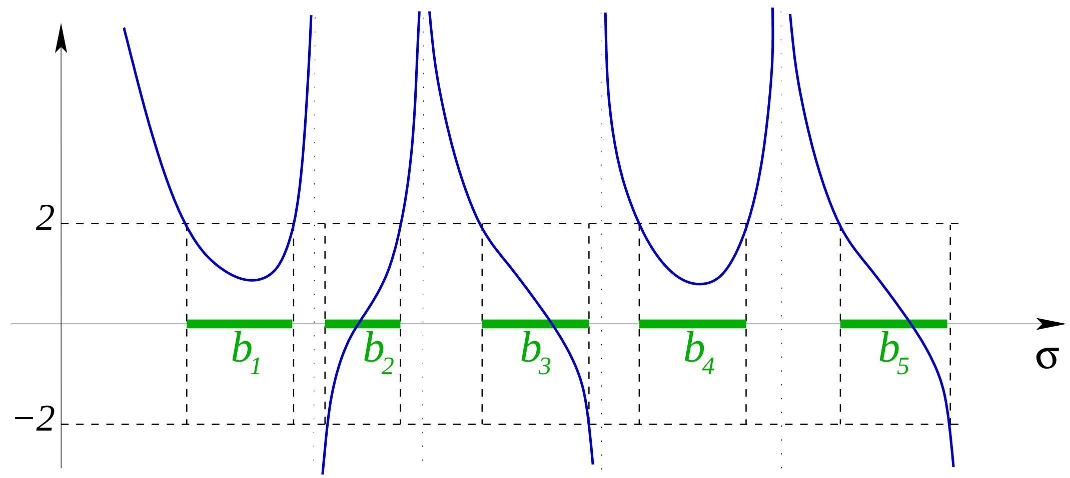
Theorem 2. *1) Matrix T_σ has the form*

$$T_\sigma = - \begin{pmatrix} \frac{m}{n} & \frac{1}{n} \\ \frac{m^2 - n^2}{n} & \frac{m}{n} \end{pmatrix}, \quad n = \langle \delta, N\delta \rangle, \quad m = c + \langle \delta, M\delta \rangle.$$

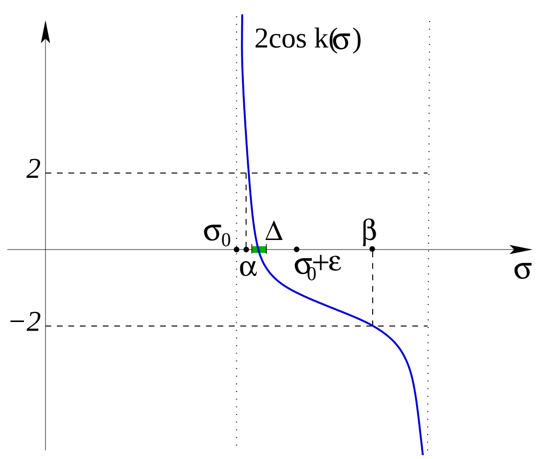
2) Matrix M_σ has the form

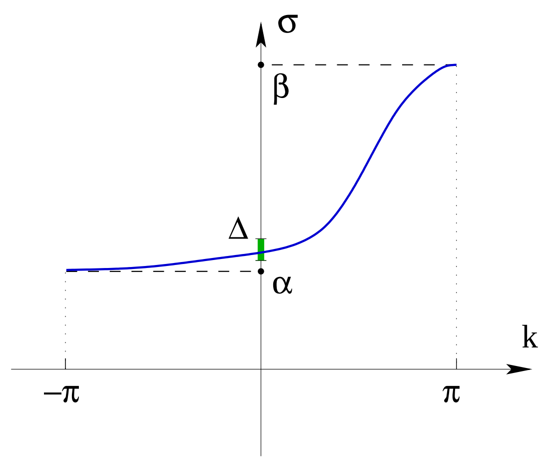
$$M_\sigma = \begin{pmatrix} \cos \sigma l_3 & \sin \sigma l_3 \\ - \sin \sigma l_3 & \cos \sigma l_3 \end{pmatrix} T_\sigma .$$

The graph of the function $\mathrm{Trace} M_\sigma = 2\cos k(\sigma), \quad \sigma = \sqrt{\omega^2 - \varepsilon^{-2}\lambda_0}$.



The function is not analytic, but meromorphic !!!, and poles can be made as close as we pleased, producing a narrow band at any place !!!





Slowing down. Let $\sigma = \sigma_0$ be the center of the frequency interval Δ of a narrow (in frequency) wave package. We will find parameters l_j such that $\cos k(\sigma_0) = 0 \quad (\cos k(\sigma) = \frac{1}{2}\mathrm{Trace} M_\sigma)$ and there is a pole of $\cos k(\sigma)$ at a very close point $\sigma = \sigma_1$. Then $|k'(\sigma_0)| >> 1$ and

$$V_g = \frac{L}{k'(\sigma)} = O(\varepsilon), \quad \sigma \in \Delta.$$

The following relation determines the poles: $n = 0$

$$T_\sigma = -\begin{pmatrix} \frac{m}{n} & \frac{1}{n} \\ \frac{m^2-n^2}{n} & \frac{m}{n} \end{pmatrix}, \quad M_\sigma = \begin{pmatrix} \cos\sigma l_3 & \sin\sigma l_3 \\ -\sin\sigma l_3 & \cos\sigma l_3 \end{pmatrix} T_\sigma.$$

Transparency. Consider a finite slab of a necklace wave guide with N periods. Δ will be in the middle of a narrow band. Then $|\sin k(\sigma)| > c > 0$ and

$$|r_N(\sigma)| \le c^{-1}|(||M_\sigma||^2 - 2)^{1/2} \le c_1 \frac{m^2 - n^2 + 1}{n}, \quad \sigma \in \Delta.$$

We will choose $l_1,\ l_2$ in such a way that $||M_\sigma||^2 = 2$ at the center σ_0 of the given frequency support Δ of the wave package. Then $r_N(\sigma_0) = 0$ and $r_N(\sigma)$ is small if $|\Delta|$ is small. And we will still have a small group velocity if there is a pole of $\cos k(\sigma)$ close enough. So in this case, we will have both the slowing down of the light and the transparency in Δ.

Since the expression for M_σ was found, we can write an explicit equation for all transparency points (where $r_N = 0$):

$$[c - \frac{d^2(2a - 2b + x^{-1} + y^{-1})}{(x^{-1} + a)(y^{-1} + a) - b^2}][c + \frac{d^2(2b - 2a + x + y)}{(x - a)(y - a) - b^2}] + 1 = 0 \quad \text{if } n \neq 0. \tag{6}$$

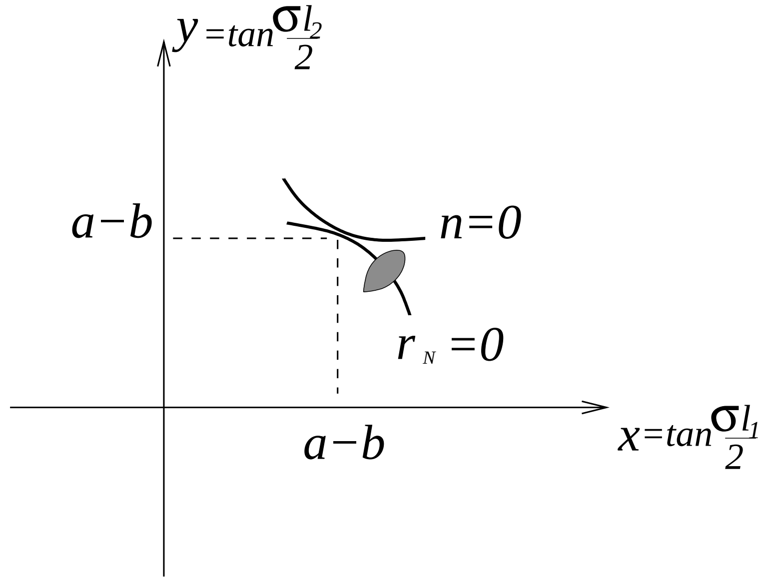
This is an algebraic curve of the forth order in (x, y)-plane.

$$x = tan\frac{\sigma l_1}{2}, \quad y = tan\frac{\sigma l_2}{2}.$$

Poles are given by the equation:

$$n = \frac{d^2(2b - 2a + x + y)}{(x - a)(y - a) - b^2} + \frac{d^2(2a - 2b + x^{-1} + y^{-1})}{(x^{-1} + a)(y^{-1} + a) - b^2} = 0.$$

The choice of parameters. Recall, that the center σ_0 in the frequency interval of the wave package is given, and we need to choose $l_1,\ l_2$ in such a way that (6) holds at σ_0. At the same time, a pole of the Hill discriminant has to be close by.



Hence $l_1,\ l_2$ have to be chosen in such a way that $x \approx y \approx a-b$. After that l_1, l_2 can be easily found:

$$\tan\frac{\sigma_0 l_1}{2} = a - b + \epsilon, \quad \tan\frac{\sigma_0 l_2}{2} \approx a - b - \epsilon + \gamma_0 \epsilon^2.$$

$$|\Delta| = O(\epsilon^2), \quad V_g = O(\varepsilon^2), \quad r_N = O(\epsilon^2), \quad \text{in} \quad \Delta.$$