Isaac Newton Institute for Mathematical Sciences
Gyrokinetics in Laboratory and Astrophysical Plasmas

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GYROKINETIC ENERGY CONSERVATION LAWS

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Three Pillars of Modern Gyrokinetic Theory

I. Gyrokinetic Vlasov equation

is written in terms of a **gyrocenter Hamiltonian** with quadratic low-frequency **ponderomotive** terms.

II. Gyrokinetic Maxwell equations

are written in terms of **gyrocenter Vlasov** distribution and contain low-frequency **polarization** (Poisson) and **magnetization** (Ampere) terms derived from quadratic nonlinearities in the gyrocenter Hamiltonian.

III. Exact Gyrokinetic Conservation Laws

exist for the **gyrokinetic Vlasov-Maxwell** equations that include linear and nonlinear coupling terms.

1. Gyrokinetic Vlasov Equation

Particle Hamiltonian Dynamics

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Guiding-center Hamiltonian Dynamics

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Gyrocenter Hamiltonian Dynamics

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Gyrokinetic Vlasov Equation

- **Guiding-center ordering** $\epsilon \equiv \rho_{\rm g}/L_B \ll 1$
  (Northrop: $\epsilon \sim m/e \sim e^{-1}$)

$$\rho_{\rm g} \equiv \frac{|\mathbf{v}_\perp|}{|\Omega|} \quad \text{vs.} \quad L_B^{-1} \equiv \left\{ \begin{array}{ll} |\widehat{\sf b} \times \nabla \ln B| & (\text{grad}_\perp B) \\ |\widehat{\sf b} \cdot \nabla \widehat{\sf b}| & (\text{curvature}) \\ |\widehat{\sf b} \cdot \nabla \ln B| & (\text{grad}_\| B) \end{array} \right.$$

a. Guiding-center Hamiltonian Dynamics

- **Guiding-center Hamiltonian**

$$H_{\mathsf{gc}}(\mathbf{X}, p_{\|}, \mu) \;=\; \frac{p_{\|}^2}{2m} \;+\; \mu\, B(\mathbf{X})$$

- **Guiding-center Poisson bracket**

$$\begin{aligned}
\{F,\, G\}_{\mathsf{gc}} \;&=\; \epsilon^{-1}\left[\frac{e}{mc}\left(\frac{\partial F}{\partial\zeta}\,\frac{\partial G}{\partial\mu} \;-\; \frac{\partial F}{\partial\mu}\,\frac{\partial G}{\partial\zeta}\right)\right] \;\rightarrow\; \text{fast} \\
&+\; \epsilon^{0}\left[\frac{\mathbf{B}^*}{B_{\|}^*}\cdot\left(\nabla F\,\frac{\partial G}{\partial p_{\|}} \;-\; \frac{\partial F}{\partial p_{\|}}\,\nabla G\right)\right] \;\rightarrow\; \text{intermediate} \\
&-\; \epsilon^{1}\left[\frac{c\widehat{\mathsf{b}}}{eB_{\|}^*}\cdot\nabla F\times\nabla G\right] \;\rightarrow\; \text{slow}
\end{aligned}$$

- Jacobian $\mathcal{J}_{\mathsf{gc}} = m\, B_{\|}^* = m\,\widehat{\mathsf{b}}\cdot\mathbf{B}^*$:

$$\mathbf{B}^* \;\equiv\; \mathbf{B} \;+\; \epsilon\left(\frac{c}{e}\,p_{\|}\right)\,\nabla\times\widehat{\mathsf{b}} \;-\; \epsilon^2\,(\cdots)$$

- **Guiding-center Hamiltonian Dynamics**

$$\frac{d_{\mathsf{gc}}\,\mathbf{X}}{dt} \;=\; \{\mathbf{X},\; H_{\mathsf{gc}}\}_{\mathsf{gc}} \;=\; v_{\|}\,\frac{\mathbf{B}^{*}}{B_{\|}^{*}} \;+\; \epsilon\,\frac{c\widehat{\mathsf{b}}}{eB_{\|}^{*}}\times\mu\,\nabla B$$

$$\frac{d_{\mathsf{gc}}\,p_{\|}}{dt} \;=\; \{p_{\|},\; H_{\mathsf{gc}}\}_{\mathsf{gc}} \;=\; -\,\mu\,\frac{\mathbf{B}^{*}}{B_{\|}^{*}}\cdot\nabla B$$

$$\frac{d_{\mathsf{gc}}\,\mu}{dt} \;=\; \{\mu,\; H_{\mathsf{gc}}\}_{\mathsf{gc}} \;=\; -\,\epsilon^{-1}\,\frac{\Omega}{B}\,\frac{\partial H_{\mathsf{gc}}}{\partial\zeta} \;\equiv\; 0$$

$$\frac{d_{\mathsf{gc}}\,\zeta}{dt} \;=\; \{\zeta,\; H_{\mathsf{gc}}\}_{\mathsf{gc}} \;=\; \epsilon^{-1}\,\Omega \;+\; \cdots$$

- Guiding-center Liouville Theorem (no attractors)

$$\nabla\cdot\left(B_{\|}^{*}\,\frac{d_{\mathsf{gc}}\,\mathbf{X}}{dt}\right) \;+\; \frac{\partial}{\partial p_{\|}}\left(B_{\|}^{*}\,\frac{d_{\mathsf{gc}}\,p_{\|}}{dt}\right) \;\equiv\; 0$$

- **Guiding-center Polarization & Magnetization**

$$\text{gyroradius}: \quad \boldsymbol{\rho}_{\rm gc} = \boldsymbol{\rho}_0 + \epsilon_B\, \boldsymbol{\rho}_1 + \cdots$$

- Guiding-center pull-back & push-forward operators

$$\mathsf{T}^{\pm}_{\rm gc} \equiv \cdots \exp\left(\mp\, \boldsymbol{\rho}_0 \cdot \nabla\right) \cdots$$

- Guiding-center polarization

$$\boldsymbol{\pi}_{\rm gc} \equiv e\, \langle \boldsymbol{\rho}_{\rm gc} \rangle = \epsilon_B\, e\, \langle \boldsymbol{\rho}_1 \rangle$$

- Guiding-center magnetization (intrinsic)

$$\boldsymbol{\mu}_{\rm gc} \equiv \frac{e}{2c} \left\langle \boldsymbol{\rho}_{\rm gc} \times \frac{d_{\rm gc}\boldsymbol{\rho}_{\rm gc}}{dt} \right\rangle = -\,\mu\, \widehat{\mathsf{b}} + \cdots$$

$$\boxed{\mathbf{J} \equiv \mathbf{J}_{\rm gc} + c\, \nabla \times \mathbf{M}_{\rm gc}}$$

b. Gyrocenter Hamiltonian Dynamics

- **Perturbed Guiding-center Lagrangian** $(\epsilon_\delta \ll 1)$

  ◦ Low-frequency $(\omega \ll \Omega)$, short perpendicular wavelength $(|k_{\|}| \ll |\mathbf{k}_\perp| \sim \rho_{\rm g}^{-1})$ field fluctuations

$$(\phi_1, \mathbf{A}_1) \;\rightarrow\; \left(\phi_{1\rm gc} \equiv \mathsf{T}_{\rm gc}^{-1}\phi_1,\; \mathbf{A}_{1\rm gc} \equiv \mathsf{T}_{\rm gc}^{-1}\mathbf{A}_1\right)$$

  ◦ Perturbed (extended) symplectic structure

$$\widehat{\Gamma}_{\rm gc} \;\equiv\; \widehat{\Gamma}_{\rm gc0} \,+\, \epsilon_\delta\,\frac{e}{c}\,\mathbf{A}_{1\rm gc}\cdot\left({\rm d}\mathbf{X} \,+\, {\rm d}\boldsymbol{\rho}_0\right)$$

  ◦ Perturbed (extended) Hamiltonian

$$\mathcal{H}_{\rm gc} \;\equiv\; \left(H_{\rm gc0} \,-\, w\right) \,+\, \epsilon_\delta\, e\, \phi_{1\rm gc}$$

Guiding-center magnetic-moment invariance is lost...but can be regained (Taylor 1967).

- **Gyrocenter Phase-space Transformation**

$$\overline{z}^a \;=\; z^a \;+\; \epsilon_\delta\, G_1^a \;+\; \epsilon_\delta^2 \left(G_2^a \;+\; \frac{1}{2}\, G_1^b\, \frac{\partial G_1^a}{\partial z^b} \right) \;+\; \cdots$$

◦ Hamiltonian Representation: $\widehat{\Gamma}_{\mathsf{gc}} \rightarrow \widehat{\overline{\Gamma}}_{\mathsf{gy}} \equiv \widehat{\Gamma}_{\mathsf{gc0}}$

$$G_1^a \;=\; \overbrace{\{S_1,\; z^a\}_{\mathsf{gc}}}^{\text{canonical}} + \frac{e}{c}\,\mathbf{A}_{1\mathsf{gc}}\cdot \left\{ \mathbf{X} + \boldsymbol{\rho}_0,\; z^a \right\}_{\mathsf{gc}}$$

$$G_2^a \;=\; \underbrace{\{S_2,\; z^a\}_{\mathsf{gc}}}_{\text{canonical}} - \frac{e}{2c}\,\boldsymbol{\rho}_{1\mathsf{gy}} \times \mathbf{B}_{1\mathsf{gc}}\cdot \left\{ \mathbf{X} + \boldsymbol{\rho}_0,\; z^a \right\}_{\mathsf{gc}}$$

"Gyrocenter" dynamics describes the motion of a gyroangle-averaged perturbed guiding-center.

- **Hamiltonian Lie-transform perturbation analysis**

$$\mathcal{H}_{\rm gc} \;\rightarrow\; \overline{\mathcal{H}}_{\rm gy} \;\equiv\; {\sf T}_{\rm gy}^{-1}\mathcal{H}_{\rm gc} \;\equiv\; \overline{H}_{\rm gy} \;-\; \overline{w}$$

∘ First-order analysis

$$\overline{H}_1 \;\equiv\; e\,\psi_{\rm 1gc} \;-\; \left(\frac{\partial S_1}{\partial t} \;+\; \{S_1,\; H_0\}_{\rm gc}\right)$$

where $\psi_{\rm 1gc} \;\equiv\; \phi_{\rm 1gc} \;-\; \mathbf{A}_{\rm 1gc}\cdot\mathbf{v}_{\rm gc}/c \;\equiv\; \langle\psi_{\rm 1gc}\rangle \;+\; \widetilde{\psi}_{\rm 1gc}$

$$\overline{H}_1 \;\equiv\; e\,\langle\psi_{\rm 1gc}\rangle \;=\; e\left\langle\phi_{\rm 1gc} \;-\; \mathbf{A}_{\rm 1gc}\cdot\frac{\mathbf{v}_{\rm gc}}{c}\right\rangle$$

$$S_1 \;\equiv\; (d_{\rm gc}/dt)^{-1}\,e\,\widetilde{\psi}_{\rm 1gc} \;=\; (e/\Omega)\int\widetilde{\psi}_{\rm 1gc}\,d\zeta \;+\; \cdots$$

∘ Second-order analysis

$$\overline{H}_2 \;=\; \frac{e^2}{2mc^2}\left\langle|\mathbf{A}_{\rm 1gc}|^2\right\rangle \;-\; \frac{e^2}{2\Omega}\left\langle\left\{S_1,\;\widetilde{\psi}_{\rm 1gc}\right\}_{\rm gc}\right\rangle$$

- **Gyrocenter Hamiltonian Dynamics**

◦ Gyrocenter Hamiltonian ($\pounds_{\rm gy}\, F \equiv G_1^a\, \partial_a F$)

$$\overline{H}_{\rm gy} \;=\; \overline{H}_{\rm gc0} \;+\; e\left(\epsilon_\delta\, e\, \langle\psi_{\rm gc}\rangle \;-\; \frac{\epsilon_\delta^2}{2}\,\left\langle \pounds_{\rm gy}\,\psi_{\rm 1gc}\right\rangle + \cdots\right)$$

$$\equiv\; \overline{H}_{\rm gc0} \;+\; e\,\delta\boldsymbol{\Psi}_{\rm gy}$$

◦ Gyrocenter pull-back & push-forward operators

$$\mathsf{T}_{\rm gy}^{\pm} \;\equiv\; \cdots\; \exp\left(\pm\,\epsilon_\delta \pounds_{\rm gy}\right)\; \cdots$$

◦ Gyrocenter Hamilton equations

$$\frac{d_{\rm gy}\,\overline{\mathbf{X}}}{dt} \;=\; \frac{d_{\rm gc}\,\overline{\mathbf{X}}}{dt} \;+\; e\,\frac{\partial\delta\boldsymbol{\Psi}_{\rm gy}}{\partial\overline{p}_\|}\,\widehat{\mathsf{b}} \;+\; \frac{c\widehat{\mathsf{b}}}{B}\times\overline{\nabla}\delta\boldsymbol{\Psi}_{\rm gy}$$

$$\frac{d_{\rm gy}\,\overline{p}_\|}{dt} \;=\; \frac{d_{\rm gc}\,\overline{p}_\|}{dt} \;-\; e\,\widehat{\mathsf{b}}\cdot\overline{\nabla}\delta\boldsymbol{\Psi}_{\rm gy}$$

c. Gyrokinetic Vlasov Equation

- **Gyrokinetic Vlasov equation**

$$0 \;=\; \frac{\partial\overline{F}}{\partial t} + \frac{d_{\mathrm{gy}}\,\overline{\mathbf{X}}}{dt}\cdot\overline{\nabla F} + \frac{d_{\mathrm{gy}}\,\overline{p}_{\|}}{dt}\,\frac{\partial\overline{F}}{\partial\overline{p}_{\|}} \;\equiv\; \frac{\partial\overline{F}}{\partial t} + \left\{\overline{F},\,\overline{H}_{\mathrm{gy}}\right\}_{\mathrm{gc}}$$

◦ Dynamical Reduction

$$\boxed{\frac{d_{\mathrm{gy}}\,\overline{\mu}}{dt} \;\equiv\; 0 \quad \text{and} \quad \frac{\partial\overline{F}}{\partial\overline{\zeta}} \;\equiv\; 0}$$

- **Extended gyrokinetic Vlasov equation**

$$\boxed{0 \;=\; \left\{\overline{\mathcal{F}},\,\overline{\mathcal{H}}\right\}_{\mathrm{gc}}}$$

◦ Extended phase-space gyrocenter Vlasov distribution

$$\overline{\mathcal{F}} \;\equiv\; c\,\delta(\overline{w} - \overline{H}_{\mathrm{gy}})\;\overline{F}(\overline{\mathbf{X}},\overline{p}_{\|},t;\overline{\mu})$$

2. Gyrokinetic Maxwell Equations

Maxwell Equations

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Gyrocenter Polarization & Magnetization

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Gyrokinetic Maxwell Equations

- Self-consistent description of low-frequency electromagnetic fluctuations produced by gyrocenter motion.

- Gyrokinetic Maxwell equations expressed in terms of moments of gyrocenter Vlasov distribution.

a. Push-forward Representation of Particle Fluid Moments: General Theory

- **Reduced Displacement** $\boldsymbol{\rho}_\epsilon \equiv \mathsf{T}_\epsilon^{-1}\mathbf{x} - \overline{\mathbf{x}}$

$$\boldsymbol{\rho}_\epsilon \;=\; -\,\epsilon\, G_1^{\mathbf{x}} \;-\; \epsilon^2\left(G_2^{\mathbf{x}} - \frac{1}{2}\,\mathsf{G}_1\cdot\mathsf{d}G_1^{\mathbf{x}}\right) + \cdots \;\equiv\; \langle\boldsymbol{\rho}_\epsilon\rangle \;+\; \widetilde{\boldsymbol{\rho}}_\epsilon$$

- **Push-forward representation of particle velocity**

$$\mathsf{T}_\epsilon^{-1}\mathbf{v} \;=\; \left[\mathsf{T}_\epsilon^{-1}\frac{d}{dt}\mathsf{T}_\epsilon\right]\left(\mathsf{T}_\epsilon^{-1}\mathbf{x}\right) \;\equiv\; \frac{d_\epsilon\overline{\mathbf{x}}}{dt} \;+\; \frac{d_\epsilon\boldsymbol{\rho}_\epsilon}{dt}$$

◦ Reduced displacement & **polarization-drift** velocities

$$\frac{d_\epsilon\boldsymbol{\rho}_\epsilon}{dt} \;\equiv\; \left\{\boldsymbol{\rho}_\epsilon,\,\overline{\mathcal{H}}\right\}_\epsilon \;=\; \frac{\partial\boldsymbol{\rho}_\epsilon}{\partial t} \;+\; \left\{\boldsymbol{\rho}_\epsilon,\,\overline{H}\right\}_\epsilon \;\;\Rightarrow\;\; \frac{d_\epsilon\langle\boldsymbol{\rho}_\epsilon\rangle}{dt} \;\neq\; 0$$

- **Reduced Maxwell Equations**

$$\nabla\cdot\mathbf{E} \;=\; 4\pi\,\rho \;\;\text{and}\;\; \nabla\times\mathbf{B} \;-\; \frac{1}{c}\,\frac{\partial\mathbf{E}}{\partial t} \;=\; \frac{4\pi}{c}\,\mathbf{J}$$

$$\Downarrow$$

$$\nabla\cdot\mathbf{D} \;=\; 4\pi\,\overline{\rho} \;\;\text{and}\;\; \nabla\times\mathbf{H} \;-\; \frac{1}{c}\,\frac{\partial\mathbf{D}}{\partial t} \;=\; \frac{4\pi}{c}\,\overline{\mathbf{J}}$$

- Reduced (macroscopic) electromagnetic fields

$$(\mathbf{D},\mathbf{H}) \;\equiv\; (\mathbf{E} + 4\pi\,\mathbf{P}_\epsilon,\; \mathbf{B} - 4\pi\,\mathbf{M}_\epsilon)$$

- Reduced four-current

$$\overline{J}^{\mu} \;=\; \left(c\overline{\rho},\overline{\mathbf{J}}\right) \;=\; e\int d^4\overline{p}\;\overline{\mathcal{F}}\;\left(c,\;\frac{d_\epsilon\overline{\mathbf{x}}}{dt}\right)$$

- Reduced Polarization and Magnetization $(\mathbf{P}_\epsilon,\mathbf{M}_\epsilon)$

- **Push-forward representation of the charge density**

$$\boxed{\rho \;\equiv\; \overline{\rho} \;-\; \nabla\cdot\mathbf{P}_\epsilon}$$

◦ Polarization

$$\mathbf{P}_\epsilon \;\equiv\; e\int d^4\overline{p}\left[\underbrace{\boldsymbol{\rho}_\epsilon\,\overline{\mathcal{F}}}_{\text{dipole}} \;-\; \underbrace{\frac{1}{2}\nabla\cdot\left(\boldsymbol{\rho}_\epsilon\,\boldsymbol{\rho}_\epsilon\,\overline{\mathcal{F}}\right)}_{\text{quadrupole}} \;+\;\cdots\right]$$

- **Push-forward representation of the current density**

$$\boxed{\mathbf{J} \;=\; \overline{\mathbf{J}} \;+\; \frac{\partial\mathbf{P}_\epsilon}{\partial t} \;+\; c\,\nabla\times\mathbf{M}_\epsilon}$$

◦ Magnetization (intrinsic + moving-electric-dipole)

$$\mathbf{M}_\epsilon \;=\; \frac{e}{c}\int d^4\overline{p}\left(\frac{1}{2}\,\boldsymbol{\rho}_\epsilon\times\frac{d_\epsilon\boldsymbol{\rho}_\epsilon}{dt} \;+\; \boldsymbol{\rho}_\epsilon\times\frac{d_\epsilon\overline{\mathbf{x}}}{dt}\right)\overline{\mathcal{F}}$$

b. Gyrocenter Polarization & Magnetization

- **Gyrocenter displacement**

$$\boldsymbol{\rho}_{1\mathrm{gy}} \equiv \left\{ \mathbf{X} + \boldsymbol{\rho}_0, S_1 \right\}_{\mathrm{gc}}$$

◦ Gyroangle-independent part → Gyrocenter Polarization

$$\langle \boldsymbol{\rho}_{1\mathrm{gy}} \rangle \equiv \left\langle \left\{ \boldsymbol{\rho}_0, S_1 \right\}_{\mathrm{gc}} \right\rangle = -\frac{\Omega}{B} \frac{\partial}{\partial \mu} \left\langle \boldsymbol{\rho}_0 \frac{\partial S_1}{\partial \zeta} \right\rangle$$

◦ Gyroangle-dependent part → Gyrocenter Magnetization

$$\widetilde{\boldsymbol{\rho}}_{1\mathrm{gy}} = \{\mathbf{X},\, S_1\}_{\mathrm{gc}} + \Big(\{\boldsymbol{\rho}_0,\, S_1\}_{\mathrm{gc}} - \langle \{\boldsymbol{\rho}_0,\, S_1\}_{\mathrm{gc}} \rangle \Big)$$

- **Gyrocenter dipole moments**

  - Gyrocenter electric-dipole moment

$$\boldsymbol{\pi}_{1\mathrm{gy}} \;\equiv\; e\,\langle\boldsymbol{\rho}_{1\mathrm{gy}}\rangle \;=\; -\,\frac{e^2}{B}\,\frac{\partial}{\partial\mu}\left\langle\boldsymbol{\rho}_0\,\widetilde{\psi}_{1\mathrm{gc}}\right\rangle$$

  - Gyrocenter magnetic-dipole moment

$$\begin{aligned}\boldsymbol{\mu}_{1\mathrm{gy}} \;&\equiv\; \frac{e}{c}\left\langle\boldsymbol{\rho}_{1\mathrm{gy}}\times\left(\frac{p_\|}{m}\,\widehat{\mathsf{b}} \;+\; \Omega\,\frac{\partial\boldsymbol{\rho}_0}{\partial\zeta}\right)\right\rangle \\ &=\; \frac{e}{c}\left\langle\widetilde{\boldsymbol{\rho}}_{1\mathrm{gy}}\times\Omega\,\frac{\partial\boldsymbol{\rho}_0}{\partial\zeta}\right\rangle \;+\; e\,\langle\boldsymbol{\rho}_{1\mathrm{gy}}\rangle\times\frac{p_\|\,\widehat{\mathsf{b}}}{mc}\end{aligned}$$

  - Zero-Larmor-Radius (ZLR) limit ($\boldsymbol{\rho}_0\rightarrow 0$)

$$\boldsymbol{\pi}_{1\mathrm{gy}} \;\rightarrow\; \frac{mc^2}{B_0^2}\left(\mathbf{E}_{1\perp} \;+\; \frac{p_\|\widehat{\mathsf{b}}_0}{mc}\times\mathbf{B}_{1\perp}\right) \;+\; \frac{e\widehat{\mathsf{b}}_0}{B_0}\times\mathbf{A}_{1\perp}$$

$$\boldsymbol{\mu}_{1\mathrm{gy}} \;\rightarrow\; -\,\mu\,\frac{\mathbf{B}_1}{B_0} \;+\; \frac{p_\|}{mB_0}\left(\frac{e}{c}\,\mathbf{A}_{1\perp} \;+\; m\,\mathbf{E}_{1\perp}\times\frac{c\widehat{\mathsf{b}}_0}{B_0} \;+\; p_\|\,\mathbf{B}_{1\perp}\right)$$

3. Exact Gyrokinetic Energy Conservation Laws

Gyrokinetic Variational Formulation

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Noether Method

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Gyrokinetic Energy-Momentum Conservation Laws

$$\mathcal{A}_{\rm gy} \equiv \int \frac{d^4x}{8\pi} \left(\epsilon^2 \, |\mathbf{E}_1|^2 \, - \, |\mathbf{B}_0 + \epsilon \, \mathbf{B}_1|^2 \right) - \int d^8\overline{z} \, \overline{\mathcal{F}} \, \overline{\mathcal{H}}(\cdots; \Phi_1, \mathbf{A}_1; \mathbf{E}_1, \mathbf{B}_1; \cdots)$$

a. Full Gyrokinetic Vlasov-Maxwell Equations

- **Full gyrokinetic Lagrangian density**

$$\mathcal{L}_{\rm gy} \;\equiv\; \frac{1}{8\pi}\left(\epsilon^2\,|\mathbf{E}_1|^2 \;-\; |\mathbf{B}_0 + \epsilon\,\mathbf{B}_1|^2\right) - \int \overline{\mathcal{F}}\;\left(\overline{H}_{\rm gy} \;-\; \overline{w}\right)\; d^4\overline{p}$$

◦ Nonlinear gyrokinetic Vlasov equation

$$0 \;=\; \frac{d_{\rm gy}\overline{F}}{dt} \;\equiv\; \frac{\partial \overline{F}}{\partial t} \;+\; \left\{\overline{F},\; \overline{H}_{\rm gy}\right\}_{\rm gc}$$

◦ Gyrokinetic Maxwell's equations

$$\epsilon\,\nabla\cdot\mathbf{E}_1 \;=\; 4\pi \int_z e\,\overline{F}\;\left\langle \mathsf{T}_{\rm gy}^{-1}\,\delta^3_{\rm gc}\right\rangle$$

$$\nabla\times\mathbf{B} \;-\; \frac{\epsilon}{c}\,\frac{\partial \mathbf{E}_1}{\partial t} \;=\; 4\pi \int_z e\,\overline{F}\;\left\langle \mathsf{T}_{\rm gy}^{-1}\left(\frac{\mathbf{v}_{\rm gc}}{c}\,\delta^3_{\rm gc}\right)\right\rangle$$

- **Exact gyrokinetic energy conservation**

$$E_{\mathsf{gy}} \equiv \int \frac{d^3x}{8\pi}\left(\epsilon^2\,|\mathbf{E}_1|^2 + |\mathbf{B}_0 + \epsilon\,\mathbf{B}_1|^2\right) - \int_z \overline{F}\,\left(\overline{H}_{\mathsf{gy}} - \epsilon\,e\,\left\langle \mathsf{T}_{\mathsf{gy}}^{-1}\phi_{1\mathsf{gc}}\right\rangle\right)$$

- Energy exchange terms

$$\frac{dE_{\mathsf{gy}}}{dt} \equiv \mathcal{P}_F + \mathcal{P}_H + \mathcal{P}_\phi + \mathcal{P}_A$$

- Vlasov exchange term

$$\mathcal{P}_F \equiv \int_z \frac{\partial \overline{F}}{\partial t}\,\overline{H}_{\mathsf{gy}} = -\int_z \{(\overline{F}\,\overline{H}_{\mathsf{gy}}),\,\overline{H}_{\mathsf{gy}}\}_{\mathsf{gc}} \equiv 0$$

Commutator : $\mathsf{C}_{\mathrm{gy}} \equiv \left[\mathsf{T}_{\mathrm{gy}}^{-1},\, \frac{\partial}{\partial t}\right]$

- Poisson exchange term

$$\mathcal{P}_{\phi} \equiv \int d^3x\, \frac{\epsilon^2 \phi_1}{4\pi} \left(\nabla\cdot\frac{\partial \mathbf{E}_1}{\partial t}\right) - \epsilon \int_z \frac{\partial \overline{F}}{\partial t}\, e\, \left\langle \mathsf{T}_{\mathrm{gy}}^{-1}\phi_{1\mathrm{gc}}\right\rangle$$

$$= -\,\epsilon \int_z e\,\overline{F}\,\langle\, \mathsf{C}_{\mathrm{gy}}\, \phi_{1\mathrm{gc}}\rangle$$

- Ampère exchange term

$$\mathcal{P}_{A} \equiv \int d^3x\, \frac{\epsilon}{4\pi}\frac{\partial \mathbf{A}_1}{\partial t}\cdot\left(\nabla\times\mathbf{B} - \frac{\epsilon}{c}\frac{\partial \mathbf{E}_1}{\partial t}\right) - \epsilon\int_z \overline{F}\, e\, \left\langle \frac{\partial}{\partial t}\left[\mathsf{T}_{\mathrm{gy}}^{-1}\left(\mathbf{A}_{1\mathrm{gc}}\cdot\frac{\mathbf{v}_0}{c}\right)\right]\right\rangle$$

$$= \epsilon \int_z e\,\overline{F}\,\left\langle\, \mathsf{C}_{\mathrm{gy}}\left(\mathbf{A}_{1\mathrm{gc}}\cdot\frac{\mathbf{v}_0}{c}\right)\right\rangle$$

- Hamiltonian exchange term

$$\mathcal{P}_H \;\equiv\; \int_z \overline{F} \left(\frac{\partial \overline{H}_{\mathsf{gy}}}{\partial t} \;-\; \epsilon\, e \left\langle \frac{\partial}{\partial t} \left(\mathsf{T}_{\mathsf{gy}}^{-1} \psi_{\mathsf{1gc}} \right) \right\rangle \right)$$

$$= \; \epsilon \int_z e\, \overline{F} \; \langle \, \mathsf{C}_{\mathsf{gy}}\, \psi_{\mathsf{1gc}} \rangle$$

$$\equiv \; -\, \mathcal{P}_\phi \;-\; \mathcal{P}_A \;\; \Rightarrow \;\; \frac{dE_{\mathsf{gy}}}{dt} \;\equiv\; 0$$

- Gyrokinetic energy transfer processes

$$\boxed{\mathcal{P}_\phi \;\Leftrightarrow\; \mathcal{P}_H \;\Leftrightarrow\; \mathcal{P}_A}$$

b. Truncated Gyrokinetic Vlasov-Maxwell Equations

- **Truncated gyrokinetic Lagrangian Density**

$$\mathcal{L}_{\rm tgy} \;\equiv\; \frac{1}{8\pi}\left(\epsilon^2\,|\nabla\phi_1|^2 \;-\; |\mathbf{B}|^2\right) \;+\; \frac{\epsilon^2}{2}\int e\,\overline{F}_0\;\langle \pounds_{\rm gy}\,\psi_{\rm 1gc}\rangle\;d^4\overline{p}$$
$$-\int \overline{\mathcal{F}}\left(H_{\rm gc} \;+\; \epsilon\,e\,\langle\psi_{\rm 1gc}\rangle \;-\; \overline{w}\right)\,d^4\overline{p}$$

◦ Truncated gyrocenter Vlasov distribution

$$\overline{\mathcal{F}} \;\equiv\; (\overline{F}_0 + \epsilon\,\overline{F}_1)\;c\,\delta\left(\overline{w} - H_{\rm gc} - \epsilon\,e\,\langle\psi_{\rm 1gc}\rangle\right)$$

- **Truncated gyrokinetic (delta-f) Vlasov equation**

$$\boxed{0 \;\equiv\; \epsilon\,\frac{d_{\rm gc}\overline{F}_1}{dt} \;+\; \epsilon\left\{\left(\overline{F}_0 \;+\; \epsilon\,\overline{F}_1\right),\; e\,\langle\psi_{\rm 1gc}\rangle\right\}_{\rm gc}}$$

Important numerical applications in gyrokinetic codes

- **Truncated gyrokinetic Maxwell equations**

◦ Truncated gyrokinetic Poisson equation

$$\epsilon\,\nabla^2\phi_1 \;=\; -\,4\pi \int_z e \left[\left(\overline{F}_0 + \epsilon\,\overline{F}_1\right) \left\langle \delta^3_{\rm gc} \right\rangle \;-\; \epsilon\,\overline{F}_0 \left\langle \pounds_{\rm gy}\delta^3_{\rm gc} \right\rangle \right]$$

◦ Truncated gyrokinetic Ampère equation

$$\nabla\times\mathbf{B} \;=\; 4\pi \int_z e \left[\left(\overline{F}_0 + \epsilon\,\overline{F}_1\right) \left\langle \frac{\mathbf{v}_{\rm gc}}{c}\,\delta^3_{\rm gc} \right\rangle \right.$$

$$\left. -\;\epsilon\,\overline{F}_0 \left\langle \pounds_{\rm gy} \left(\frac{\mathbf{v}_{\rm gc}}{c}\,\delta^3_{\rm gc} \right) \right\rangle \right]$$

◦ **Polarization and magnetization:**

$$\left\langle \frac{v^\alpha_{\rm gc}}{c}\,\delta^3_{\rm gc} \right\rangle \qquad \& \qquad \left\langle \pounds_{\rm gy} \left(\frac{v^\alpha_{\rm gc}}{c}\,\delta^3_{\rm gc} \right) \right\rangle$$

Guiding-center **Gyrocenter**

- **Exact Truncated Gyrokinetic Energy Conservation**

$$E_{\mathrm{tgy}} \;\equiv\; \int_z \left[\left(\overline{F}_0 + \epsilon\,\overline{F}_1\right) \left(H \;-\; \epsilon\, e\, \langle \phi_{1\mathrm{gc}} \rangle\right) \;+\; \epsilon^2\, \overline{F}_0\, K_{2\mathrm{gy}} \right] \;+\; \int \frac{d^3x}{8\pi} \left(\epsilon^2\, |\nabla\phi_1|^2 \;+\; |\mathbf{B}|^2\right)$$

∘ Ponderomotive gyrocenter kinetic energy

$$K_{2\mathrm{gy}} \;\equiv\; H_{2\mathrm{gy}} + e \left\langle \left\{ S_1,\, \phi_{1\mathrm{gc}} \right\}_{\mathrm{gc}} \right\rangle \;=\; e\, \left\langle \pounds_{\mathrm{gy}} \left(\phi_{1\mathrm{gc}} \;-\; \frac{1}{2}\psi_{1\mathrm{gc}} \right) \right\rangle$$

- Energy exchange terms

$$\boxed{\frac{dE_{\mathrm{tgy}}}{dt} \equiv \mathcal{P}_{tF} + \mathcal{P}_{tH} + \mathcal{P}_{t\phi} + \mathcal{P}_{tA} \equiv 0}$$

- Vlasov term

$$\mathcal{P}_{tF} \equiv \epsilon \int_z \frac{\partial \overline{F}_1}{\partial t} \overline{H}_{\mathrm{gy}} \equiv 0$$

- Poisson term

$$\mathcal{P}_{t\phi} \equiv \epsilon^2 \int_z e\, \overline{F}_0 \left\langle \left[\pounds_{\mathrm{gy}},\, \frac{\partial}{\partial t} \right] \phi_{1\mathrm{gc}} \right\rangle$$

- Ampère term

$$\mathcal{P}_{tA} \equiv -\,\epsilon^2 \int_z e\, \overline{F}_0 \left\langle \pounds_{\mathrm{gy}} \left(\frac{\partial \mathbf{A}_{1\mathrm{gc}}}{\partial t} \cdot \frac{\mathbf{v}_0}{c} \right) \right\rangle$$

- Hamiltonian term

$$\mathcal{P}_{tH} \equiv \epsilon^2 \int_z \overline{F}_0 \frac{\partial K_{2\mathrm{gy}}}{\partial t} \equiv -\,\mathcal{P}_{t\phi} - \mathcal{P}_{tA}$$

Summary

- **Dynamical reduction** introduces **ponderomotive** effects into reduced Hamiltonian dynamics.
- **Dynamical reduction** introduces reduced **polarization** and **magnetization** effects in reduced Maxwell equations.
- **Dynamical reduction** introduced within variational formulation yields **exact** energy conservation laws.
- **Exact gyrofluid** and **gyrokinetic** (with **Tronko**) **momentum** conservation laws for can be similarly derived by **Noether method**.

Recent Review Papers

Theory: Brizard & Hahm
Foundations of nonlinear gyrokinetic theory
Reviews of Modern Physics **79**, 412 (2007).

Simulations: Garbet, Idomura, Villard, & Watanabe
Gyrokinetic simulations of turbulent transport
Nuclear Fusion **50**, 043002 (2010).

Guiding-center: Cary & Brizard
Hamiltonian theory of guiding-center motion
Reviews of Modern Physics **81**, 693 (2009).

Upcoming Textbook

Brizard, *Fundamental Principles of Gyrokinetic Theory*
(World Scientific, 201**x**)