The spectra of a general class of stochastic climate models

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Relevant Reference Material

R. Kleeman. Spectral analysis of multi-dimensional stochastic geophysical models with an application to decadal ENSO variability. J. Atmos. Sci., 2010. submitted.

R. Kleeman. Stochastic theories for the irregularity of ENSO. Phil. Trans. Roy. Soc. A., 366:2509-2524, 2008.

Stochastic Modeling and Occams Razor

Three principles for influential models

I. Model should be based solidly on the dominant physical mechanisms operating within the system under consideration.

II. Model should demonstrate robust qualitative agreement with observations of the dominant form of variability being considered.

III. Model should be as simple as possible consistent with the first two principles.

The simplest stochastic models are arguably the class of linear systems with additive stochastic forcing referred to generically as (multivariate) Ornstein Uhlenbeck processes. Do such models meet the test of the first two principles?

Sometimes........

Hasselmann Climate Model

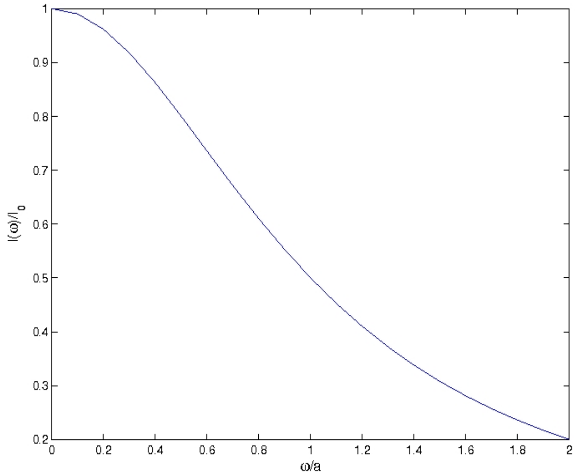
Consider the mixed layer of the ocean where temperature change is dominated by heat flux. If we linearize the bulk relations for heat flux we obtain the SST anomaly equation:

$$T_t^{'} = -kT' + rW'$$

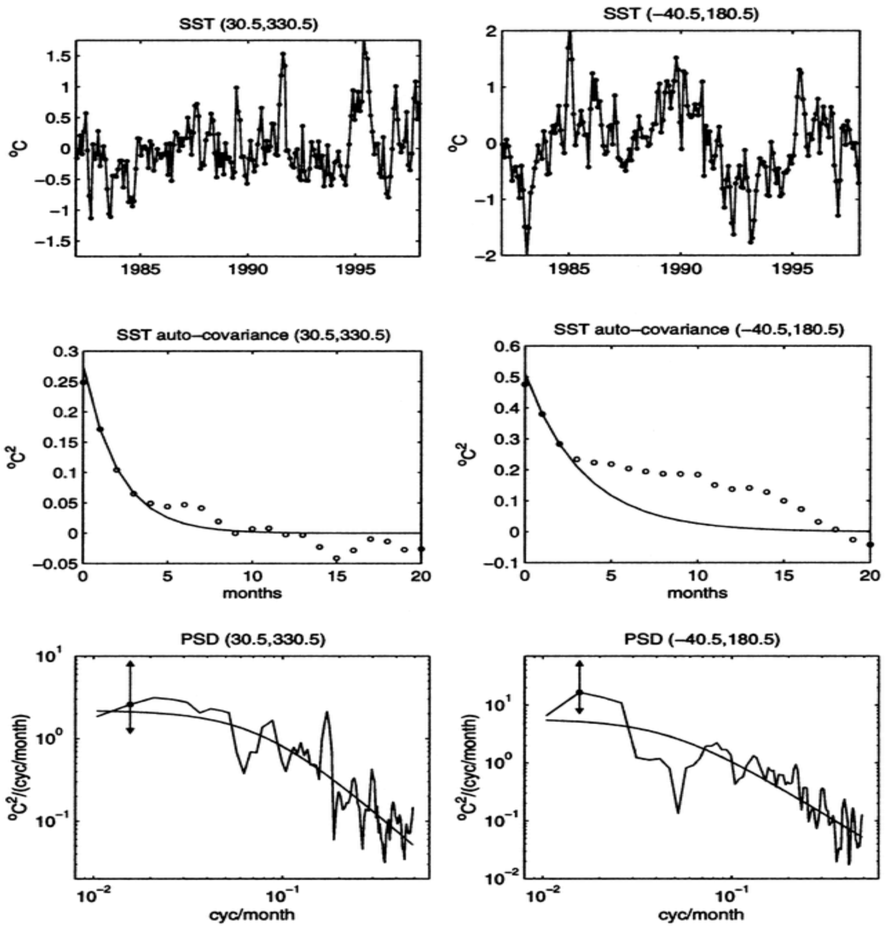
Heatflux acts as a negative SST feedback (first term) and windspeed fluctuations act as a forcing which is stochastic in character on long time scales. This is the simplest OU process and it has a spectrum given by

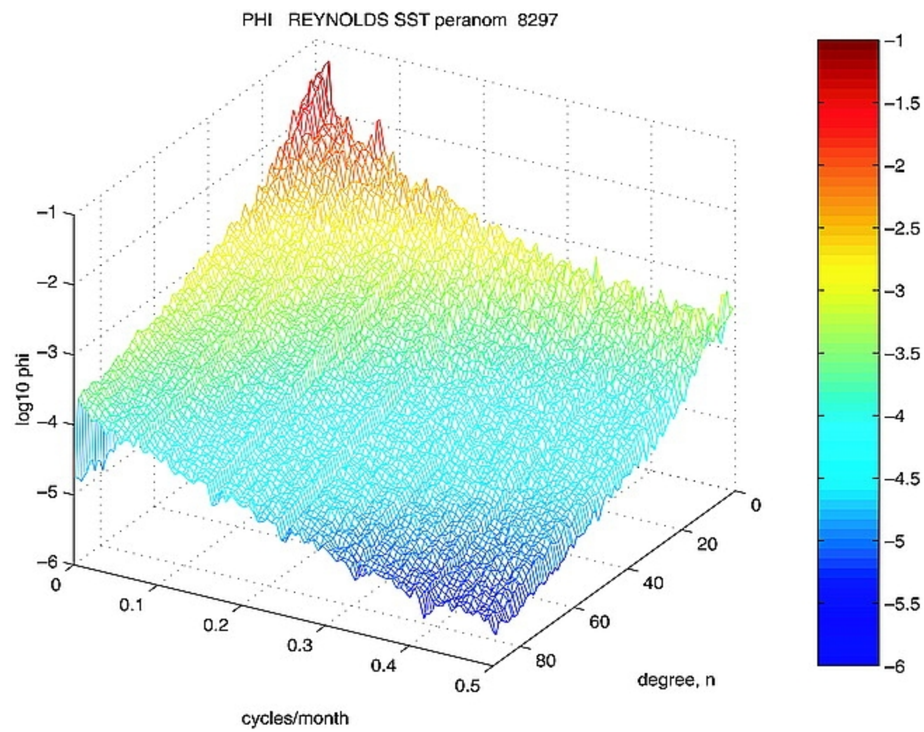
$$S(\omega) = S_0 \frac{k}{k^2 + \omega^2}$$

This functional form is called a Lorentzian curve and the positive frequency piece looks like



Hasselmann Climate Model





How good is this model? Both for single locations (left) and globally (right) it works rather well as a first order explanation of observed data. The best fit spectrum is obtained with realistic values of the decay parameter k.

Thus this model appears to meet all three principles for stochastic modeling.

Simple ENSO model

ENSO is a coupled ocean atmosphere phenomenon. Linear instability analysis (Jin and Neelin (1993) and Moore et. al. (2003)) suggest that with respect to damping time, the modal spectrum is strongly peaked. Observations of ENSO EOF variability supports such a view as only two spatial patterns are required to explain most variability and moreover such EOFs correspond well with a model normal mode (the first two EOFs of heat content are a Principal Oscillation Pattern pair). Give this, a minimal model of ENSO is a stochastically forced damped oscillator as such an oscillator represents the dominant normal mode. Many studies have shown that the stochastic forcing is provided by convective atmospheric transients. The equation for this minimal model can be written as

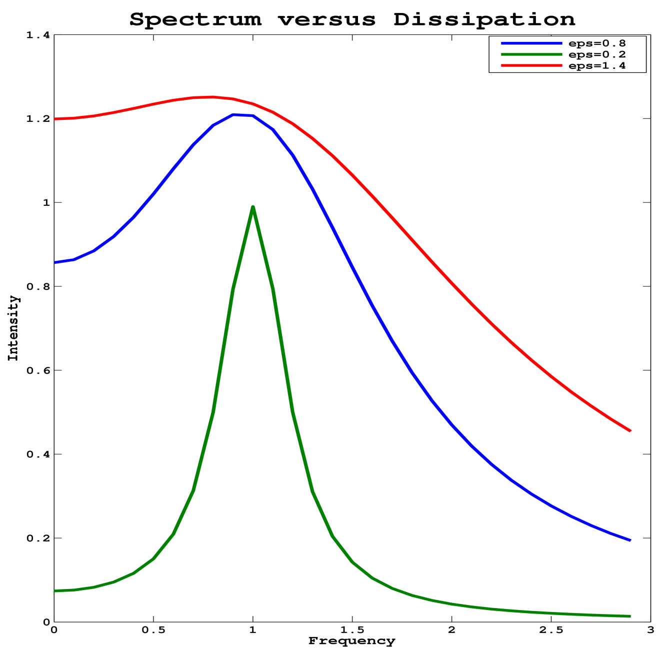
$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t = \begin{pmatrix} -\epsilon & \eta \\ -\eta & -\epsilon \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$

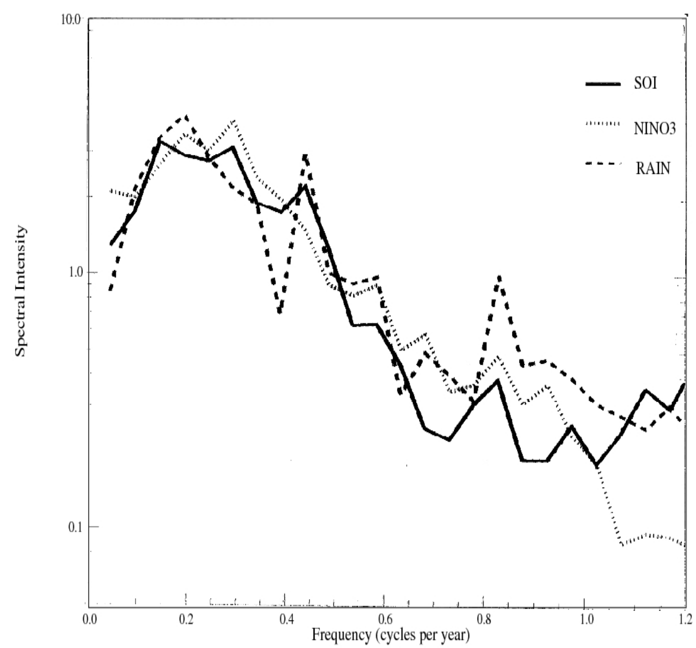
Where there are two natural time scales: The damping time and the oscillation period which are related to the system by

$$\epsilon = \frac{1}{\tau} \qquad \eta = \frac{2\pi}{T}$$

Simple ENSO model

How good is this model? The spectrum can be calculated exactly and for reasonable assumptions for the stochastic forcing (see later) it has the form shown on the left. Shown are spectra for a variety of decay times. The observed spectrum of the peak phase of ENSO can be obtained from existing datasets extending back some 130 years. The estimates on the right are from NINO3 SST; SOI and Pacific rain gauge data.

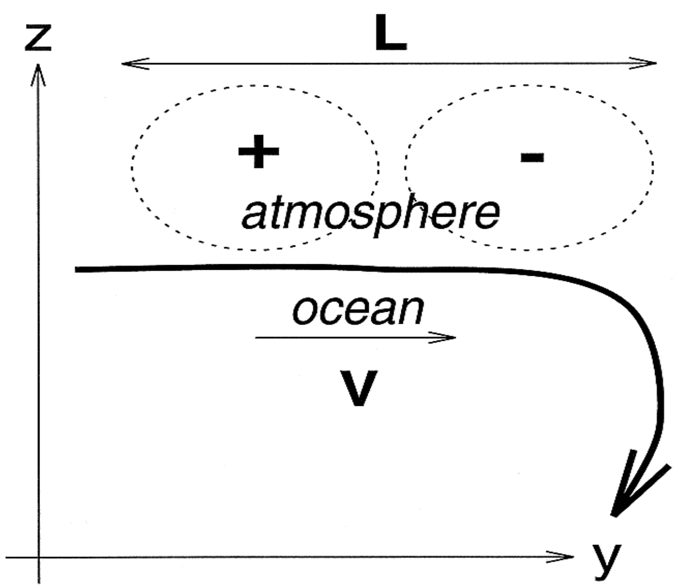


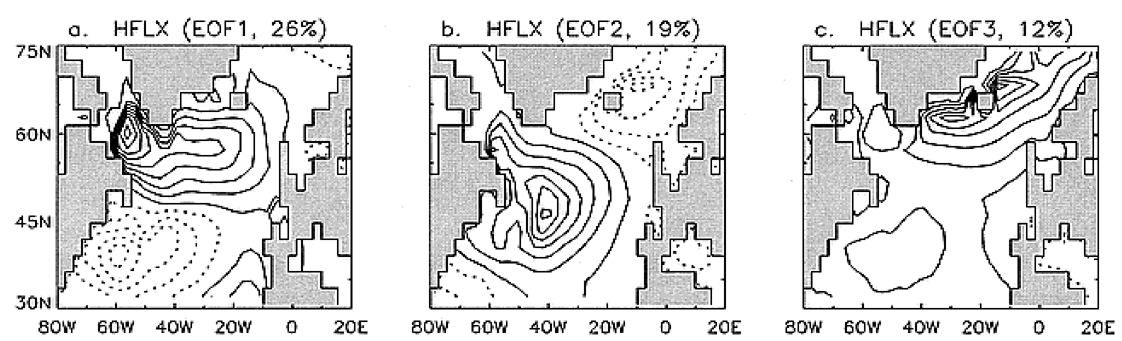


Again a reasonable qualitative fit is seen. Best values for the damping time are perhaps around one year. The ENSO spectral peak is at around 4 years.

Decadal variability model

In certain mid-latitude regions a spectral peak at decadal time scales is evident. The study of such variability is almost always hampered by insufficiently long time series however Saravanan and McWilliams (1997) suggest a very simple stochastic model. This consists of a simple model of the North Atlantic/Pacific in which ocean temperature is influenced deterministically by advection by western boundfary currents (Gulf stream and Kuroshio) and stochastically by heatflux in the manner of Hasselmann. The latter tends to be dominated by a few large scale patterns (EOFs). This model can be mathematically transformed easily to the the same one considered in the ENSO case with the stochastic forcing components corresponding with heat flux EOFs. The authors assumed only the lead EOF was important so only one stochastic component is non-zero.





A general multivariate Ornstein Uhlenbeck stochastic process

The success of the previous three models suggests a study of the spectra of general linear systems with additive stochastic forcing. In such a study it would be useful to explicitly see the importance of both the (linear) dynamics as well as the statistics of the stochastic forcing. Consider the (Langevin) system

$$\mathbf{x}_t = -\boldsymbol{A}\mathbf{x} + \mathbf{F}$$

Where the matrix **A** is constant and has eigenvalues whose real part is positive in order to ensure an equilibrium solution. The vector stocastic forcing **F** is assumed to be white but has general statistics between the components. It is a standard result in the theory of stochastic processes that the spectral matrix of this system is

$$\mathbf{S}(\omega) = \frac{1}{2\pi}\left(\mathbf{A} + i\omega I\right)^{-1}\mathbf{R}\left(\mathbf{A}^* - i\omega I\right)^{-1}$$

Where **R** is the covariance matrix for the stochastic forcing **F**.

A general multivariate Ornstein Uhlenbeck stochastic process

Suppose the spectrum of a linear combination y of the components of **x** is desired. Then we have

$$y = \sum_{i=1}^{n} c_i x_i \qquad S_y(\omega) = \sum_{i,j=1}^{n} c_i \boldsymbol{S}(\omega) c_j^*$$

The spectral matrix expression above is most conveniently solved for in the normal mode basis for the matrix **A** because the inverse matrices in the expression above are diagonal. In such a basis the spectrum of a general linear combination of normal modes is given by

$$S_y(\omega) = \sum_{k,l} \frac{t_k}{q_k + i\omega} R^N_{kl} \frac{t_l^*}{q_l^* - i\omega}$$

Where the complex eigenvalues of **A** are

$$q_l \equiv \epsilon_l + i\eta_l$$

A general multivariate Ornstein Uhlenbeck stochastic process

The stochastic forcing covariance matrix entries in the normal mode basis are R^N_{kl}

While the components of the variable y whose spectrum we desire is given in the normal mode basis by the components t_l

It is easily seen that the spectrum of an individual normal mode is given by

$$S_i(\omega) = \frac{R^N_{ii}}{(\eta_i + \omega)^2 + \epsilon_i^2}$$

Which is a Lorentzian curve centered on (minus) the normal mode frequency. The spread and strength of this spectrum are controlled by the normal mode damping as well as the variance of the stochastic forcing of the particular normal mode. In the 1D case (Hasselmann model) clearly the frequency of the normal mode is zero so we get the solution discussed earlier. We consider the 2D case later.....

If we are interested (as we usually are) in linear combinations of normal modes then the spectrum is not simply the sum of Lorentzians for each normal mode but also includes cross terms from the above sum which are commonly called the cross spectra.

Cross Spectra

By straightforward complex arithmetic manipulation involving polar decomposition we can write the (two) cross terms between two normal modes labelled by k and l as

$$2\sqrt{S_k(\omega)S_l(\omega)}f_{kl}\cos\left(\theta_{kl}(\omega)\right)$$

Where the so called Wiener coherence is given by in this case (OU stochastic process) by

$$f_{kl} = \frac{\left|R_{kl}^N\right|}{\sqrt{R_{kk}^N R_{ll}^N}}$$

Notice that the Wiener coherence always lies between 0 and 1 and is determined only by the stochastic forcing statistics and is not frequency dependent. It is also obvious that the cross spectrum between any two normal modes is always less than or equal to the sum of the two diagnonal (Lorentzian) spectra with equality only possible when the Wiener coherence reaches 1.

The remaining functional form θ_{kl} is called the *phase* spectrum. It is more conveniently discussed in the context of an application.

ENSO Application

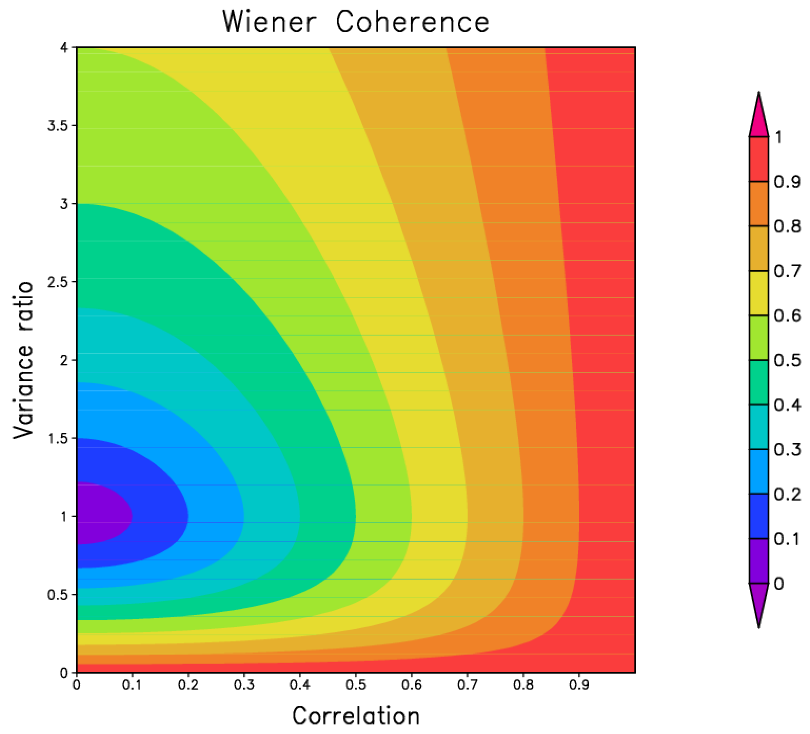
A 2D OU process with a real **A** can always be brought into the matrix form discussed previously for ENSO by a suitable basis change. The eigenvalues are complex conjugates of each other which implies that the diagonal (Lorentzian) parts of the spectrum are centered on $\pm\eta$ the oscillator frequency. Such Lorentzians reinforce each other in the region between peaks as can be seen in the theoretical spectra shown earlier (and also in the observations).

What happens to the cross spectrum? Let us assume that the covariance matrix for the ENSO stochastic forcing is of the form:

$$\boldsymbol{R} = c\begin{pmatrix} 1 & r\sqrt{\alpha} \\ r\sqrt{\alpha} & \alpha \end{pmatrix}$$

The two components in this model represent the orthogonal phases of ENSO (the first two EOFs of heat content) so this matrix specifies the ratio of the stochastic forcing onto the two phases as well as their correlation (r). Given this it is a straightforward calculation to determine the Wiener coherency.

ENSO Application



So it is clear that the cross spectrum only vanishes when the correlation between the stochastic forcing onto the orthogonal ENSO phases vanishes. Otherwise we need to consider how it modifies the Lorentzian spectrum.

ENSO Application

The phase spectrum may be decomposed for this problem into three parts. The first depends on the phase of ENSO whose spectrum we are calculating. The second depends only on the stochastic forcing statistics while the third part depends only on the normal mode eigenvalue and frequency.

$$\theta(\omega) = 2\theta^1 + \theta^2 + \theta^3_+(\omega) - \theta^3_-(\omega)$$

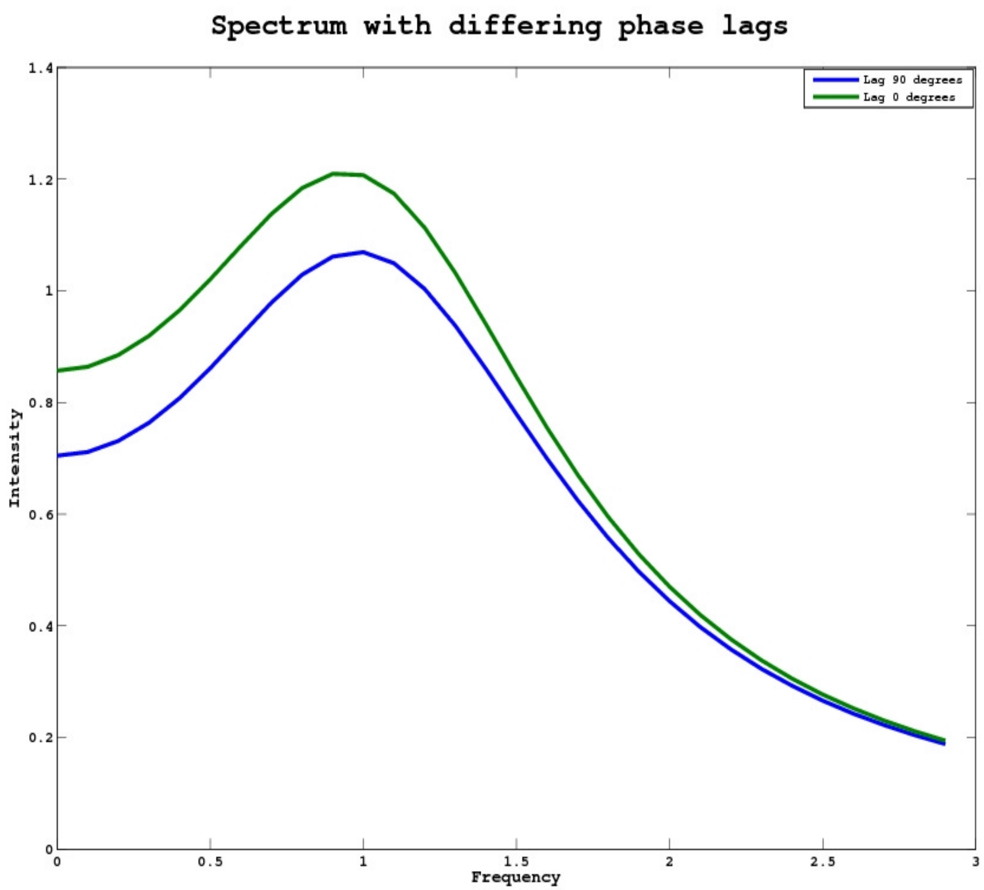
$$\cos\theta^2 = \frac{1-\alpha}{\sqrt{(1-\alpha)^2 + 4r^2\alpha}}$$

$$\cos\theta^3_\pm(\omega) = \frac{\epsilon}{\sqrt{\epsilon^2 + (\omega \pm \eta)^2}}$$

Note that the first two phases angles do not depend on frequency so can be regarded as offset angles.

Note also that the first phase angle θ^1 which is the ENSO phase under consideration, causes the cross spectrum to change sign as it varies from 0 to pi/2. Thus it follows that phases of ENSO separated by pi/2 have opposite signed cross spectra.

ENSO Application



If we compare the spectra at different ENSO phases the biggest difference occurs, as noted, when the phase difference is pi/2 but also usually occurs for frequencies less that the peak ENSO frequency i.e. in the decadal and longer range. Shown is a case where the Wiener coherency is set to 0.1 (realistic?) and the dissipation time scale is somewhat less than a year.

Thus if this (very simple) model is a correct explanation for ENSO low frequency variability, we would expect such variability to be phase dependent and thus exhibit a characteristic spatial structure. This is indeed observed.

Conclusions

- Linear stochastic models with additive stochastic forcing are able to provide a physically plausible explanation for observed climate spectra. Their appeal also lies in their simplicity (Occams Razor).
- A general formalism is derived for analysing the spectra for such models. Results are related to widely known concepts from linear resonance and multivariate time series analysis.
- An application to the ENSO stochastic model shows that the cross spectrum can provide an explanation for decadal variability of this phenomena. This explanation predicts that the variability should have a characteristic spatial structure associated with a particular phase of ENSO.