Deformation Theory for Vector Bundles

Nitin Nitsure

Abstract

These expository notes give an introduction to the elements of deformation theory which is meant for graduate students interested in the theory of vector bundles and their moduli. The original version appeared in the volume ‘Moduli spaces and vector bundles’ LMS lecture note series 359 (2009) in honour of Peter Newstead.

Contents

1 Introduction : Basic examples

For simplicity, we will work over a fixed base field k which may be assumed to be algebraically closed. All schemes and all morphisms between them will be assumed to be over the base k, unless otherwise indicated. In this section we introduce four examples which are of basic importance in deformation theory, with special emphasis on vector bundles.

Basic example 1: Deformations of a point on a scheme

We begin by setting up some notation. Let $\mathbf{Art}_k$ be the category of all artin local k-algebras, with residue field k. In other words, the objects of $\mathbf{Art}_k$ are local k-algebras with residue field k which are finite-dimensional as k-vector spaces, and morphisms are all k-algebra homomorphisms. Note that k is both an initial and a final object of $\mathbf{Art}_k$. By a **deformation functor** we will mean a covariant functor $F : \mathbf{Art}_k \to \mathbf{Sets}$ for which $F(k)$ is a singleton point. As k is an initial object of $\mathbf{Art}_k$, this condition means that we can as well regard F to be a functor to the category of pointed sets.

For any A in $\mathbf{Art}_k$, let $h_A : \mathbf{Art}_k \to \mathbf{Sets}$ be the deformation functor defined by taking $h_A(B) = Hom_{k\text{-alg}}(A, B)$. Recall the well-known Yoneda lemma, which asserts that there is a natural bijection $Hom(h_A, F) \to F(A)$ under which a natural transformation $\alpha : h_A \to F$ is identified with the element $\alpha(id_A) \in F(A)$. To simplify notation, given any natural transformation $\alpha : h_A \to F$, we denote again by $\alpha \in F(A)$ the element $\alpha(id_A) \in F(A)$. Any element $\alpha \in F(A)$ will be called a **family** parametrised by A (the reason for this nomenclature will be clear from the examples). Given $f : B \to A$ and $\beta \in F(B)$, we denote the family $F(f)\beta \in F(A)$ simply by $\beta|_A$, when f is understood.

Let $\widehat{\mathbf{Art}}_k$ denote the category of complete local noetherian k-algebras with residue field k as objects and all k-algebra homomorphisms as morphisms. Given any R

in $\widehat{\mathbf{Art}_k}$, we denote by $h_R : \mathbf{Art}_k \to \mathbf{Sets}$ the deformation functor defined by taking $h_R(A) = Hom_{k\text{-alg}}(R, A)$. A deformation functor F will be called **pro-representable** if there exists a natural isomorphism $r : h_R \to F$ where R is in $\widehat{\mathbf{Art}_k}$. The pair (R, r) will be called a **universal pro-family** for F. (To understand this name, see Lemma 2.7.)

If X is any scheme over k and $x \in X$ a k-rational point, we define a deformation functor $h_{X,x}$ by taking for any A in $\mathbf{Art}_k$ the set $h_{X,x}(A)$ to be the set of all morphisms $\mathrm{Spec}\, A \to X$ over k, for which the closed point of $\mathrm{Spec}\, A$ maps to x. Any such morphism is the same as a k-algebra homomorphism $\mathcal{O}_{X,x} \to A$. As the maximal ideal of A is nilpotent, such homomorphisms are in a natural bijection with k-algebra homomorphisms $R \to A$ where R denotes the completion of the local ring $\mathcal{O}_{X,x}$ at its maximal ideal. This shows that the above functor $h_{X,x}$ is naturally isomorphic to h_R, hence is pro-representable.

The functor $h_{X,x}$ is the ultimate example of a deformation functor in the sense that it is the simplest and has the best possible properties. One of the aims of the general theory is to determine when a given deformation functor is of this kind, at least, to determine whether it shares some nice properties with $h_{X,x}$.

Basic example 2: Deformations of a coherent sheaf

Let X be a proper scheme over a field k, and let E be a coherent sheaf of $\mathcal{O}_X$-modules. The **deformation functor $\mathcal{D}_E$ of** E is defined as follows. For any A in $\mathbf{Art}_k$, we take $\mathcal{D}_E(A)$ to be the set of all equivalence classes of pairs $(\mathcal{F}, \theta)$ where $\mathcal{F}$ is a coherent sheaf on $X_A = X \otimes_k A$ which is flat over A, and $\theta : i^*\mathcal{F} \to E$ is an isomorphism where $i : X \hookrightarrow X_A$ is the closed embedding induced by the residue map $A \to k$, with $(\mathcal{F}, \theta)$ and $(\mathcal{F}', \theta')$ to be regarded as equivalent when there exists some isomorphism $\eta : \mathcal{F} \to \mathcal{F}'$ such that $\theta' \circ (\eta|_X) = \theta$. It can be seen that $\mathcal{D}_E(A)$ is indeed a set. Given any homomorphism $f : B \to A$ in $\mathbf{Art}_k$ and an equivalence class $(\mathcal{F}, \theta)$ in $\mathcal{D}_E(B)$, we define $f(\mathcal{F}, \theta)$ in $\mathcal{D}_E(A)$ to be its pull-back under the induced morphism $X_A \to X_B$ (by applying $- \otimes_B A$). This preserves equivalences, and so we get a functor $\mathcal{D}_E : \mathbf{Art}_k \to \mathbf{Sets}$.

If we assume that E is a vector bundle (that is, locally free), then flatness of $\mathcal{F}$ over A just amounts to assuming that $\mathcal{F}$ is a vector bundle on X_A.

Basic example 3: Deformations of a quotient

Let X be a proper scheme over k, E be a coherent $\mathcal{O}_X$-module over X, and $q_0 : E \to F_0$ be a coherent quotient $\mathcal{O}_X$-module. For any A in $\mathbf{Art}_k$, let E_A denote the pull-back of E to $X_A = X \otimes_k A$. Let $i : X \hookrightarrow X_A$ be the inclusion of the special fiber of X_A. We consider all $\mathcal{O}_{X_A}$-linear surjections $q : E_A \to \mathcal{F}$ such that $\mathcal{F}$ is flat over A and the kernel of $q|_X : E \to \mathcal{F}|_X$ equals $\ker(q_0)$. For any such, there exists a unique isomorphism $\theta : i^*\mathcal{F} \to F_0$ such that the following square commutes.

$$\begin{array}{ccc} i^*E_A & = & E \\ {\scriptstyle i^*q}\downarrow & & \downarrow{\scriptstyle q_0} \\ i^*\mathcal{F} & \xrightarrow{\theta} & F_0 \end{array}$$

Two such surjections $q : E_A \to \mathcal{F}$ and $q' : E_A \to \mathcal{F}'$ will be called equivalent if $\ker(q) = \ker(q')$. For any object A of $\mathbf{Art}_k$, let $Q(A)$ be the set of all equivalence

classes of such $q : E_A \to \mathcal{F}$ (it can be seen that $Q(A)$ is indeed a set). For any morphism $B \to A$ in $\mathbf{Art}_k$, we get by pull-back (by applying $- \otimes_B A$) a well-defined set map $Q(B) \to Q(A)$, so we have a deformation functor $Q : \mathbf{Art}_k \to \mathbf{Sets}$ (note that $Q(k)$ is clearly a singleton). Sending $(q : E_A \to \mathcal{F}) \mapsto (\mathcal{F}, \theta)$, where $\theta : i^*\mathcal{F} \to F_0$ is defined as above, defines a natural transformation $Q \to \mathcal{D}_{F_0}$.
In the special case when $E = \mathcal{O}_X$, a coherent quotient $q_0 : E \to F_0$ is the same as a closed subscheme $Y_0 \subset X$, and the functor Q becomes the functor of its proper flat deformations of Y_0 inside X.
If X is projective over X then by a fundamental theorem of Grothendieck there exists a k-scheme $Z = Quot_{E/X}$ (called the quot scheme of E or the Hilbert scheme of X when $E = \mathcal{O}_X$), and q_0 corresponds to a k-valued point z on Z. The functor Q in this case becomes just $h_{Z,z}$, the deformation functor of a point on a scheme which was our basic example 1 introduced above. However, even in the projective case, it is useful to study the deformation theory of Q from a general functorial point of view, as we will do later.

Basic example 4: Deformations of a scheme

Given a scheme X of finite type over a field k, let the deformation functor $\mathbf{Def}_X : \mathbf{Art}_k \to \mathbf{Sets}$ be defined as follows. For any $A \in \mathbf{Art}_k$, consider pairs $(p : \mathfrak{X} \to \operatorname{Spec} A, i : X \to X_0)$ where p is a flat morphism of k-schemes, $X_0 = p^{-1}(\operatorname{Spec} k) = \mathfrak{X}|_{\operatorname{Spec} k}$ is the schematic special fibre of p, and i is an isomorphism. Denoting again by i the composite $X \to X_0 \hookrightarrow \mathfrak{X}$, this means that the following square is cartesian.

$$\begin{array}{ccc} X & \stackrel{i}{\to} & \mathfrak{X} \\ \downarrow & & \downarrow {\scriptstyle p} \\ \operatorname{Spec} k & \to & \operatorname{Spec} A \end{array}$$

We say that two such pairs (p, i) and (p', i') are equivalent if there exists an A-isomorphism between $\mathfrak{X}$ and $\mathfrak{X}'$ which takes i to i'. We take $\mathbf{Def}_X(A)$ to be the set of all equivalence classes of pairs (p, i). It can be seen that this is indeed a set, and moreover it is clear that a morphism $A \to B$ in $\mathbf{Art}_k$ gives by pull-back a well-defined set map $\mathbf{Def}_X(A) \to \mathbf{Def}_X(B)$ which indeed gives a functor $\mathbf{Def}_X : \mathbf{Art}_k \to \mathbf{Sets}$.
The above example and its variants and special cases quickly bring out all the possible complications in deformation theory, and have historically led to its major developments. In these notes, which are designed to be a short introduction aimed at graduate students interested in vector bundles, we will not treat this example in any detail, but will just mention some basic results.

Relation with moduli functors

The theory of moduli may suggest that rather than the deformation functor $\mathcal{D}_E$ of basic example 2, we should consider the functor $\mathcal{M}_E$ defined as follows. For any A in $\mathbf{Art}_k$, we take $\mathcal{M}_E(A)$ to be the set of isomorphism classes $[\mathcal{F}]$ of coherent sheaves $\mathcal{F}$ on X_A that are flat over A, such that the restriction $\mathcal{F}|_X$ to the special fiber of $\operatorname{Spec} A$ is *isomorphic* to E. Note that this does not involve a choice of a specific isomorphism $\theta : \mathcal{F}|_X \to E$, so this functor differs from the deformation functor $\mathcal{D}_E$.

Clearly, there is a natural transformation $\mathcal{D}_E \to \mathcal{M}_E$, which forgets the choice of θ. If the sheaf E has the special property that for each A and each $(\mathcal{F}, \theta)$ in $\mathcal{D}_E(A)$ every automorphism of E is the restriction of an automorphism of $\mathcal{F}$, then the natural transformation $\mathcal{D}_E \to \mathcal{M}_E$ is an isomorphism. For example, this may happen when E is stable in a certain sense, where the stability condition ensures that all automorphisms of E are just scalars. If moreover a fine moduli scheme M for stable sheaves exists, with E defining a point $[E] \in M$, then $\mathcal{M}_E$ is the same as the corresponding local moduli functor $h_{M,[E]}$ which is a case of the deformation functor of our basic example 1. Hence in this case, $\mathcal{D}_E$ will just be $h_{M,[E]}$, and the study of $\mathcal{D}_E$ will shed light on the local structure of M around $[E]$.
But even when the above condition (that automorphisms of E must be extendable to any infinitesimal family $\mathcal{F}$ around it) is not fulfilled, the study of the deformation functor $\mathcal{D}_E$ continues to be of importance, for it sheds light on the local structure of the corresponding moduli stacks. On the other hand, the functor $\mathcal{M}_E$, which may at first sight look more natural than $\mathcal{D}_E$, does not have good properties in general. The Example 3.9 illustrates this point.
Similar remarks apply to the functor $\mathbf{Def}_X$ of basic example 4 vis à vis the corresponding local moduli functor.

2 General theory

Tangent space to a functor

2.1 Let $\mathbf{Vect}_k$ be the category of all vector spaces over k, and let $\mathbf{FinVect}_k$ be its full subcategory consisting of all finite dimensional vector spaces. Let $\varphi : \mathbf{FinVect}_k \to \mathbf{Sets}$ be a functor into the category of sets which satisfies the following:
(T0) For the zero vector space 0, the set $\varphi(0)$ is a singleton set.
(T1) The natural map $\beta_{V,W} : \varphi(V \times W) \to \varphi(V) \times \varphi(W)$ induced by applying φ to the projections $V \times W \to V$ and $V \times W \to W$ is bijective.
Then for each V in $\mathbf{FinVect}_k$, there exists a unique structure of a k vector space on the set $\varphi(V)$ which gives a lift of φ to a k-linear functor which we again denote by $\varphi : \mathbf{FinVect}_k \to \mathbf{Vect}_k$. The addition map $\varphi(V) \times \varphi(V) \to \varphi(V)$ is the composite

$$\varphi(V) \times \varphi(V) \stackrel{\beta_{V,V}^{-1}}{\to} \varphi(V \times V) \stackrel{\varphi(+)}{\to} \varphi(V)$$

where $\beta_{V,V}^{-1}$ is the inverse of the natural isomorphism given by the assumption on φ, and $+ : V \times V \to V$ is the addition map of V. Also, for any $\lambda \in k$, the scalar multiplication map $\lambda_{\varphi(V)} : \varphi(V) \to \varphi(V)$ is just $\varphi(\lambda_V)$.

2.2 The **tangent space** T_φ to a functor $\varphi : \mathbf{FinVect}_k \to \mathbf{Sets}$ which satisfies **(T0)** and **(T1)** is defined to be the vector space $\varphi(k)$. This may not necessarily be finite dimensional. We now show that there exists a linear isomorphism

$$\Psi_{\varphi,V} : \varphi(V) \to T_\varphi \otimes_k V$$

which is functorial in both V and φ. For this, choose an isomorphism $g : V \to k^n$, and apply **(T1)** repeatedly to get a composite isomorphism

$$\varphi(V) \overset{\varphi(g)}{\to} \varphi(k^n) \overset{\beta}{\to} \varphi(k)^n = \varphi(k) \otimes k^n \overset{id\otimes\varphi(g)^{-1}}{\to} \varphi(k) \otimes V = T_\varphi \otimes V$$

which can be verified to be independent of the choice of g.
Thus, a functor φ : $\mathbf{FinVect}_k \to \mathbf{Sets}$ satisfying **(T0)** and **(T1)** is completely described by its tangent space T_φ. Conversely, given any vector space T, the association $V \mapsto T \otimes V$ defines a functor φ that satisfies **(T0)** and **(T1)**, for which T_φ is just T.

Artin local algebras

If A_1 and A_2 are local k-algebras with residue field k, that is, the composite $k \to A_i \to A_i/\mathfrak{m}_i$ is an isomorphism where $\mathfrak{m}_i \subset A_i$ is its maximal ideal, then any k-algebra homomorphism $f : A_1 \to A_2$ is necessarily *local*, that is, $f^{-1}(\mathfrak{m}_2) = \mathfrak{m}_1$.
If $f : B \to A$ and $g : C \to A$ are homomorphisms in $\mathbf{Art}_k$, the fibred product

$$B \times_A C = \{(b, c)|f(b) = g(c) \in A\}$$

with component-wise operations is again an object in $\mathbf{Art}_k$ (Exercise). Also, for homomorphisms $A \to B$ and $A \to C$ in $\mathbf{Art}_k$, the tensor product $B \otimes_A C$ is again an object in $\mathbf{Art}_k$ (Exercise). Thus, $\mathbf{Art}_k$ admits both fibred products (pull-backs) $B \times_A C$ and tensor products (push-outs) $B \otimes_A C$.
As k is the final object in $\mathbf{Art}_k$, the fibred product $A \times_k B$ serves as the direct product in the category $\mathbf{Art}_k$, and as k is the initial object in $\mathbf{Art}_k$, the tensor product $B \otimes_A C$ serves as the Co-product in the category $\mathbf{Art}_k$.
For a k-vector space V, let $k\langle V\rangle = k \oplus V$ with ring multiplication defined by putting $(a, v)(b, w) = (ab, aw + bv)$, and obvious k-algebra structure. Note that $k\langle V\rangle$ is artinian if and only if V is finite dimensional. It can be seen that $V \mapsto k\langle V\rangle$ defines a fully faithful functor $\mathbf{FinVect}_k \to \mathbf{Art}_k$, and its image consists of all A in $\mathbf{Art}_k$ with $\mathfrak{m}_A^2 = 0$, as such an A is naturally isomorphic to $k\langle\mathfrak{m}_A\rangle$. The functor $V \mapsto k\langle V\rangle$ takes the zero vector space (which is both an initial and final object of $\mathbf{FinVect}_k$) to the algebra k (which is both an initial and final object of $\mathbf{Art}_k$). If $V \to U$ and $W \to U$ are morphisms in $\mathbf{FinVect}_k$, then it can be seen that the natural map

$$k\langle V \times_U W\rangle \to k\langle V\rangle \times_{k\langle U\rangle} k\langle W\rangle$$

(which is induced by the projections from $V \times_U W$ to V and W) is an isomorphism. Therefore the functor $\mathbf{FinVect}_k \to \mathbf{Art}_k : V \mapsto k\langle V\rangle$ preserves all finite inverse limits, in particular, it preserves equalisers.
For any deformation functor $F : \mathbf{Art}_k \to \mathbf{Sets}$, let the composite $\mathbf{FinVect}_k \to \mathbf{Art}_k \to \mathbf{Sets}$ be denoted by φ. As $F(k)$ is a singleton, $\varphi(0) = F(k\langle 0\rangle) = F(k) = 0$, so φ satisfies the condition **(T0)** on functors $\mathbf{FinVect}_k \to \mathbf{Sets}$. We now introduce another condition **(Hϵ)** on a deformation functor F, which just amounts to demanding that φ should satisfy **(T1)**.

2.3 Deformation condition (Hϵ) For any two A and B in $\mathbf{Art}_k$ with $\mathfrak{m}_A^2 = 0$ and $\mathfrak{m}_B^2 = 0$, the map $F(A \times_k B) \to F(A) \times F(B)$ that is induced by applying F to the projections of $A \times_k B$ on A and B is a bijection.
Note that $k\langle k\rangle$ is just the ring $k[\epsilon]/(\epsilon^2)$ of dual numbers over k.

2.4 The **tangent space T_F to a deformation functor** F that satisfies **(Hϵ)** is defined to be the resulting k-vector space $T_F = F(k[\epsilon]/(\epsilon^2)) = \varphi(k) = T_\varphi$.

Exercise 2.5 Let X be a k-scheme, $x \in X$ a k-valued point, and let F be the deformation functor $h_{X,x}$ of basic example 1. Then T_F equals the tangent space T_xX, which is the dual to $\mathfrak{m}_x/\mathfrak{m}_x^2$ where $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ is the maximal ideal of x. This is the motivation for defining T_F for deformation functors.

Exercise 2.6 Universal first-order family Let a deformation functor F satisfy **(Hϵ)**, and moreover let the resulting tangent vector space T_F be finite dimensional. Let T_F^* be the dual vector space of T_F, and let $A = k\langle T_F^*\rangle \in \mathbf{Art}_k$. Note that $T_A = T_F$. The identity endomorphism $\theta \in End(T_F) = T_F \otimes T_F^* = F(A)$ defines a family (A, θ). Show that this family has the following properties.
(i) The map $\theta : h_A \to F$ induces the identity isomorphism $T_F \to T_F$.
(ii) Let (R, r) be any pro-family for F parametrised by $R \in \widehat{\mathbf{Art}_k}$. Let $R_1 = R/\mathfrak{m}_R^2$ and let $r_1 = r|_{R_1}$. Then there exists a unique k-homomorphism $A \to R_1$ such that $r_1 \in F(R_1)$ is the image of $\theta \in F(A)$.
In view of property (ii), the family (A, θ) is called as the universal first-order family for F.

Pro-families and limit Yoneda lemma

Recall that the Yoneda lemma asserts the following. If $\mathcal{C}$ is any category, A is any object of $\mathcal{C}$ and $h_A = Hom(A, -)$ the corresponding representable functor, and $F : \mathcal{C} \to \mathbf{Sets}$ another functor, then there is a natural bijection $Hom(h_A, F) \to F(A)$, under which a natural transformation $\alpha : h_A \to F$ is mapped to the element $\alpha(\mathrm{id}_A) \in F(A)$. By abuse of notation, we denote $\alpha(\mathrm{id}_A)$ just by $\alpha \in F(A)$.
A **pro-family** for a deformation functor $F : \mathbf{Art}_k \to \mathbf{Sets}$ is a pair (R, r) where R is in $\widehat{\mathbf{Art}_k}$ and $r \in \widehat{F}(R)$ where by definition

$$\widehat{F}(R) = \lim_{\leftarrow} F(R/\mathfrak{m}^n)$$

where $\mathfrak{m} \subset R$ is the maximal ideal. By the following lemma, r is same as a morphism of functors $h_R \to F$.

Lemma 2.7 (Limit Yoneda Lemma)
Let $F : \mathbf{Art}_k \to \mathbf{Sets}$ be a deformation functor, and let $\widehat{F} : \widehat{\mathbf{Art}_k} \to \mathbf{Sets}$ be its prolongation as constructed above. Let $\alpha_R : Hom(h_R, F) \to \widehat{F}(R)$ be the map defined as follows. Given $f \in Hom(h_R, F)$, and $n \geq 1$, let $f_{R/\mathfrak{m}^n}(q_n) \in F(R/\mathfrak{m}^n)$ denote the image of the quotient $q_n \in Hom_{k\text{-}alg}(R, R/\mathfrak{m}^n)$. This defines an inverse system as n varies, so gives an element $\alpha_R(f) = (f(R/\mathfrak{m}^n)(q_n))_{n\in\mathbb{N}} \in \widehat{F}(R)$. The

map $\alpha_R : Hom(h_R, F) \to \widehat{F}(R)$ *so defined is a bijection, functorial in both* R *and* F. $\square$

We leave the proof of this lemma (which is a straight-forward generalisation of the usual Yoneda lemma) as an exercise.

Versal, miniversal, universal families

For a quick review of basic notions about smoothness and formal smoothness, see for example Milne [Mi].

Let $F : \mathbf{Art}_k \to \mathbf{Sets}$ and $G : \mathbf{Art}_k \to \mathbf{Sets}$ be functors. Recall that a morphism of functors $\phi : F \to G$ is called **formally smooth** if given any surjection $q : B \to A$ in $\mathbf{Art}_k$ and any elements $\alpha \in F(A)$ and $\beta \in G(B)$ such that

$$\phi_A(\alpha) = G(q)(\beta) \in F(A),$$

there exists an element $\gamma \in F(B)$ such that

$$\phi_B(\gamma) = \beta \in G(B) \text{ and } F(q)(\gamma) = \alpha \in F(A)$$

In other words, the following diagram of functors commutes, where the diagonal arrow $h_B \to F$ is defined by γ.

$$\begin{array}{ccc} h_A & \stackrel{\alpha}{\to} & F \\ {\scriptstyle q}\downarrow & \nearrow & \downarrow{\scriptstyle \phi} \\ h_B & \stackrel{\beta}{\to} & G \end{array}$$

The morphism $\phi : F \to G$ is called **formally étale** if it is formally smooth, and moreover the element $\gamma \in F(B)$ is unique.

Caution If the functors F and G are of the form h_R and h_S for rings R and S, then ϕ is formally étale if and only if it is formally smooth and the tangent map $T_R \to T_S$ is an isomorphism. However, if F and G are not both of the above form, then a functor ϕ can be formally smooth, and moreover the map $T_F \to T_G$ can be an isomorphism, yet ϕ need not be formally étale. It is because of this subtle difference that a miniversal family can fail to be universal, as we will see in examples later.

2.8 A **versal family** for a deformation functor $F : \mathbf{Art}_k \to \mathbf{Sets}$ is a pro-family (R, r) (where R is a complete local noetherian k-algebra with residue field k, and $r \in \widehat{F}(R)$) such that the morphism of functors $r : h_R \to F$ is formally smooth.

If (R, r) is a versal family, then for any A in $\mathbf{Art}_k$, the induced set map $r(A) : h_R(A) \to F(A)$ is surjective. For, given any $v \in F(A)$, we can regard it as a morphism $v : h_A \to F$. Now consider the following commutative square.

$$\begin{array}{ccc} h_k & \longrightarrow & h_R \\ \downarrow & & \downarrow \\ h_A & \stackrel{v}{\longrightarrow} & F \end{array}$$

By formal smoothness of $h_R \to F$, there exists a morphism $u : h_A \to h_R$ which makes the above diagram commute. But such a morphism is just an element of $h_R(A)$

which maps to $v \in F(A)$, which proves that $r(A) : h_R(A) \to F(A)$ is surjective. In other words, every family over A is a pull-back of the versal family over R, under a morphism $u : \operatorname{Spec} A \to \operatorname{Spec} R$. However, the morphism u need not be unique.
For any deformation functor $F : \mathbf{Art}_k \to \mathbf{Sets}$, the pointed set

$$T_F = F(k[\epsilon]/(\epsilon^2))$$

is called the **tangent set** to F, or the set of **first order deformations** under F.

2.9 A **minimal versal ('miniversal') family** (also called as a **hull**) for a deformation functor $F : \mathbf{Art}_k \to \mathbf{Sets}$ is a versal family for which the set map

$$dr : T_R = h_R(k[\epsilon]/(\epsilon^2)) \to F(k[\epsilon]/(\epsilon^2)) = T_F$$

is a bijection.

Exercise 2.10 If $r : h_R \to F$ is a hull for a deformation functor F, then show that F satisfies the deformation condition **(Hϵ)**, and the bijection of sets $dr : T_R \to T_F$ is in fact a linear isomorphism.

2.11 A **universal family** for a deformation functor $F : \mathbf{Art}_k \to \mathbf{Sets}$ is a pro-family (R, r) such that $r : h_R \to F$ is a natural bijection. If a universal family exists, it is clearly unique up to a unique isomorphism. A deformation functor $F : \mathbf{Art}_k \to \mathbf{Sets}$ is called **pro-representable** if a universal family exists. (The reason for the prefix 'pro-' is that R need not be in the subcategory $\mathbf{Art}_k$ of $\widehat{\mathbf{Art}_k}$.)

Exercise 2.12 Show the following.
(i) A pro-family (R, r) is universal if and only if the morphism of functors $r : h_R \to F$ is formally étale.
(ii) If F is pro-representable, then each hull pro-represents it.
(iii) **A miniversal family that is not universal.** Let $F : \mathbf{Art}_k \to \mathbf{Sets}$ be the functor $A \mapsto \mathfrak{m}_A/\mathfrak{m}_A^2$. Show that a hull (R, r) for F is given by $R = k[[t]]$ with $r = dt \in \mathfrak{m}_R/\mathfrak{m}_R^2 = \widehat{F}(R)$, but F is not pro-representable.
(iv) If a deformation functor F admits a hull and moreover if $T_F = 0$ then $F(A)$ is the singleton set $F(k)$ for all A in $\mathbf{Art}_k$.
(v) Let a deformation functor F have a versal family $(R, r : h_R \to \varphi)$, such that h_R is formally smooth. Then F is formally smooth. Conversely, if F is formally smooth, then each versal family is formally smooth.

Grothendieck's pro-representability theorem

The following condition on a deformation functor F is obviously satisfied by any pro-representable functor h_R.

2.13 Deformation condition (Lim) The functor F preserves fibred products: the induced map $F(B \times_A C) \to F(B) \times_{F(A)} F(C)$ is bijective for any pair of homomorphisms $B \to A$ and $C \to A$ in $\mathbf{Art}_k$.
As $\mathbf{Art}_k$ has a final object and admits fibred products, this is equivalent to the condition that F preserves all finite inverse limits in $\mathbf{Art}_k$, hence the name **(Lim)**. As **(Lim)** implies the deformation condition **(Hϵ)**(see 2.3 above), the set $T_F = F(k[\epsilon]/(\epsilon^2))$ is naturally a k-vector space whenever **(Lim)** is satisfied.

Theorem 2.14 (Grothendieck) *A deformation functor F is pro-representable if and only if the following two conditions* **(Lim)** *and* **(H3)** *are satisfied.*
(Lim) *The deformation functor F preserves fibred products.*
(H3) *The k-vector space T_F is finite dimensional.*

Obviously, a pro-representable functor satisfies the above conditions **(Lim)** and **(H3)**. The sufficiency of these conditions follows from Schlessinger's theorem (Theorem 2.19), in which the conditions **(Lim)** and **(H3)** are weakened to the conditions **(H1)**, **(H2)**, **(H3)** and **(H4)**. In practice, the conditions **(Hϵ)**and **(H3)** are the easiest to verify, the conditions **(H1)** and **(H2)** are of intermediate difficulty, while the condition **(Lim)** is quite difficult to check in most examples. Hence Schlessinger's theorem is more useful in actual practice than Theorem 2.14.
The interested reader can take the proof of Schlessinger's theorem, and shorten and simplify it using the stronger hypothesis **(Lim)** to get a proof of Theorem 2.14. Though this exercise makes a reversal of the actual history, it helps us understand the Schlessinger theorem better.

Schlessinger's conditions and the resulting group action

2.15 A **small extension** e in $\mathbf{Art}_k$ is a surjective homomorphism $B \to A$ whose kernel I satisfies $\mathfrak{m}_B I = 0$. The small extension e is called a **principal small extension** if moreover I is principal. We often use the notation $e = (0 \to I \to B \to A \to 0)$ for a small extension.

2.16 We now state the famous Schlessinger conditions **(H1)**, **(H2)**, **(H3)** and **(H4)** on a deformation functor, and their variants **(H1')** and **(H2')**.
(H1) For any homomorphisms $B \to A$ and $C \to A$ in $\mathbf{Art}_k$ such that $C \to A$ is a principal small extension, the induced map $F(B \times_A C) \to F(B) \times_{F(A)} F(C)$ is surjective.
The condition **(H1)** is equivalent to the following seemingly stronger condition:
(H1') For any homomorphisms $B \to A$ and $C \to A$ in $\mathbf{Art}_k$ such that $C \to A$ is surjective, the induced map $F(B \times_A C) \to F(B) \times_{F(A)} F(C)$ is surjective.
To see this, first note that a surjective homomorphism $p : C \to A$ can be factored in $\mathbf{Art}_k$ as the composite of a finite sequence of surjections $C = C_n \to C_{n-1} \to \ldots \to C_1 \to C_0 = A$ where $n \geq 1$ is an integer such that $\mathfrak{m}_C^n = 0$, and $C_j = C/\mathfrak{m}^j I$ where I is the kernel of $C \to A$. Then each $C_i \to C_{i-1}$ is a small extension. Moreover, a small extension $0 \to I \to C \to A \to 0$ can be factored as $C = C_n \to C_{n-1} \to \ldots \to$

$C_1 \to C_0 = A$ where $n = \dim_k I$, and each $C_i \to C_{i-1}$ is a principal small extension. Hence **(H1')** follows by applying **(H1)** successively to a finite sequence of principal small extensions.
(H2) For any B in $\mathbf{Art}_k$, the induced map $F(B \times_k k[\epsilon]/(\epsilon^2)) \to F(B) \times F(k[\epsilon]/(\epsilon^2))$ is bijective.
Similarly, the condition **(H2)** is equivalent to the following:
(H2') Let B be any object in $\mathbf{Art}_k$, and let $C = k\langle V\rangle$ where V is a finite dimensional k-vector space. Then the induced map $F(B \times_k C) \to F(B) \times F(C)$ is bijective.
Note As **(H2')** implies **(Hϵ)**, the set $T_F = F(k[\epsilon]/(\epsilon^2))$ gets a natural k-vector space structure whenever **(H2)** is satisfied. Hence the condition **(H3)** below makes sense whenever **(H2)** is satisfied.
(H3) The k-vector space T_F is finite dimensional.
(H4) If $B \to A$ is a principal small extension, then the induced map $F(B \times_A B) \to F(B) \times_{F(A)} F(B)$ is a bijection.

2.17 (**Definition of an action**) Let $F : \mathbf{Art}_k \to \mathbf{Sets}$ be a functor with $F(k)$ a singleton, which satisfies the Schlessinger conditions **(H1)** and **(H2)**. Let $0 \to I \to B \to A \to 0$ be a small extension in $\mathbf{Art}_k$. We now define an action of the group $T_F \otimes I$ on the set $F(B)$. Note that we have a k-algebra isomorphism $f : B \times_k k\langle I\rangle \to B \times_A B$ defined by $(b, \overline{b} + u) \mapsto (b, b + u)$ where $\overline{b} \in k$ is the residue class of b. Applying F and using **(H1')** and **(H2')**, the following composite map is a surjection:

$$F(B) \times T_F \otimes I = F(B) \times F(k\langle I\rangle) \stackrel{F(f)}{\to} F(B \times_A B) \to F(B) \times_{F(A)} F(B).$$

From its definition, the above map sends any pair (β, x) to a pair of the form (β, γ). We define a map $F(B) \times T_F \otimes I \to F(B)$ by $(\beta, x) \mapsto \gamma$, and this can be verified to define an action of the abelian group $T_F \otimes I$ on the set $F(B)$.

Proposition 2.18 *Let F be a deformation functor which satisfies the Schlessinger conditions* **(H1)** *and* **(H2)**. *Then for any small extension $0 \to I \to B \to A \to 0$ in $\mathbf{Art}_k$, the induced action of the abelian group $T_F \otimes I$ on the set $F(B)$ has the following properties.*
(i) The orbits of $T_F \otimes I$ in $F(B)$ are exactly the fibers of the map $F(B) \to F(A)$.
(ii) When $A = k$, the action is free.
(iii) The action is functorial in small extensions: given a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & I & \to & B & \to & A & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & I' & \to & B' & \to & A' & \to & 0 \end{array}$$

where the rows are small extensions, the induced map $F(B) \to F(B')$ is equivariant w.r.t. the induced group homomorphism $T_F \otimes I \to T_F \otimes I'$ and the actions of $T_F \otimes I$ and $T_F \otimes I'$ on $F(B)$ and $F(B')$.
(iv) The action is functorial in F: if $F \to G$ is a natural transformation of deformation functors which satisfy **(H1)** *and* **(H2)**, *then the induced set-map $F(B) \to G(B)$ is equivariant w.r.t. the induced group homomorphism $T_F \otimes I \to T_G \otimes I$.*

Proof Assertion (i) is just the surjectivity of $F(B) \times T_F \otimes I \to F(B) \times_{F(A)} F(B)$. (ii) amounts to the bijectivity of $F(B) \times T_F \otimes I \to F(B) \times_{F(A)} F(B)$ when $A = k$, and so follows from **(H2)**. (iii) and (iv) can be verified in a straight-forward manner from the definition of the action. $\square$

Schlessinger's theorem

Theorem 2.19 (Schlessinger) *A deformation functor F admits a hull if and only if the conditions* **(H1)**, **(H2)**, **(H3)** *are satisfied. Moreover, F is pro-representable if and only if the conditions* **(H1)**, **(H2)**, **(H3)** *and* **(H4)** *are satisfied.*

Proof It is a simple exercise to show that if R is in $\widehat{\mathbf{Art}_k}$ then h_R satisfies **(H1)**, **(H2)**, **(H3)** and **(H4)**, and if $r : h_R \to F$ is a hull for F then F satisfies **(H1)**, **(H2)** and **(H3)**. We now prove the reverse implications.
Existence of hull together with (H4) implies pro-representability : We will show that if **(H4)** is satisfied then any hull (R, r) is in fact a universal family. Let $0 \to I \to B \to A \to 0$ be a small extension in $\mathbf{Art}_k$, and consider the action of $T_F \otimes I$ on $F(B)$, which satisfies the properties given by Proposition 2.18. If **(H4)** holds, then $F(B \times_A B) \to F(B) \times_{F(A)} F(B)$ is bijective, so the surjective map $F(B) \times (T_F \otimes I) \to F(B) \times_{F(A)} F(B)$ is actually a bijection, which means that each fibre of $F(B) \to F(A)$ is a principal set (possibly empty) under the group $T_F \otimes I$.
To show that a miniversal family (R, r) is universal, we must show that the map $r(B) : h_R(B) \to F(B)$ is a bijection for each object B of $\mathbf{Art}_k$. This is clear for $B = k$. So now we proceed by induction on the smallest positive integer $n(B)$ for which $\mathfrak{m}_B^{n(B)} = 0$ (for $B = k$ we have $n = 1$). For a given B, suppose $n(B) \geq 2$. Let $I = \mathfrak{m}_B^{n(B)-1}$ so that $\mathfrak{m}_B I = 0$. Let $A = B/I$, so that $n(A) = n(B) - 1$, which by induction gives a bijection $r(A) : h_R(A) \to F(A)$. Consider the commutative square

$$\begin{array}{ccc} h_R(B) & \to & F(B) \\ \downarrow & & \downarrow \\ h_R(A) & = & F(A) \end{array}$$

Note that each fiber of $h_R(B) \to h_R(A)$ is a principal $T_R \otimes I$-set over $h_R(A)$ (possibly empty) and the map $r(B) : h_R(B) \to F(B)$ is $T_F \otimes I$-equivariant, where we identify T_R with T_F via $r : h_R \to F$. It follows that $r(B) : h_R(B) \to F(B)$ is injective. As $r(B) : h_R(B) \to F(B)$ is already known to be surjective by versality, this shows that $r(B)$ is bijective, thus (R, r) pro-represents F.

(H1), (H2), (H3) imply the existence of a hull : The proof will go in two stages. First, we will construct a family (R, r), which will be our candidate for a hull. Next, we prove that the family (R, r) is indeed a hull.
Construction of a family (R, r) **:** Let S be the completion of the local ring at the origin of the affine space $\operatorname{Spec} \operatorname{Sym}_k(T_F^*)$. If $x_1, \ldots, x_d$ is a linear basis for T_F^*, then $S = k[[x_1, \ldots, x_d]]$. Let $\mathfrak{n} = (x_1, \ldots, x_d) \subset S$ denote the maximal ideal of S.

We will construct a versal family (R, r) where $R = S/J$ for some ideal J. The ideal J will be constructed as the intersection of a decreasing chain of ideals

$$\mathfrak{n}^2 = J_2 \supset J_3 \supset J_4 \supset \ldots \supset \cap_{q=2}^{\infty} J_q = J$$

such that at each stage we will have $J_q \supset J_{q+1} \supset \mathfrak{n}J_q$. Consequently, we will have $J_q \supset \mathfrak{n}^q$ which in particular means $R/J_q \in \mathbf{Art}_k$, and J_q/J is a fundamental system of open neighbourhoods in $R = S/J$ for the $\mathfrak{m}$-adic topology on R, where $\mathfrak{m} = \mathfrak{n}/J$ is the maximal ideal of R. Note that R will be automatically complete for the $\mathfrak{m}$-adic topology.

Starting with $q = 2$, we will define for each q an ideal J_q and a family (R_q, r_q) parametrised by $R_q = S/J_q$, such that $r_{q+1}|_{R_q} = r_q$. We take $J_2 = \mathfrak{n}^2$. On $R_2 = S/\mathfrak{n}^2 = k\langle T_F^* \rangle$ we take q_2 to be the 'universal first order family' θ (see Exercise 2.6 above). Having already constructed (R_q, r_q), we next take J_{q+1} to be the unique smallest ideal in the set Ψ of all ideals $I \subset S$ which satisfy the following two conditions:

(1) We have inclusions $J_q \supset I \supset \mathfrak{n}J_q$.

(2) There exists a family α (need not be unique) parametrised by R/I which prolongs r_q, that is, $\alpha|_{R_q} = r_q$.

Note that Ψ is non-empty as $J_q \in \Psi$, and Ψ has at least one minimal element as $S/\mathfrak{n}J_q$ is artinian being a quotient of $S/\mathfrak{n}^{q+1}$. We next show that if $I_1, I_2 \in \Psi$ then $I_0 = I_1 \cap I_2 \in \Psi$, hence the minimal element of Ψ is unique.

Consider the vector space $J_q/\mathfrak{n}J_q$ and its subspaces $I_1/\mathfrak{n}J_q$, $I_2/\mathfrak{n}J_q$, and $I_0/\mathfrak{n}J_q$. Let $u_1, \ldots, u_a, v_1, \ldots, v_b, w_1, \ldots, w_c, z_1, \ldots, z_d \in J_q$ be elements such that (i) $u_1, \ldots, u_a$ (mod $\mathfrak{n}J_q$) is a linear basis of $I_0/\mathfrak{n}J_q$, (ii) $u_1, \ldots, u_a, v_1, \ldots, v_b$ (mod $\mathfrak{n}J_q$) is a linear basis of $I_1/\mathfrak{n}J_q$, (iii) $u_1, \ldots, u_a, w_1, \ldots, w_c$ (mod $\mathfrak{n}J_q$) is a linear basis of $I_2/\mathfrak{n}J_q$, and (iv) $u_1, \ldots, u_a, v_1, \ldots, v_b, w_1, \ldots, w_c, z_1, \ldots, z_d$ (mod $\mathfrak{n}J_q$) is a linear basis of $J_q/\mathfrak{n}J_q$. Let $I_3 = (u_1, \ldots, u_a, w_1, \ldots, w_c, z_1, \ldots, z_d) + \mathfrak{n}J_q$. Then we have $I_2 \subset I_3$, $I_1 \cap I_3 = I_0$ and $I_1 + I_3 = J_q$. Note that we have

$$\frac{S}{I_1} \times_{\left(\frac{S}{I_1+I_3}\right)} \frac{S}{I_3} = \frac{S}{I_1 \cap I_3}$$

As $I_1 + I_3 = J_q$ and $I_1 \cap I_3 = I_0$, this reads

$$\frac{S}{I_1} \times_{\left(\frac{S}{\mathfrak{n}J_q}\right)} \frac{S}{I_3} = \frac{S}{I_0}$$

As **(H1)** is satisfied, this gives surjection

$$F\left(\frac{S}{I_0}\right) = F\left(\frac{S}{I_1} \times_{\left(\frac{S}{J_q}\right)} \frac{S}{I_3}\right) \to F\left(\frac{S}{I_1}\right) \times_{F\left(\frac{S}{J_q}\right)} F\left(\frac{S}{I_3}\right)$$

Let $\alpha_1 \in F(S/I_1)$ and $\alpha_2 \in F(S/I_2)$ be any prolongation of $r_q \in F(S/J_q)$, which exist as $I_1, I_2 \in \Psi$. Let $\alpha_3 = \alpha_2|_{S/I_3}$. This defines an element

$$(\alpha_1, \alpha_3) \in F\left(\frac{S}{I_1}\right) \times_{F\left(\frac{S}{J_q}\right)} F\left(\frac{S}{I_3}\right)$$

Therefore by **(H1)** there exists $\alpha_0 \in F(S/I_0)$ which prolongs both α_1 and α_3 (it might not prolong α_2). This means α_0 prolongs r_q, so $I_0 \in \Psi$ as was to be shown.
Therefore Ψ has a unique minimal element J_{q+1}, and
We now choose $r_{q+1} \in F(S/J_{q+1})$ to be an arbitrary prolongation of r_q (not claimed to be unique).
Let J be the intersection of all the J_n, and let $R = S/J$. We want to define an element $r \in \widehat{F}(R)$ which restricts to r_q on S/J_q for each q. This makes sense and is indeed possible, as we will show using the following lemma, whose proof we leave as an exercise in the application of the familiar Mittag-Leffler condition for exactness of inverse limits.

Lemma 2.20 *Let R be a complete noetherian local ring with with maximal ideal $\mathfrak{m}$. Let $I_1 \supset I_2 \supset \ldots$ be a decreasing sequence of ideals such that (i) the intersection $\cap_{n\ge 1} I_n$ is 0, and (ii) for each $n \geq 1$, we have $I_n \supset \mathfrak{m}^n$. Then the natural map $f : R \to \lim_{\leftarrow} R/I_n$ is an isomorphism. Moreover, for any $n \geq 1$ there exists an $q \geq n$ such that $\mathfrak{m}^n \supset I_q$.*

Let $R = S/J$ as before, which is a complete noetherian local ring with with maximal ideal $\mathfrak{m} = \mathfrak{n}/J$, and let $I_q = J_q/J$ for $q \geq 2$. By construction, we have $J_q \supset J_{q+1} \supset \mathfrak{n}J_q$, which means $I_q \supset I_{q+1} \supset \mathfrak{m}I_q$. In particular, this means $I_q \supset \mathfrak{m}^q$. As $J = \cap J_q$, we get $\cap I_q = 0$. Therefore by Lemma 2.20, for each $n \geq 1$ there exists a $q \geq n$ with $I_n \supset \mathfrak{m}^n \supset I_q$, and in particular the natural map $R \to \lim_{\leftarrow} R/I_n$ is an isomorphism. Recall that we have already chosen an inverse system of elements $r_q \in F(R/I_q)$, as $R/I_q = S/J_q$. For each n choose the smallest $q_n \geq n$ such that $\mathfrak{m}^n \supset I_{q_n}$. We have a natural surjection $R/I_{q_n} \to R/\mathfrak{m}^n$. Let $\theta_n = r_{q_n}|_{R/\mathfrak{m}^n}$. Then from its definition it follows that under $R/\mathfrak{m}^{n+1} \to R/\mathfrak{m}^n$, we have $\theta_n = \theta_{n+1}|_{R/\mathfrak{m}^n}$. Therefore (θ_n) defines an element

$$r = (\theta_n) \in \lim_{\leftarrow} F(R/\mathfrak{m}^n) = \widehat{F}(R)$$

Verification that (R, r) **is a hull for** F **:** By its construction, the map $T_R \to T_F$ is an isomorphism. So all that remains is to show that $h_R \to F$ is formally smooth. This means given any surjection $p : B \to A$ in $\mathbf{Art}_k$ and a commutative square

$$\begin{array}{ccc} h_A & \stackrel{h_u}{\to} & h_R \\ {\scriptstyle h_p}\downarrow & & \downarrow {\scriptstyle r} \\ h_B & \stackrel{b}{\to} & F \end{array}$$

there exists a diagonal morphism $h_v : h_B \to h_R$ (that is, a homomorphism $v : R \to B$) which makes the resulting diagram (the above square together with a diagonal) commute. If $\dim_k(B) = \dim_k(A)$ as k-vector space, then the surjection $B \to A$ is an isomorphism, and we are done. Otherwise, we can reduce to the case where $\dim_k(B) = \dim_k(A)+1$ (the case of a small extension) by factoring $p : B \to A$ as the composite of a finite sequence of surjections $B = B_n \to B_{n-1} \to \ldots \to B_1 \to B_0 = A$ where each $B_i \to B_{i-1}$ is a principal small extension, and lifting step-by-step, making the analogue of the above square commute at each step.

Suppose there exists a homomorphism $w : R \to B$ such that $u = p \circ w : R \to B \to A$. Using such a w, we will construct a homomorphism $v : R \to B$ as needed in the proof of formal smoothness of $h_R \to F$, which satisfies both $u = p \circ v : R \to B \to A$ and $r \circ h_v = b : h_B \to F(B) \to F(A)$.
Consider the following commutative square:

$$\begin{array}{ccc} h_R(B) & \stackrel{r(B)}{\to} & F(B) \\ {\scriptstyle h_R(p)} \downarrow & & \downarrow {\scriptstyle F(p)} \\ h_R(A) & \stackrel{r(A)}{\to} & F(A) \end{array}$$

As $B \to A$ is a small extension, and as both h_R and F satisfy **(H1)** and **(H2)**, and as $T_R = T_F$, there is a natural transitive action of the additive group $T_F \otimes I$ on each fibre of the set maps $h_R(B) \to h_R(A)$ and $F(B) \to F(A)$. By Proposition 2.18.(iv), the top map $r(B) : h_R(B) \to F(B)$ in the above square is $T_F \otimes I$-equivariant. As $u = p \circ w$, the elements $r(B)w$ and b both lie in the same fibre of $F(B) \to F(A)$, over $r(A)u \in F(A)$. Therefore, there exists some $\alpha \in G$ (not necessarily unique) such that $b = r(B)w + \alpha$. Let $v = w + \alpha \in h_R(B)$. By G-equivariance of $r(B)$, we get $r(B)v = r(B)(w + \alpha) = r(B)w + \alpha = b$. Also, as the action of G preserves the fibers of $h_R(B) \to h_R(A)$, we have $p \circ v = p \circ (w + \alpha) = p \circ w = u$. Therefore v has the desired property. It therefore just remains to show the existence of $w : R \to B$ with $p \circ w = u$. For this we first make the following elementary observation.

Remark 2.21 Let $B \to A$ be a surjection in $\mathbf{Art}_k$ such that $\dim_k(B) = \dim_k(A) + 1$ (equivalently, the kernel I of the surjection satisfies $\mathfrak{m}_B I = 0$ and $\dim_k(I) = 1$). Suppose that $B \to A$ does not admit a section $A \to B$. Then for any k-algebra homomorphism $C \to B$, the composite $C \to B \to A$ is surjective (if and) only if $C \to B$ is surjective.

We now show the existence of $w : R \to B$ with $p \circ w = u$. As A is artinian, the homomorphism $u : R \to A$ must factor via $R_q = R/\mathfrak{m}^q$ for some $q \geq 1$, giving a homomorphism $u_q : R_q \to A$. We are given a diagram

$$\begin{array}{ccc} \operatorname{Spec} A & \stackrel{u_q^*}{\to} \operatorname{Spec} R_q \to \operatorname{Spec} R \hookrightarrow & \operatorname{Spec} S \\ \downarrow & & \downarrow \\ \operatorname{Spec} B & \to & \operatorname{Spec} k \end{array}$$

The morphism $\operatorname{Spec} S \to \operatorname{Spec} k$ is formally smooth, therefore, there exists a diagonal homomorphism $f^* : \operatorname{Spec} B \to \operatorname{Spec} S$ which makes the resulting diagram commute. Equivalently, there exists a k-algebra homomorphism $f : S \to B$ such that $p \circ f = u \circ \pi : S \to A$ where $\pi : S \to R = S/J$ is the quotient map. Therefore, we get a commutative square

$$\begin{array}{ccc} S & \stackrel{f}{\to} & B \\ \downarrow & & \downarrow \\ R_q & \stackrel{u_q}{\to} & A \end{array}$$

where the vertical maps are the quotient maps $\pi_q : S \to S/J_q = R_q$, and $p : B \to A$. This defines a k-homomorphism

$$\varphi = (\pi_q, f) : S \to R_q \times_A B$$

The composite $S \to R_q \times_A B \to R_q$ is π_q which is surjective. As by assumption $\dim_k(B) = \dim_k(A) + 1$, it follows that

$$\dim_k(R_q \times_A B) = \dim_k(R_q) + 1$$

Therefore by Remark 2.21, at least one of the following holds:
(1) The projection $R_q \times_A B \to R_q$ admits a section $(\mathrm{id}, s) : R_q \to R_q \times_A B$, in other words, there exists some $s : R_q \to B$ such that $p \circ s = u_q : R_q \to A$.
(2) The homomorphism $\varphi : S \to R_q \times_A B$ is surjective.
If (1) holds, then we immediately get a lift

$$v : R \to R_q \xrightarrow{s} B$$

of $u : R \to A$, completing the proof.
If (2) holds, then we claim that $\varphi : S \to R_q \times_A B$ factors through $S \to S/J_{q+1} = R_{q+1}$, thereby giving a homomorphism $s' : R_{q+1} \to B$ such that $p \circ s' = u_{q+1} : R_{q+1} \to A$. This would immediately give a lift

$$v : R \to R_{q+1} \xrightarrow{s'} B$$

of $u : R \to A$, again completing the proof.
Therefore, all that remains is to show that if $\varphi : S \to R_q \times_A B$ is surjective, then it must factor through $S \to S/J_{q+1} = R_{q+1}$. To see this, let $K = ker(\varphi) \subset S$, so that $R_q \times_A B$ gets identified with S/K by surjectivity of φ. We have the families $r_q \in F(R_q)$, $a \in F(A)$ and $b \in F(B)$ such that both r_q and b map to a under $R_q \to A$ and $B \to A$. By **(H1)** the map $F(R_q \times_A B) \to F(R_q) \times_{F(A)} F(B)$ is surjective, so there exists a family $\mu \in F(R_q \times_A B) = F(S/K)$ which restricts to $r_q \in F(R_q)$. This means the ideal K is in the set of ideals Ψ defined earlier while constructing the nested sequence $J_2 \supset J_3 \supset \ldots$ of ideals in S. By minimality and uniqueness of J_{q+1}, we have $K \supset J_{q+1}$. Therefore $\varphi : S \to R_q \times_A B = S/K$ factors through $S \to S/J_{q+1} = R_{q+1}$ as desired.
This completes the proof of Schlessinger's theorem. $\square$

Obstruction theory

A deformation functor F is called **formally smooth** or **unobstructed** if for each surjection $B \to A$ in $\mathbf{Art}_k$ the map $F(B) \to F(A)$ is surjective.
Note in particular that the deformation functor $h_{X,x}$ of basic example 1 is unobstructed if and only X is smooth at x.
The above notion is generalised by the notion of an obstruction theory, defined below. In these terms, F will be formally smooth if it admits an obstruction theory $(O_F, (o_e))$ where $O_F = 0$ (or more generally where each $o_e = 0$).

2.22 An **obstruction theory** $(O_F, (o_e))$ for a deformation functor F is a k-vector space O_F together with additional data (o_e) consisting of a set-map $o_e : F(A) \to O_F \otimes_k I$ associated to each small extension $e = (0 \to I \to B \to A \to 0)$ in $\mathbf{Art}_k$ such that the following conditions **(O1)** and **(O2)** are satisfied.

(O1) An element $\alpha \in F(A)$ lifts to an element of $F(B)$ if and only if $o_e(\alpha) = 0$.

(O2) The map o_e is functorial in e in the sense that given any commutative diagram

$$\begin{array}{ccccccc} 0 \to & I & \to B \to & A & \to 0 \\ & \downarrow & \downarrow & \downarrow & \\ 0 \to & I' & \to B' \to & A' & \to 0 \end{array}$$

with rows e and e' small extensions in $\mathbf{Art}_k$, the following induced square commutes:

$$\begin{array}{ccc} F(A) & \stackrel{o_e}{\to} & O_F \otimes I \\ \downarrow & & \downarrow \\ F(A') & \stackrel{o_{e'}}{\to} & O_F \otimes I' \end{array}$$

Let R be a complete noetherian local k-algebra with residue field k, so that R can be expressed as a quotient S/J where $S = k[[t_1, \ldots, t_n]]$ is the power-series ring in n variables where $n = \dim_k \mathfrak{m}/\mathfrak{m}^2$, $\mathfrak{m} \subset R$ is its maximal ideal, and $J \subset S$ is an ideal with $J \subset \mathfrak{n}^2$ where $\mathfrak{n} = (t_1, \ldots, t_n)$ is the maximal ideal of S. Given any small extension $e = (0 \to I \to B \to A \to 0)$ in $\mathbf{Art}_k$ and a homomorphism $\alpha : R \to A$, by arbitrarily lifting the generators t_i we get a homomorphism $\alpha' : S \to B$. This induces a homomorphism $\alpha'' : J \to I$, such that the following diagram commutes.

$$\begin{array}{ccccccc} 0 \to & J & \to S \to & R & \to 0 \\ & {\scriptstyle \alpha''}\downarrow & {\scriptstyle \alpha'}\downarrow & {\scriptstyle \alpha}\downarrow & \\ 0 \to & I & \to B \to & A & \to 0 \end{array}$$

As $\alpha'(\mathfrak{n}) \subset \mathfrak{m}_B$ for the maximal ideal $\mathfrak{n} \subset S$, it follows that $\alpha''(\mathfrak{n}J) \subset \mathfrak{m}_B I = 0$, hence α'' induces a map $\overline{\alpha} : J/\mathfrak{n}J \to I$. We can regard this as an element $\overline{\alpha} \in (J/\mathfrak{n}J)^* \otimes_k I$. This defines a linear map

$$o_e : F(A) \to (J/\mathfrak{n}J)^* \otimes_k I : \alpha \mapsto \overline{\alpha}$$

which can be seen to be well-defined (independent of the intermediate choice of α') and functorial in e. As the map $\alpha : R \to A$ admits a lift to a map $\beta : R \to B$ if and only if $\alpha'' = 0$ (equivalently, $\overline{\alpha} = 0$), the following proposition is proved.

Proposition 2.23 *For any complete noetherian local k-algebra R with residue field k, the above data $((J/\mathfrak{n}J)^*, (o_e))$ is an obstruction theory for h_R.*

Remark 2.24 The above obstruction theory for h_R is *minimal* in the sense that given any other obstruction theory (V, v_e) there exists a unique linear injection $(J/\mathfrak{n}J)^* \to V$ making the obvious diagrams commute. Consequently, if the deformation functor $F = h_{X,x}$ has an obstruction theory $(O_F, (o_e))$ with $\dim_x X = \dim T_x X - \dim O_F$, then X is a local complete intersection at x. (See for example [F-G] Theorem 6.2.4 and its corollaries.)

2.25 A **tangent-obstruction theory** $(T^1, T^2, (\phi_e), (o_e))$ for a deformation functor F consists of finite-dimensional k-vector spaces T^1 and T^2 together with the following additional data. For each small extension $e = (0 \to I \to B \to A \to 0)$ in $\mathbf{Art}_k$, we are given an action $\phi_e : F(B) \times T^1 \otimes I \to F(B)$ satisfying the conclusions (i)-(iii) of Proposition 2.18, and a set map $o_e : F(A) \to T^2 \otimes I$ such that $(T^2, (o_e))$ is an obstruction theory for F. Then such an F automatically satisfies **(H1)**, **(H2)** and **(H3)**, and $(T^1, (\phi_e))$ is isomorphic to T_F together with its natural action (exercise). Moreover, the action of $T^1 \otimes I$ on $F(B)$ is free if an only if F also satisfies **(H4)**. Hence the following implications hold for any deformation functor F.

Pro-representability $\Rightarrow$ Existence of a tangent-obstruction theory $\Rightarrow$ Existence of a hull.

3 Calculations for basic examples

Preliminaries on flatness and base-change

All our examples involve flat families over base A, where $A \in \mathbf{Art}_k$, and so tools for verification of flatness are important. Here we have gathered together all the flatness statements we need. The reader in a hurry can skip this part and return to it as needed.

Exercise 3.1 **(i) (Nilpotent Nakayama)** Let A be a ring and $J \subset A$ a nilpotent ideal (means $J^n = 0$ for $n \gg 0$). If M is any A-module (not necessarily finitely generated) with $M = JM$, then show that $M = 0$.
(ii) (Schlessinger [S] Lemma 3.3) Apply (i) to show the following. Let A be a ring and $J \subset A$ a nilpotent ideal. Let $u : M \to N$ be a homomorphism of A-modules where N is flat over A. If $\overline{u} : M/JM \to N/JN$ is an isomorphism, then u is an isomorphism.
(iii) (Flat equivalent to free) Deduce from (ii) that flatness is equivalent to freeness for any module over an artin local ring.
(iv) (Tor vanishing implies flatness) Let A be an artin local ring, and M an A-module (not necessarily finitely generated). Then M is flat if and only if $Tor_1^A(A/\mathfrak{m}, M) = 0$.
(v) Let B be a commutative ring and I an ideal which satisfies $I^2 = 0$. For any B-module N, the sequence $0 \to IN \to N \to N/IN \to 0$ is exact. Using this, show that a B-module M is flat if and only if $Tor_1^B(K, M) = 0$ for all B-modules K that are annihilated by I.
(vi) (Flatness over arbitrary square-zero extensions) Let $A = B/I$ where the ideal $I \subset B$ satisfies $I^2 = 0$. It follows from the well-known 'local criterion for flatness' that an R-module M is flat over B if and only if the following two conditions are satisfied: (1) $A \otimes_B M$ is flat over A, and (2) $Tor_1^B(A, M) = 0$ (the second condition is equivalent to the injectivity of the scalar multiplication map $I \otimes_B M \to M$).
Proof: Choose a short-exact sequence $0 \to Z \to F \to M \to 0$ where F is a free B-module. By condition (2), the sequence $0 \to A \otimes_B Z \to A \otimes_B F \to A \otimes_B M \to 0$

is exact. For any A-module K, by applying $K \otimes_A -$ to the above exact sequence and using the condition (1), the sequence $0 \to K \otimes_B Z \to K \otimes_B F \to K \otimes_B M \to 0$ is exact. Hence by freeness of F, we must have $Tor_1^B(K, M) = 0$. The result now follows by applying the statement **(v)**. $\square$

The following lemma is an example of **non-flat descent**: even though $\operatorname{Spec} A' \to \operatorname{Spec} B$ and $\operatorname{Spec} A'' \to \operatorname{Spec} B$ is not necessarily a flat cover of $\operatorname{Spec} B$, we get a flat B-module N by gluing together over A flat modules M' and M'' over A' and A''.

Lemma 3.2 (Schlessinger [S] Lemma 3.4) *Let $A' \to A$ and $A'' \to A$ be ring homomorphisms, such that $A'' \to A$ is surjective with its kernel a nilpotent ideal $J \subset A''$. Let $B = A' \times_A A''$, with $B \to A'$ and $B \to A''$ the projections. Let M, M' and M'' be modules over A, A', A'', together with A'-linear homomorphism $u' : M' \to M$ and A''-linear homomorphism $u'' : M'' \to M$ which give isomorphisms $M' \otimes_{A'} A \to M$ and $M'' \otimes_{A''} A \to M$. Let N be the B-module defined by $N = M' \times_M M'' = \{(x', x'') \in M' \times M'' \,|\, u'(x') = u''(x'') \in M\}$, where scalar multiplication by elements $(a', a'') \in B$ is defined by $(a', a'') \cdot (x', x'') = (a'x', a''x'')$. If M' and M'' are flat modules over A' and A'' respectively, then N is flat over B. Moreover, the projection maps $N \to M'$ and $N \to M''$ induce isomorphisms $N \otimes_B A' \xrightarrow{\sim} M'$ and $N \otimes_B A'' \xrightarrow{\sim} M''$.*

Proof We will prove this only in the case where M' is a free A'-module. Note that if A' is artin local, then we are automatically in this case by Exercise 3.1.(iii). This is therefore the only case which we need in these notes.
Let $(x'_i)_{i \in I}$ be a free basis for M' over A'. As $M' \otimes_{A'} A \to M$ is an isomorphism, this gives a free basis $u'(x'_i)$ of M over A. As $A'' \to A$ is surjective, any element $\sum y''_j \otimes a''_j$ of $M'' \otimes_{A''} A$ equals $x'' \otimes 1$ for some (not necessarily uniquely determined) element $x'' \in M''$. Therefore the assumption of surjectivity of $M'' \otimes_{A''} A \to M$ tells us that $u'' : M'' \to M$ must be surjective. Hence we can choose elements $x''_i \in M''$ such that $u''(x''_i) = u'(x'_i)$. Let $N = \oplus_I A''$ be the free A''-module on the set I, with standard basis denoted by $(e_i)_{i \in I}$, and let $u : N \to M''$ be defined by $e_i \mapsto x''_i$. Then $\overline{u} : N/JN \to M''/JM'' = M$ is an isomorphism. Therefore by Exercise 3.1.(ii), u is an isomorphism, which shows M'' is free with basis $(x''_i)_{i \in I}$. It follows that N is free over B, with basis $(x'_i, x''_i)_{i \in I}$. It is now immediate that the projections $N \to M'$ and $N \to M''$ induce isomorphisms $N \otimes_B A' \xrightarrow{\sim} M'$ and $N \otimes_B A'' \xrightarrow{\sim} M''$. $\square$

Corollary 3.3 (Schlessinger [S] Corollary 3.6) *With hypothesis and notation as in the above lemma, let L be a B-module, and $q' : L \to M'$ and $q'' : L \to M''$ be B-linear homomorphisms, such that the following diagram commutes:*

$$\begin{array}{ccc} L & \stackrel{q''}{\to} & M'' \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle u''} \\ M' & \stackrel{u'}{\to} & M \end{array}$$

Suppose that q' induces an isomorphism $L \otimes_B A' \to M'$. Then the map $(q', q'') : L \to N = M' \times_M M''$ is an isomorphism of B-modules.

Proof The kernel of the projection $B \to A'$ is the ideal $I = 0 \times J \subset A' \times_A A'' = B$. The ideal I is nilpotent as by assumption J is nilpotent. The desired result follows by applying Exercise 3.1.(ii) to the B-homomorphism $u = (q', q'') : L \to N$, which becomes the given isomorphism $L \otimes_B A' \to M'$ on going modulo the nilpotent ideal $I \subset B$. $\square$

We will need the following well-known base change result of Grothendieck [EGA] (for an exposition also see Hartshorne [H] Theorem 12.11).

Theorem 3.4 *Let $S = \mathrm{Spec}(A)$ where A is a noetherian local ring. Let $\pi : \mathfrak{X} \to S$ be a proper morphism and $\mathcal{F}$ a coherent $\mathcal{O}_{\mathfrak{X}}$-module which is flat over S. Let $s \in S$ be the closed point with residue field denoted by k. Let $\mathfrak{X}_s$ be the fiber over s and let $\mathcal{F}_s = \mathcal{F}|_{\mathfrak{X}_s}$ denote the restriction of $\mathcal{F}$ to $\mathfrak{X}_s$. Let i be an integer, such that the natural map $H^i(\mathfrak{X}, \mathcal{F}) \otimes_A k \to H^i(\mathfrak{X}_s, \mathcal{F}_s)$ is surjective. Then for any A-module M, the induced map $H^i(\mathfrak{X}, \mathcal{F}) \otimes_A M \to H^i(\mathfrak{X}, \mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{X}}} \pi^* M)$ is an isomorphism. In particular if $H^i(\mathfrak{X}_s, \mathcal{F}_s) = 0$ then $H^i(\mathfrak{X}, \mathcal{F}) = 0$.*

Both [EGA] and [H] give rather complicated proofs of the above, involving inverse limits over modules of finite length (which in [H] is done by invoking the theorem on formal functions). These can be replaced by a simple application of Nakayama lemma to the semi-continuity complex.

The following lemma will be used in the deformation theory for a coherent sheaf E which is simple (that is, $End(E) = k$), to prove the theorem that the deformation functor $\mathcal{D}_E$ of such a sheaf is pro-representable.

Lemma 3.5 *Let A be a noetherian local ring, let $S = \mathrm{Spec}\, A$, and let $\pi : \mathfrak{X} \to S$ be a proper morphism. Let X denote the schematic fiber of π over the closed point* $\mathrm{Spec}\, k$*, where k is the residue field of A. Let $\mathcal{E}$ be a coherent sheaf on $\mathfrak{X}$ such that $\mathcal{E}$ is flat over S. Assume that there exists an exact sequence $\mathcal{F}_1 \to \mathcal{F}_0 \to \mathcal{E} \to 0$ of $\mathcal{O}_{\mathfrak{X}}$-modules, where $\mathcal{F}_1$ and $\mathcal{F}_0$ are locally free (note that this condition is automatically satisfied when $\mathcal{E}$ itself is locally free, or when $\pi : \mathfrak{X} \to S$ is a projective morphism). Let $E = \mathcal{E}|_X$ be the restriction of $\mathcal{E}$ to X. If the ring homomorphism $k \to End_X(E)$ (under which k acts on E by scalar multiplication) is an isomorphism, then for any morphism $f : T \to S$, the natural ring homomorphism*

$$H^0(T, \mathcal{O}_T) \to End_{\mathfrak{X}_T}((\mathrm{id} \times f)^* \mathcal{E})$$

(under which $H^0(T, \mathcal{O}_T)$ acts on $(\mathrm{id} \times f)^* \mathcal{E}$ by scalar multiplication) is an isomorphism.

Proof Consider the contravariant functor $\mathrm{End}(\mathcal{E})$ from S-schemes to sets, which associates to any S-scheme $f : T \to S$ the set $\mathrm{End}(\mathcal{E})(T) = End_{\mathfrak{X}_T}((\mathrm{id} \times f)^* \mathcal{E})$. Then by a fundamental theorem of Grothendieck (EGA III 7.7.8, 7.7.9), there exists a coherent sheaf $\mathcal{Q}$ on S and a functorial $H^0(T, \mathcal{O}_T)$-module isomorphism $\alpha_T : End_{\mathfrak{X}_T}((\mathrm{id} \times f)^* \mathcal{E}) \to Hom_T(f^* \mathcal{Q}, \mathcal{O}_T)$. As $S = \mathrm{Spec}\, A$, the coherent sheaf $\mathcal{Q}$ corresponds to the finite A-module $Q = H^0(S, \mathcal{Q})$. Consider the isomorphism $\alpha_S : End_{\mathfrak{X}}(\mathcal{E}) \to Hom_S(\mathcal{Q}, \mathcal{O}_S) = Hom_A(Q, A)$. Let $\theta : Q \to A$ be the image of $1_{\mathcal{E}}$

under α_S. By functoriality, the restriction $\theta_k : Q \otimes_A k \to k$ of θ to $\mathrm{Spec}\, k$ is the image of 1_E under the isomorphism $\alpha_k : End_X(E) \to Hom_k(Q \otimes_A k, k) = Hom_A(Q, A)$. As by assumption $k \to End_X(E)$ is an isomorphism, by composing with α_k we get an isomorphism $k \mapsto Hom_k(Q \otimes_A k, k)$ under which $1 \mapsto \theta_k$. Hence $Hom_k(Q \otimes_A k, k)$ is 1-dimensional as a k-vector space with basis θ_k. Therefore θ_k is surjective, and so by Nakayama it follows that $\theta : Q \to A$ is surjective. Hence we have a splitting $Q = A \oplus N$ where $N = \ker(\theta)$, under which the map $\theta : Q \to A$ becomes the projection $p_1 : A \oplus N \to A$ on the first factor. But as θ_k is an isomorphism, it again follows by Nakayama that $N = 0$. This shows that $\theta : Q \to A$ is an isomorphism. Identifying $\mathcal{Q}$ with $\mathcal{O}_S$ under θ, for any $f : T \to S$ we have $Hom_T(f^*\mathcal{Q}, \mathcal{O}_T) = H^0(T, \mathcal{O}_T)$, and so we get a functorial $H^0(T, \mathcal{O}_T)$-module isomorphism $\alpha_T : End_{\mathfrak{X}_T}((\mathrm{id} \times f)^*\mathcal{E}) \to H^0(T, \mathcal{O}_T)$ which maps $1 \mapsto 1$. The composite map $H^0(T, \mathcal{O}_T) \to End_{\mathfrak{X}_T}((\mathrm{id} \times f)^*\mathcal{E}) \to H^0(T, \mathcal{O}_T)$ is identity, so it follows that $H^0(T, \mathcal{O}_T) \to End_{\mathfrak{X}_T}((\mathrm{id} \times f)^*\mathcal{E})$ is an isomorphism. $\square$

Deformations of a coherent sheaf

Let X be a k-scheme of finite type, and E a coherent sheaf of $\mathcal{O}_X$-modules. We now return to the deformation functor $\mathcal{D}_E$ introduced earlier as our basic example 2. The calculation of the tangent space to $\mathcal{D}_E$ in the special case where E is locally free is the first exposure many of us have had to deformation theory. So let us begin with this illuminating case. Let the vector bundle E be described by transition functions $g_{i,j}$ w.r.t an affine open cover U_i of X. If $(\mathcal{F}, \theta)$ is a deformation of E over $k[\epsilon]/(\epsilon^2)$, then $\mathcal{F}$ is again a vector bundle, which is trivial over each $U_i[\epsilon] = U_i \times \mathrm{Spec}\, k[\epsilon]/(\epsilon^2)$. Choose a trivialization for $\mathcal{F}|_{U_i[\epsilon]}$ which restricts under θ to the chosen trivialization for $E|_{U_i}$, so that $\mathcal{F}$ is described by transition functions of the form $g_{i,j} + \epsilon h_{i,j}$. The cocycle condition on the transition function now reads

$$h_{i,k} = g_{i,j} h_{j,k} + h_{i,j} g_{j,k}$$

over $U_{i,j,k}$, which means the $h_{i,j}$ regarded as an endomorphism of E over $U_{i,j}$ – going from the basis restricted from U_j to the basis restricted from U_i – define a Cech 1-cocycle $(h_{i,j}) \in Z^1((U_i), \underline{End}(E))$. The corresponding cohomology class $h \in H^1(X, \underline{End}(E))$ can be seen to be independent of the choice of open cover and local trivializations, and so we get a map $\mathcal{D}_E(k[\epsilon]/(\epsilon^2)) \to H^1(X, \underline{End}(E))$. We can define an inverse to this map by sending $(h_{i,j})$ to the pair $(\mathcal{F}, \theta)$ consisting of the bundle $\mathcal{F}$ on $X[\epsilon]$ defined by transition functions $g_{i,j} + \epsilon h_{i,j}$, with θ induced by identity. Hence the map $\mathcal{D}_E(k[\epsilon]/(\epsilon^2)) \to H^1(X, \underline{End}(E))$ is a bijection. An exact analogue of the above argument, where we replace $k[\epsilon]/(\epsilon^2)$ by $k\langle V \rangle$ for a finite-dimensional k-vector space V, gives a bijection $\mathcal{D}_E(k\langle V \rangle) \to H^1(X, \underline{End}(E)) \otimes V$. If $\phi : V \to W$ is a linear map, then the following square commutes.

$$\begin{array}{ccc}
\mathcal{D}_E(k\langle V \rangle) & \to & H^1(X, \underline{End}(E)) \otimes V \\
{\scriptstyle k\langle\phi\rangle} \downarrow & & \downarrow {\scriptstyle id \otimes \phi} \\
\mathcal{D}_E(k\langle W \rangle) & \to & H^1(X, \underline{End}(E)) \otimes W
\end{array}$$

Hence $\mathcal{D}_E$ satisfies **(Hϵ)**, and its tangent space is $H^1(X, \underline{End}(E))$. If X is proper over k, this is finite dimensional, so $\mathcal{D}_E$ satisfies **(H3)** in that case.

Theorem 3.6 *Let X be a proper scheme over a field k. Let E be a coherent sheaf on X. Then the deformation functor $\mathcal{D}_E$ admits a hull, with tangent space $Ext^1(E,E)$.*

Proof We will show that the conditions **(H1)**, **(H2)**, **(H3)** in the Schlessinger Theorem 2.19 are satisfied by our functor $\mathcal{D}_E$.

Verification of (H1): An element of $\mathcal{D}_E(A') \times_{\mathcal{D}_E(A)} \mathcal{D}_E(A'')$ is an ordered tuple $(\mathcal{F}', \theta', \mathcal{F}'', \theta'')$ where $(\mathcal{F}', \theta') \in \mathcal{D}_E(A')$ and $(\mathcal{F}'', \theta'') \in \mathcal{D}_E(A'')$, such that there exists an isomorphism $\eta : \mathcal{F}'|_A \to \mathcal{F}''|_A$ which makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{F}'|_X & \stackrel{i^*(\eta)}{\to} & \mathcal{F}''|_X \\ \theta' \downarrow & & \downarrow \theta'' \\ E & = & E \end{array}$$

We fix one such η. Let $\mathcal{F} = \mathcal{F}''|_A$, let $u'' : \mathcal{F}'' \to \mathcal{F}$ be the quotient and let $u' : \mathcal{F}' \to \mathcal{F}$ be induced by η. Let $B = A' \times_A A''$, and let $\mathcal{G}$ be the sheaf of $\mathcal{O}_{X_B}$-modules defined by

$$\mathcal{G} = \mathcal{F}' \times_{u', \mathcal{F}, u''} \mathcal{F}''$$

This is clearly coherent, as the construction can be done on each affine open and glued. By Lemma 3.2 applied stalk-wise, the sheaf $\mathcal{G}$ is flat over B. By Lemma 3.3 applied stalk-wise, this is up to isomorphism the only coherent sheaf on X_B, flat over B, which comes with homomorphisms $p' : \mathcal{G} \to \mathcal{F}'$ and $p'' : \mathcal{G} \to \mathcal{F}''$ which make the following square commute:

$$\begin{array}{ccc} \mathcal{G} & \stackrel{p''}{\to} & \mathcal{F}'' \\ p' \downarrow & & \downarrow u'' \\ \mathcal{F}' & \stackrel{u'}{\to} & \mathcal{F} \end{array}$$

This shows that $\mathcal{D}_E(B) \to \mathcal{D}_E(A') \times_{\mathcal{D}_E(A)} \mathcal{D}_E(A'')$ is surjective, as desired. Thus, Schlessinger condition **(H1)** is satisfied.

Caution: If we choose another η, we might get a different $\mathcal{G}$, and so the map $\mathcal{D}_E(B) \to \mathcal{D}_E(A') \times_{\mathcal{D}_E(A)} \mathcal{D}_E(A'')$ may not be injective.

Verification of (H2): If we take A to be k in the above verification of the condition **(H1)**, then η would be unique, and so we will get a bijection $\mathcal{D}_E(A' \times_k A'') \to \mathcal{D}_E(A') \times_{\mathcal{D}_E(k)} \mathcal{D}_E(A'')$. In particular, this implies that **(H2)** is satisfied.

Verification of (H3): We have already seen in the special case when E is locally free that the finite dimensional vector space $H^1(X, \underline{End}(E)) = Ext^1(E,E)$ is the tangent space to $\mathcal{D}_E$. Now we give a proof that for a general coherent E, the tangent space is $Ext^1(E,E)$. This proof is very different in spirit, and in particular it gives another proof in the vector bundle case. For any finite dimensional vector space V over k, we define a map

$$f_V : V \otimes_k Ext^1(E,E) = Ext^1(V \otimes_k E, E) \to \mathcal{D}_E(k\langle V \rangle)$$

as follows. An element of $Ext^1(V \otimes_k E, E)$ is represented by a short exact sequence $S = (0 \to V \otimes_k E \stackrel{i}{\to} F \stackrel{j}{\to} E \to 0)$ of $\mathcal{O}_X$-modules. We give F the structure of an

$\mathcal{O}_{X[V]}$-module (where $X[V] = X \otimes_k k\langle V\rangle$) by defining the scalar-multiplication map $V \otimes_k F \to F$ as the composite $V \otimes_k F \stackrel{(\mathrm{id}_V, j)}{\to} V \otimes_k E \stackrel{i}{\to} F$. We denote the resulting $\mathcal{O}_{X[V]}$-module by $\mathcal{F}_S$. Note that the induced homomorphism $V \otimes_k (\mathcal{F}_S/V\mathcal{F}_S) \to V\mathcal{F}_S$ is an isomorphism, as it is just the identity map on $V \otimes_k E$. Hence by Exercise 3.1.(vi), it follows that $\mathcal{F}_S$ is flat over $k\langle V\rangle$. Hence we indeed get an element of $\mathcal{D}_E(k\langle V\rangle)$, which completes the definition of the map $f_V : V \otimes_k Ext^1(E,E) \to \mathcal{D}_E(k\langle V\rangle)$.

It can be seen from its definition that f is functorial in V, that is, if $\phi : V \to W$ is a linear map, then the following square commutes.

$$\begin{array}{ccc} Ext^1(E,E) \otimes V & \to & \mathcal{D}_E(k\langle V\rangle) \\ {\scriptstyle id\otimes\phi}\downarrow & & \downarrow{\scriptstyle k\langle\phi\rangle} \\ Ext^1(E,E) \otimes W & \to & \mathcal{D}_E(k\langle W\rangle) \end{array}$$

Next, we give an inverse $g_V : \mathcal{D}_E(k\langle V\rangle) \to V \otimes_k Ext^1(E,E)$ to f_V as follows. Given any $(\mathcal{F},\theta) \in \mathcal{D}_E(k\langle V\rangle)$, let $F = \pi_*(\mathcal{F})$ where $\pi : X[V] \to X$ is the projection induced by the ring homomorphism $k \hookrightarrow k\langle V\rangle$. Let $j : F \to E$ be the $\mathcal{O}_X$-linear map which is obtained from the $\mathcal{O}_X[V]$-linear map $\mathcal{F} \to \mathcal{F}|_X \stackrel{\theta}{\to} E$ by forgetting scalar multiplication by V. By flatness of $\mathcal{F}$ over $k\langle V\rangle$, the sequence $0 \to V \otimes_{k\langle V\rangle} \mathcal{F} \to \mathcal{F} \to \mathcal{F}|_X \to 0$ obtained by applying $- \otimes_{k\langle V\rangle} \mathcal{F}$ to $0 \to V \to k\langle V\rangle \to k \to 0$ is again exact. As $V \otimes_{k\langle V\rangle} \mathcal{F} = V \otimes_k (\mathcal{F}/V\mathcal{F})$, by composing with θ (and its inverse) this gives an exact sequence $S = (0 \to V \otimes_k E \stackrel{i}{\to} \mathcal{F} \stackrel{j}{\to} E \to 0)$. We define $g_V : \mathcal{D}_E(k\langle V\rangle) \to V \otimes_k Ext^1(E,E)$ by putting $g_V(\mathcal{F},\theta) = S$.

Hence $\mathcal{D}_E$ satisfies **(Hϵ)**, and its tangent space is $Ext^1(E,E)$. If X is proper over k, this is finite dimensional, so $\mathcal{D}_E$ satisfies **(H3)** in that case. This completes the proof of the Theorem 3.6 in the general case of coherent sheaves. $\square$

Pro-Representability for a simple sheaf

Theorem 3.7 *Let X be a proper scheme over a field k, and let F be a coherent sheaf on X. Assume that there exists an exact sequence $E_1 \to E_0 \to F \to 0$ of $\mathcal{O}_X$-modules, where E_1 and E_0 are locally free (note that this condition is automatically satisfied when F itself is locally free, or when X is projective over k). If the ring homomorphism $k \to End(F)$ (under which k acts on F by scalar multiplication) is an isomorphism, then the deformation functor $\mathcal{D}_F$ is pro-representable.*

Proof Let A be artin local, and let I be a proper ideal. Let $(\mathcal{F},\theta) \in \mathcal{D}_F(A)$, and let $(\mathcal{F}',\theta')$ denote its restriction to A/I. By Lemma 3.5, the natural ring homomorphisms $A \to End_{X\otimes A/I}(\mathcal{F})$ and $A/I \to End_{X\otimes A/I}(\mathcal{F}')$, under which A and A/I act respectively on $\mathcal{F}$ and $\mathcal{F}'$ by scalar multiplication, are isomorphisms. In particular, we get induced group isomorphisms $A^\times \to Aut(\mathcal{F})$ and $(A/I)^\times \to Aut(\mathcal{F}')$. The subgroups $1 + \mathfrak{m}_A \subset A^\times$ and $1 + \mathfrak{m}_{A/I} \subset (A/I)^\times$ therefore map isomorphically onto $Aut(\mathcal{F},\theta)$ and $Aut(\mathcal{F}',\theta')$ respectively. As the homomorphism $1+\mathfrak{m}_A \to 1+\mathfrak{m}_{A/I}$ is surjective, the restriction map $Aut(\mathcal{F},\theta) \to Aut(\mathcal{F}',\theta')$ is again surjective. From this, it follows that the Schlessinger condition **(H4)** is satisfied, and so the functor $\mathcal{D}_F$ is pro-representable by Theorem 2.19. $\square$

Obstruction theory for deformations of a coherent sheaf

Theorem 3.8 *Let X be a projective scheme over a field k, and let F be a coherent sheaf on X. Then the deformation functor $\mathcal{D}_F$ admits an obstruction theory $(Ext^2(F,F),(o_e))$. In particular when $Ext^2(F,F)=0$ the functor $\mathcal{D}_F$ is smooth, that is, for any surjection $B \to A$ in $\mathbf{Art}_k$, the induced map $\mathcal{D}_F(B) \to \mathcal{D}_F(A)$ is surjective.*

Proof Let $\mathcal{O}_X(1)$ be a chosen very ample line bundle on X. Then for for all n sufficiently large, the higher cohomologies $H^i(X,F(n))$ vanish, and by evaluating global sections we get a surjection $q_0 : H^0(X,F(n)) \otimes_k \mathcal{O}_X(-n) \to F$. Choose a large enough n, and let E be the corresponding vector bundle $E = H^0(X,F(n)) \otimes_k \mathcal{O}_X(-n)$. Let Q be the deformation functor of basic example 3, which keeps E fixed and deforms the quotient q_0. For A in $\mathbf{Art}_k$, given any element $q : E \otimes_k A \to \mathcal{F}$ of $Q(A)$, the sheaf $\mathcal{F}$ together with the unique isomorphism $\theta : F \to \mathcal{F} \otimes_A k$ which takes q_0 to $q \otimes_A k$ defines an element $(\mathcal{F},\theta)$ of $\mathcal{D}_F(A)$. This association is functorial, and so we have a forgetful functor $f : Q \to \mathcal{D}_F$. Let A be in $\mathbf{Art}_k$, and let $(\mathcal{F},\theta) \in \mathcal{D}_F(A)$. The surjectivity of $q_0 : H^0(X,F(n)) \otimes_k \mathcal{O}_X(-n) \to F$ implies the surjectivity of the evaluation map $p : H^0(X_A,\mathcal{F}(n)) \otimes_A \mathcal{O}_{X_A}(-n) \to \mathcal{F}$. As $\mathcal{F}$ is flat over A and as higher cohomologies of $F(n)$ are zero, it follows that $H^0(X_A,\mathcal{F}(n))$ is a free A-module, of the same rank as $\dim_k H^0(X,F(n))$. Hence we can choose an isomorphism $\phi : H^0(X,F(n)) \otimes_k A \to H^0(X_A,\mathcal{F}(n))$ which restricts to identity modulo $\mathfrak{m}_A$. Consider the composite surjection $q = p \circ \phi : E \otimes_k A \to \mathcal{F}$. Then $q \in Q(A)$ maps to $(\mathcal{F},\theta) \in \mathcal{D}_F(A)$ under the forgetful functor $f : Q \to \mathcal{D}_F$. This shows that the forgetful functor $f : Q \to \mathcal{D}_F$ is formally smooth. We have shown later (Theorem 3.11) that Q has an obstruction theory taking values in $Ext^1_X(G,F)$ where G is the kernel of $q : E \to F$. We have $Ext^i_X(E,F) = H^i(X,\underline{Hom}(E,F)) = H^i(X,F(n)) \otimes H^0(X,F(n))^* = 0$ for all $i \geq 1$. Applying $Hom_X(-,F)$ to the short-exact sequence $0 \to G \to E \to F \to 0$ therefore gives an isomorphism $\partial : Ext^1_X(G,F) \to Ext^2_X(F,F)$. If $e = (0 \to I \to B \to A \to 0)$ is a small extension in $\mathbf{Art}_k$, we have an obstruction map $o_e : Q(A) \to Ext^1_X(G,F) \otimes I$.
Let $q' : E_A \to \mathcal{F}$ be another homomorphism with $q'|_X = q_0 = q|_X$. Then q' is necessarily surjective, and all such q' form the fiber of $Q(A) \to \mathcal{D}_F(A)$ containing q. Let $\mathcal{G} = \ker(q)$ and $\mathcal{G}' = \ker(q')$. These are flat over A, with $\mathcal{G}|_X = G = \mathcal{G}'_X$. The corresponding obstruction classes ω_A and ω'_A for lifting these to $Q(B)$ (see the proof of Theorem 3.11) respectively lie in the vector spaces $Ext^1_{X_A}(I \otimes_k F, \mathcal{G})$ and $Ext^1_{X_A}(I \otimes_k F, \mathcal{G}')$. But under the isomorphisms of these spaces with $Ext^1_X(G,F) \otimes I$, it follows from $q'|_X = q_0 = q|_X$ and $\mathcal{G}|_X = G = \mathcal{G}'_X$ that ω_A and ω'_A map to the same element of $Ext^1_X(G,F) \otimes I$, that is, $o_e(q) = o_e(q') \in Ext^1_X(G,F) \otimes I$. Therefore o_e is constant on fibers of $Q(A) \to \mathcal{D}_F(A)$, so we get a unique map

$o'_e : \mathcal{D}_F(A) \to Ext^2_X(F,F) \otimes I$ such that the following diagram commutes:

$$\begin{array}{ccccc} Q(B) & \to & Q(A) & \stackrel{o_e}{\to} & Ext^1_X(G,F)\otimes I \\ {\scriptstyle f_B}\downarrow & & {\scriptstyle f_A}\downarrow & & \downarrow {\scriptstyle \partial\otimes \mathrm{id}_I} \\ \mathcal{D}_F(B) & \to & \mathcal{D}_F(A) & \stackrel{o'_e}{\to} & Ext^2_X(F,F)\otimes I \end{array}$$

From the commutativity of the above diagram, the surjectivity of the first two vertical maps and the fact that the last vertical map is an isomorphism, it follows that the lower row is exact. We leave the verification of the functoriality condition on o'_e (using the functoriality of o_e) as an exercise to the reader. Thus, $(Ext^2_X(F,F), (o'_e))$ is an obstruction theory for $\mathcal{D}_F$. $\square$

Example 3.9 Consider the projective line $\mathbf{P}^1$ over k with standard open cover $U_0 = \operatorname{Spec} k[z]$ and $U_\infty = \operatorname{Spec} k[z^{-1}]$. Let $E = \mathcal{O}(-1) \oplus \mathcal{O}(1)$. Then the tangent space $Ext^1(E,E)$ to $\mathcal{D}_E$ is 1-dimensional, and $\mathcal{D}_E$ is formally smooth as $Ext^2(E,E) = 0$. Hence E admits a hull parametrised by the formal power series ring $k[[t]]$. The transition function

$$g_{0,\infty}(t) = \left(\begin{array}{cc} z^{-1} & t \\ 0 & z \end{array} \right)$$

over the open cover of $\mathbf{P}^1 \times \mathbf{A}^1$ given by $U_0 \times \mathbf{A}^1$ and $U_\infty \times \mathbf{A}^1$ defines a vector bundle $\mathcal{E}(t)$ on $\mathbf{P}^1 \times \mathbf{A}^1$. As $g_{0,\infty}(0) = \mathrm{diag}(z^{-1}, z)$, this comes with an isomorphism $\theta : \mathcal{E}(t)|_{\mathbf{P}^1} \to E$. (This is actually the universal family of extensions of $\mathcal{O}(1)$ by $\mathcal{O}(-1)$.) Going modulo (t^2), the restriction $(\mathcal{E}(\epsilon), \theta)$ gives a universal first order family for $\mathcal{D}_E$ parametrised by $k[\epsilon]/(\epsilon^2)$. Hence by inverse function theorem for $k[[t]]$, the pro-family $(\mathcal{E}(t), \theta)$ (obtained by restrictions to each $k[t]/(t^n)$) is miniversal. The pro-family $(\mathcal{E}(t+t^2), \theta)$ defined by transition matrix $g(t+t^2)$ is isomorphic to the original family, as $g(t+t^2) = h(t)g(t)h(t)^{-1}$ where $h(t) = diag(1+t, 1)$ is invertible over $k[[t]]$, with $h(0) = I$. Hence the non-trivial automorphism $k[[t]] \to k[[t]]$ defined $t \mapsto t + t^2$ pulls back the miniversal family to another such. Hence the family is not universal. Hence the condition of simplicity in Theorem 3.7 is not superfluous.
Finally, for the moduli functor $\mathcal{M}_E$, the set $\mathcal{M}_E(k[\epsilon]/(\epsilon^2))$ has exactly two elements, namely $[\mathcal{E}(\epsilon)]$ and $[\mathcal{E}(0)]$. This cannot be a vector space when k is larger than $\mathbb{Z}/(2)$. In particular the functor $\mathcal{M}_E$ does not have the good properties of $\mathcal{D}_E$.

Homological preliminaries for the Quot functor

Consider a short-exact sequence $s = (0 \to M' \to M \to M'' \to 0)$ in an abelian category, together with monomorphisms $u' : N' \to M'$ and $u'' : N'' \to M''$. An **exact filler** for (s, u', u'') will mean a monomorphism $u : N \hookrightarrow M$ such that we have morphisms $N' \to N$ and $N \to N''$ (necessarily unique) which give the following commutative diagram with short-exact rows.

$$\begin{array}{ccccccc} 0 \to & N' & \to & N & \to & N'' & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & M' & \to & M & \to & M'' & \to 0 \end{array}$$

Lemma 3.10 *Let $\mathcal{C}$ be an abelian category and let $s = (0 \to M' \xrightarrow{i} M \xrightarrow{j} M'' \to 0)$ be a short exact sequence in $\mathcal{C}$. Let $u' : N' \hookrightarrow M'$ and $u'' : N'' \hookrightarrow M''$ be given sub-objects. Then the following holds.*
(1) Under the natural map $Ext^1_{\mathcal{C}}(M'', M') \to Ext^1_{\mathcal{C}}(N'', M'/N')$, the image of the class s is the class ω of the induced short exact sequence

$$\omega = (0 \to M'/N' \to j^{-1}N''/N' \to N'' \to 0).$$

There exists an exact filler for (s, u', u'') if and only if $\omega = 0$.
(2) The set $\mathbf{S}$ of all isomorphism classes of exact fillers for the above diagram is in a natural bijection φ (described within the proof) with the set $\mathbf{L}$ of all lifts $h : N'' \to M/N'$ of $N'' \hookrightarrow M''$. The set $\mathbf{L}$ admits a natural action of the abelian group $Hom_{\mathcal{C}}(N'', M'/N')$, under which an element $\alpha \in Hom_{\mathcal{C}}(N'', M'/N')$ acts by $h \mapsto h+\alpha$. This action makes $\mathbf{L}$ (and hence also $\mathbf{S}$ via φ) a principal $Hom_{\mathcal{C}}(N'', M'/N')$-set (which by (1) is non-empty if and only if $\omega = 0$).
*(3)***(Equivariance)** *With notation as before, suppose we have a commutative square*

$$\begin{array}{ccc} N' & \to & L' \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle v'} \\ M' & \to & K' \end{array}$$

where $v' : L' \to K'$ is monic. Let $f : Hom_{\mathcal{C}}(N'', M'/N') \to Hom_{\mathcal{C}}(N'', K'/L')$ denote the homomorphism induced by the above commutative square. Let $K = K' \coprod_{M'} M$ be the push-out, so that we have the following commutative diagram with short-exact rows.

$$\begin{array}{ccccccc} 0 \to & M' & \to & M & \to & M'' & \to 0 \\ & \downarrow & & \downarrow & & || & \\ 0 \to & K' & \to & K & \to & M'' & \to 0 \end{array}$$

Let s' denote the bottom row of the above diagram, and let $\mathbf{S}'$ be the set of isomorphism classes of exact fillers for (s', v', u'') which by (2) is in natural bijection with the set $\mathbf{L}'$ of section over N'' of $K/L' \to M''$. Given any $h : N'' \to M/N'$ in $\mathbf{L}$, we get an element $h' : N'' \to K/L'$ in $\mathbf{L}'$ by composing with the homomorphism $M/N' \to K/L'$. This defines a natural map $\mathbf{S} \to \mathbf{S}'$, which is equivariant under the homomorphism $f : Hom_{\mathcal{C}}(N'', M'/N') \to Hom_{\mathcal{C}}(N'', K'/L')$.

Proof Let $j^{-1}N'' \subset M$ denote the pull-back of $N'' \subset M''$ under $j : M \to M''$. By definition, the image ω of e in $Ext^1_{\mathcal{C}}(N'', M'/N')$ is the extension class of the short exact sequence $0 \to M'/N' \to j^{-1}N''/N' \to N'' \to 0$. Therefore $\omega = 0$ if and only if there exists a 'lift' $h : N'' \hookrightarrow M/N'$ with $\overline{j} \circ h = u'' : N'' \hookrightarrow M''$ where $\overline{j}$ is induced by j.
An exact filler $u : N \hookrightarrow M$ induces a sub-object $\overline{u} : N/N' \hookrightarrow M/N'$. As we have an isomorphism $N/N' \to N''$, this gives a lift $h : N'' \to M/N'$ of $u'' : N'' \hookrightarrow M''$. We define $\varphi : \mathbf{S} \to \mathbf{L}$ by $u \mapsto h$. Conversely, the pull-back of the sub-object (N'', h) of M/N' under the quotient morphism $M \to M/N'$ is a sub-object (N, u) of M, which defines an inverse for φ, showing it is a bijection.
It is clear that $\mathbf{L}$ is a principal $Hom_{\mathcal{C}}(N'', M'/N')$-set under the given action. The rest of the lemma is now a simple exercise. □

Pro-representability and tangent space for the Quot functor

Let X be a proper scheme over k. Let E be a coherent $\mathcal{O}_X$-module over X, and let $q_0 : E \to F_0$ be a coherent quotient $\mathcal{O}_X$-module. Let Q be the deformation functor for the above quotient, which was introduced as our basic example 3.
The following result is essentially due to Grothendieck, though re-cast in the language of Schlessinger.

Theorem 3.11 *Let k be any field, X proper over k, and let $q_0 : E \to F_0$ be a surjective morphism of coherent $\mathcal{O}_X$-modules. Let Q denote the corresponding deformation functor. Then we have the following.*
(1) The functor Q is pro-representable.
(2) It has tangent space $T_Q = Hom_X(G_0, F_0)$ where $G_0 = \ker(q_0)$.
(3) There exists a deformation theory for Q taking values in $Ext^1_X(G_0, F_0)$. In particular if $Ext^1_X(G_0, F_0) = 0$, then the functor Q is formally smooth.

We begin with the proof of pro-representability.
If $X \to \operatorname{Spec} k$ is projective, then as proved by Grothendieck (see for example [Ni]), there exists a scheme $Quot_{E/X}$ (the **quot scheme**) of locally finite type over k, whose S-valued points for any k-scheme S are equivalence classes of S-flat families of coherent quotients of E over $X \times_k S$. The given quotient $E \to F_0$ defines a k-rational point $q_0 \in Quot_{E/X}$. Hence the functor Q is just the functor of deformations of the point q_0 in $Quot_{E/X}$, so this is a case of basic example 1, hence is representable (which is more than being pro-representable).
The general case, where the proper morphism $X \to \operatorname{Spec} k$ need not be projective, is treated via Schlessinger's theorem for pro-representability, which has the following obvious corollary.

Theorem 3.12 *A deformation functor φ is pro-representable if and only if the following two conditions are satisfied.*
(1) For any morphisms $A' \to A$ and $A'' \to A$ in $\mathbf{Art}_k$ such that $A'' \to A$ is surjective, the induced map $\varphi(A' \times_A A'') \to \varphi(A') \times_{\varphi(A)} \varphi(A'')$ is a bijection of sets.
(2) The tangent vector space $T_\varphi = \varphi(k[\epsilon]/(\epsilon)^2)$ is finite dimensional (this is indeed a vector space when (1) is satisfied).

We now show that the condition (1) in the Theorem 3.12 is satisfied by our functor Q. An element of $Q(A') \times_{Q(A)} Q(A'')$ has the form (q', q''), where $q' : E_{A'} \to \mathcal{F}'$ and $q'' : E_{A''} \to \mathcal{F}''$ are coherent quotients over $X_{A'}$ and $X_{A''}$ respectively, with $\mathcal{F}'$ flat over A' and $\mathcal{F}''$ flat over A'', such that there exists an isomorphism $\eta : \mathcal{F}'|_A \to \mathcal{F}''|_A$ such that the following diagram commutes:

$$\begin{array}{ccc} E_A & = & E_A \\ {\scriptstyle q'|_A} \downarrow & & \downarrow {\scriptstyle q''|_A} \\ \mathcal{F}'|_A & \stackrel{\eta}{\to} & \mathcal{F}''|_A \end{array}$$

Note that if a η exists as above, it is necessarily unique by surjectivity of the vertical maps.

Note The above uniqueness of η is the reason why the functor Q is pro-representable, in contrast with the functor of deformations of a coherent sheaf, where we only had a hull. Recall that the corresponding isomorphism η was not unique in the case of the functor of deformations of a coherent sheaf.

Let $\mathcal{F} = \mathcal{F}''|_A$, let $u'' : \mathcal{F}'' \to \mathcal{F}$ be the quotient and let $u' : \mathcal{F}' \to \mathcal{F}$ be induced by η. Let $B = A' \times_A A''$, and let $\mathcal{G}$ be the sheaf of $\mathcal{O}_{X_B}$-modules defined by

$$\mathcal{G} = \mathcal{F}' \times_{u',\mathcal{F},u''} \mathcal{F}''$$

This is clearly coherent, as the construction can be done on each affine open and glued. By Lemma 3.2 applied stalk-wise, the sheaf $\mathcal{G}$ is flat over B. By Lemma 3.3 applied stalk-wise, this is up to isomorphism the only coherent sheaf on X_B, flat over B, which comes with homomorphisms $p' : \mathcal{G} \to \mathcal{F}'$ and $p'' : \mathcal{G} \to \mathcal{F}''$ which make the following square commute:

$$\begin{array}{ccc} \mathcal{G} & \stackrel{p''}{\to} & \mathcal{F}'' \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle u''} \\ \mathcal{F}' & \stackrel{u'}{\to} & \mathcal{F} \end{array}$$

Next, let $p : E_B \to \mathcal{G}$ be the $\mathcal{O}_{X_B}$-linear homomorphism induced by (q', q''). This is clearly surjective, and is the only $\mathcal{O}_{X_B}$-linear homomorphism which restricts to q' and q'' over A' and A''. This shows that $Q(B) \to Q(A') \times_{Q(A)} Q(A'')$ is bijective, as desired.

Therefore, to complete the proof of pro-representability, it only remains to verify the condition (2) of Theorem 3.12. This we do next, when we determine the space T_Q.

Remark 3.13 Let $B \to A$ be surjection of rings with kernel I, such that $I^2 = 0$, so that I is naturally an A-module. Then the natural map $I \otimes_B M \to I \otimes_A (M/IM) : \sum b_i \otimes_B x_i \mapsto \sum b_i \otimes_A \overline{x_i}$ is an isomorphism of B-modules for any B-module M.

Lemma 3.14 *Let $B \to A$ be surjection of rings with kernel I, such that $I^2 = 0$. Let M be a flat B-module (not necessarily finitely generated). Let $u'' : \mathcal{G} \hookrightarrow A \otimes_B M = M/IM$ be an A-submodule, such that the quotient $\mathcal{F} = (A \otimes_B M)/\mathcal{G}$ is a flat A-module. In particular, the induced map $u' : I \otimes_A \mathcal{G} \to I \otimes_A (A \otimes_B M) = I \otimes_B M$ is monic, where the last equality is by Remark 3.13. As M is B-flat, the sequence $s = (0 \to I \otimes_B M \to M \to A \otimes_B M \to 0)$ is exact. For any exact filler $u : N \hookrightarrow M$ of (s, u', u''), consider the resulting commutative diagram of B-modules*

$$\begin{array}{ccccccccc} 0 \to & I \otimes_A \mathcal{G} & \to & N & \to & \mathcal{G} & \to 0 \\ & {\scriptstyle u'}\downarrow & & {\scriptstyle u}\downarrow & & {\scriptstyle u''}\downarrow & \\ 0 \to & I \otimes_B M & \to & M & \to & A \otimes_B M & \to 0 \end{array}$$

where the rows are exact. Then we have the following:
(1) The submodule $I \otimes_A \mathcal{G} \subset N$ from the top row is the submodule $IN \subset N$. Consequently, the quotient map $N \to \mathcal{G}$ induces an isomorphism $A \otimes_B N \to \mathcal{G}$.
(2) The quotient module M/N is flat over B.

Proof (1) As I annihilates $\mathcal{G}$, from the surjection $N \to \mathcal{G}$ it follows that $IN \subset \ker(N \to \mathcal{G}) = I \otimes_A \mathcal{G}$. For the reverse inclusion, consider an element $b \otimes_A g$ of $I \otimes_A \mathcal{G}$. Under the inclusion $\mathcal{G} \subset A \otimes_B M$, it follows that g can be written as $1 \otimes_B x$ for some $x \in M$. Then $b \otimes_A g$ maps to the element $b \otimes_A (1 \otimes_B x) = bx \in M$ under the composite $I \otimes_A \mathcal{G} \hookrightarrow I \otimes_B M \hookrightarrow M$. As $N \to \mathcal{G}$ is surjective, there exists some element $y \in N \subset M$ which maps to $g \in \mathcal{G}$, that is, $1 \otimes_B y = g \in A \otimes_B M$. This means $1 \otimes_B x = g = 1 \otimes_B y$, so $1 \otimes_B (x-y) = 0 \in A \otimes_B M$, which means $x - y \in IM$. As by assumption $I^2 = 0$, it follows that $bx = by$. This shows $bx \in IN$, proving the desired inclusion $I \otimes_A \mathcal{G} \subset IN$.
(2) As $N \to \mathcal{G}$ is the quotient $N \to N/IN$, it follows that $M/N \to \mathcal{F}$ is the quotient $(M/N)/I(M/N)$. In other words, applying $A \otimes_B -$ to $0 \to N \to M \to M/N \to 0$ produces the exact sequence $0 \to \mathcal{G} \to M/IM \to \mathcal{F} \to 0$ of A-modules. As M is flat over B so $Tor_1^B(A, M) = 0$, the above exact sequence shows that $Tor_1^B(A, M/N) = 0$. Moreover, by assumption $A \otimes_B (M/N) = \mathcal{F}$ is flat over A. Hence by Exercise 3.1.(vi), it follows that M/N is B-flat as desired. □

Let $B \to A$ be a small extension in $\mathbf{Art}_k$ with kernel I, let $q : E_A \to \mathcal{F}$ be in $Q(A)$, and let $\mathcal{G} \hookrightarrow E_A$ be the kernel of $q : E_A \to \mathcal{F}$. Note that $I \otimes_A \mathcal{G} = I \otimes_k G_0$ as $\mathfrak{m}_B I = 0$. Hence the above lemma has the following immediate corollary.

Lemma 3.15 *If $B \to A$ is a small extension in $\mathbf{Art}_k$ with kernel I, then the fiber of $Q(B) \to Q(A)$ over $(q : E_A \to \mathcal{F}) \in Q(A)$ is in a natural bijection with the set of all exact fillers of the diagram*

$$\begin{array}{ccccccc}
 & I \otimes_k G_0 & & & \mathcal{G} & & \\
 & \downarrow & & & \downarrow & & \\
0 \to & I \otimes_k E & \to & B \otimes_k E & \to & A \otimes_k E & \to 0
\end{array}$$

To determine the tangent space T_Q, we apply the above description of the fibers of $Q(B) \to Q(A)$ in the case where $A = k$ and $B = k\langle V\rangle$ for any finite dimensional k-vector space V. As $Q(k)$ is a singleton set, this shows that $Q(k\langle V\rangle)$ is in a natural bijection with the set $\mathbf{S}_V$ of all exact fillers of the diagram

$$\begin{array}{ccccccc}
 & V \otimes_k G_0 & & & G_0 & & \\
 & \downarrow & & & \downarrow & & \\
0 \to & V \otimes_k E & \to & k\langle V\rangle \otimes_k E & \to & E & \to 0
\end{array}$$

The set $\mathbf{S}_V$ has a natural base-point $*_V$, given by the filler $k\langle V\rangle \otimes_k G_0 \hookrightarrow k\langle V\rangle \otimes_k E$. By Lemma 3.10, the set $\mathbf{S}_V$ is naturally a principal set under $Hom_X(G_0, V \otimes_k F_0)$. Therefore the base point gives a bijection $Hom_X(G_0, V \otimes_k F_0) \to \mathbf{S}_V$. Given any linear map $V \to W$ of finite dimensional k-vector spaces, by Lemma 3.10, we get an induces map $\mathbf{S}_V \to \mathbf{S}_W$, which is equivariant under the induces group homomorphism $Hom_X(G_0, V \otimes_k F_0) \to Hom_X(G_0, W \otimes_k F_0)$. Also, it maps the base point $*_V$ to the base point $*_W$. Hence the bijection $Hom_X(G_0, V \otimes_k F_0) \to \mathbf{S}_V$ is functorial on the category of finite dimensional k-vector spaces. Composing with the natural bijection $\mathbf{S}_V \to Q(k\langle V\rangle)$, we get a natural bijection $Hom_X(G_0, V \otimes_k F_0) \to Q(k\langle V\rangle)$ in the category of finite dimensional k-vector spaces.

This shows that the tangent space to Q is $Hom_X(G_0, F_0)$, proving Theorem 3.11.(2). As X is proper over k, the vector space $Hom_X(G_0, F_0)$ is finite dimensional, which completes the proof of Theorem 3.11.(1) via the Schlessinger criterion **(H3)**.

Obstruction theory for Q

We now prove Theorem 3.11.(3). Given a small extension $e = (0 \to I \to B \to A \to 0)$, we define a map $o_e : Q(A) \to Ext^1_X(G_0, F_0) \otimes_k I$ as follows. By Lemma 3.15, the fiber of $Q(B) \to Q(A)$ over an element $(E_A \to \mathcal{F})$ is the set of exact fillers of the diagram

$$\begin{array}{ccccccc} & I \otimes_A \mathcal{G} & & & \mathcal{G} & \\ & \downarrow & & & \downarrow & \\ 0 \to & I \otimes_A E_A & \to & E_B & \to & E_A & \to 0 \end{array}$$

where $\mathcal{G} = \ker(E_A \to \mathcal{F})$. Let $s \in Ext^1_{X_B}(E_A,\, I \otimes_A E_A)$ be the extension class of $0 \to I \otimes_A E_A \xrightarrow{i} E_B \xrightarrow{j} E_A \to 0$. By Lemma 3.10(1), an exact filler exists for the above diagram if and only if its image $\omega = 0$, where $\omega \in Ext^1_{X_B}(\mathcal{G},\, I \otimes_A \mathcal{F})$ is the extension class of

$$0 \to \frac{I \otimes_A E_A}{I \otimes_A \mathcal{G}} \to \frac{j^{-1}(\mathcal{G})}{i(I \otimes_A \mathcal{G})} \to \mathcal{G} \to 0$$

As $(I \otimes_A E_A)/(I \otimes_A \mathcal{G}) = I \otimes_A \mathcal{F} = I \otimes_k F_0$, we regard ω as an element of $Ext^1_{X_B}(\mathcal{G}, I \otimes_k F_0)$. As the module $j^{-1}(\mathcal{G})/i(I \otimes_A \mathcal{G})$ is annihilated by I, the above short exact sequence is a sequence of $\mathcal{O}_{X_A}$-modules, hence ω corresponds to an element

$$\omega_A \in Ext^1_{X_A}(\mathcal{G}, I \otimes_k F_0).$$

As X_A is projective over A, the module $\mathcal{G}$ admits a locally free resolution $\ldots \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{G} \to 0$. As $\mathcal{G}$ is flat over A, the above restricted to X gives a resolution $\ldots \to L_1 \to L_0 \to G_0 \to 0$. The functor Ext can be calculated by using a locally free resolution (see for example Hartshorne [H] Chapter III Proposition 6.5). We therefore have

$$\begin{aligned} Ext^1_{X_A}(\mathcal{G}, I \otimes_k F_0) &= h^1(\underline{Hom}_{\mathcal{O}_{X_A}}(\mathcal{L}_\bullet, I \otimes_k F_0)) \text{ by [H] Chapter III Prop. 6.5} \\ &= h^1(\underline{Hom}_{\mathcal{O}_X}(L_\bullet, I \otimes_k F_0)) \text{ as } I \otimes_k F_0 \text{ is annihilated by } \mathfrak{m}_A \\ &= I \otimes_k h^1(\underline{Hom}_{\mathcal{O}_X}(L_\bullet, F_0)) \\ &= I \otimes_k Ext^1_X(G_0, F_0) \text{ again by [H] Chapter III Prop. 6.5.} \end{aligned}$$

Hence the element ω associated to a given element of $Q(A)$ and a small extension e defines an element $\omega_k \in I \otimes_k Ext^1_X(G_0, F_0)$ We now define $o_e : Q(A) \to Ext^1_X(G_0, F_0) \otimes I$ by sending $q \mapsto \omega_A$. By its definition, this gives an exact sequence $Q(B) \to Q(A) \xrightarrow{o_e} Ext^1_X(G_0, F_0) \otimes I$. The reader may verify from its definition that o_e is functorial in e.

This completes the proof of the theorem. $\square$

Deformations of a variety

Though our focus in these notes has been on vector bundles, historically the main source of motivation and ideas in deformation theory has been the study of deformations a variety. We have room here only to mention some basic facts.

Given a smooth variety X over a field k, consider the deformation functor $\mathbf{Def}_X$ of our basic example 4. Then $\mathbf{Def}_X$ admits a hull, with tangent space $H^1(X, T_X)$, where T_X is the tangent bundle of X. In particular if X is affine then all its infinitesimal deformations are trivial. When C is a smooth projective curve of genus $g \geq 2$, the tangent space is of dimension $3g-3$. This is the so called 'Riemann's count', which is the historical beginning of deformation theory. When X is smooth, the functor $\mathbf{Def}_X$ admits an obstruction theory taking values in $H^2(X, T_X)$. For more on this subject, the reader can see the book by Sernesi [Se] and the notes by Vistoli [V].

Some suggestions for further reading

The above brief notes give just the beginning of the algebraic approach to deformation theory. For a graduate student wishing to study further, the next basic topic I would like to suggest is the use of the Grothendieck existence theorem to convert pro-families into 'actual' families. For this and further theoretical development of algebraic deformation theory (including the cotangent complex) with some of its important applications, a good introductory source is the lecture notes of Luc Illusie [I]. To see some examples of the application of basic deformation theory to vector bundles and moduli, the reader can consult the textbook of Huybrechts and Lehn [L-H].

Finally, I take this opportunity to express my gratitude to Peter Newstead. His lecture notes [Ne] and other writings have helped me (and in fact an entire generation of algebraic geometers) learn the foundations of vector bundles and moduli theory.

References

[EGA] A. Grothendieck : *Eléments de Géométrie Algébrique I - IV*, Pub. Math. IHES, 1960-1964.

[FGA] A. Grothendieck : *Fondements de Géométrie Algébrique*, Bourbaki Seminar talks, 1957-62.

[FGA Explained] B. Fantechi et al : *Fundamental Algebraic Geometry - Grothendieck's FGA Explained*, Math. Surveys and Monographs 123, Amer. Math. Soc., 2005.

[F-G] B. Fantechi and L. Göttsche : Elementary deformation theory. Chapter 5 of [FGA Explained].

[H] R. Hartshorne : *Algebraic Geometry*, Springer Verlag, 1977.

[I] L. Illusie : Grothendieck's existence theorem in formal geometry. Chapter 8 of [FGA Explained].

[H-L] D. Huybrechts and and M. Lehn : *The Geometry of Moduli Spaces of Sheaves*, Aspects of Mathematics 31, Vieweg Verlag, 1997.

[Mi] J.S. Milne : *Etale cohomology*, Princeton Univ. Press, 1980.

[Ne] P.E. Newstead : *Introduction to moduli problems and orbit spaces*, TIFR lect. notes, Springer 1978.

[Ni] N. Nitsure : Construction of Hilbert and Quot schemes. Chapter 2 of [FGA Explained].

[S] M. Schlessinger : Functors of Artin rings. Trans. Amer. Math. Soc. 130 (1968), 208-222.

[Se] E. Sernesi : *Deformations of Algebraic Schemes*, Springer Verlag, 2006.

[V] A. Vistoli : The deformation theory of local complete intersections. arXiv:alg-geom/9703008v2

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400 005, India. e-mail: `nitsure@math.tifr.res.in`

Revised: December 2010