A short proof of the Göttsche conjecture.

Joint with M Kool + V Shende.

- Steiner 1848
- Cayley
- Salmon

$\Big\}\ \mathbb{P}^2$

⋮

- Ran, Choi $\mathbb{P}^2$
- Vainsencher S
- Di Francesco – Itzykson $\mathbb{P}^2$
- Caporaso – Harris $\mathbb{P}^2$
- Yau – Zaslow (Beauville, Bryan – Leung) K3
- Göttsche S
- Kleiman – Piene S
- A. Liu (S, ω)
- Kazarian S
- Fomin – Mikhalkin $\mathbb{P}^2$
- Y. Tzeng 2010 S

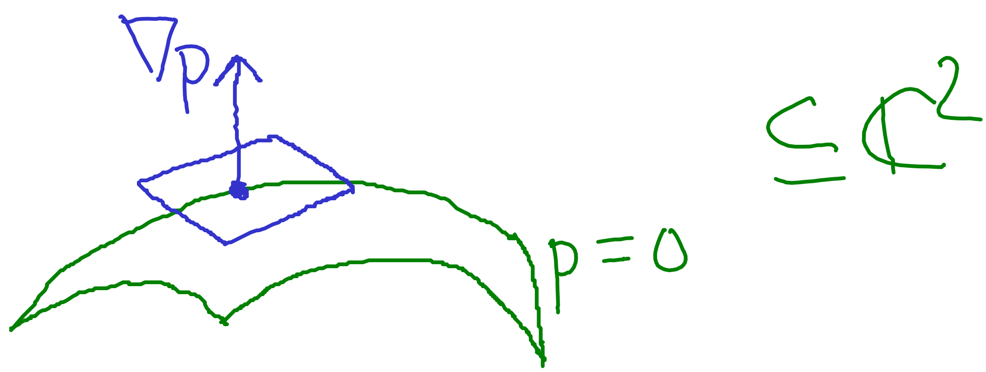
$p(x,y)$ degree n poly in 2 complex variables

Zero locus is complex curve

$$\boxed{\{p(x,y)=0\} \subseteq \mathbb{C}^2}$$

Generically smooth

modelled on copy of $\mathbb{C}$ $\{\underline{v} : \underline{v}.\nabla p = 0\}$ near zeros of p where $\nabla p \neq 0$.



(i.e. local model just $x=0$)

Singular in codimension-1

3 eqn's $p = 0 = \partial_x p = \partial_y p$

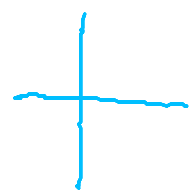
in 2 unknowns x, y

$\Rightarrow$ In a generic 1-parameter family of curves,

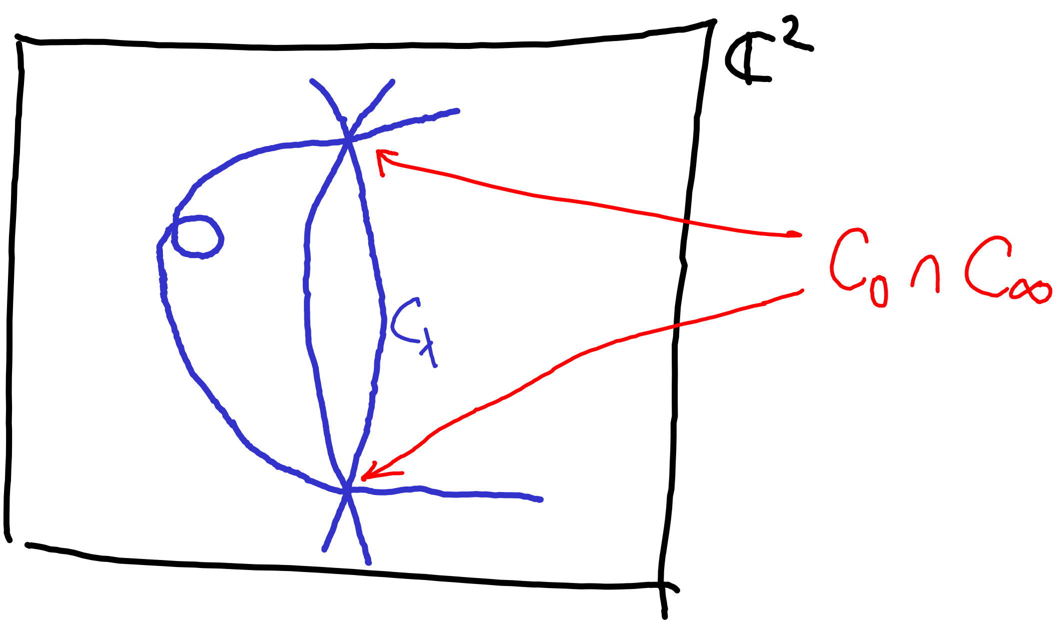
↑ linear, for simplicity

$$C_\lambda := \{ p_\lambda := p_0 + \lambda p_\infty = 0 \}$$

expect a finite number of simplest singularities – <u>nodes</u> local model $\{xy = 0\}$

 $\subseteq \mathbb{C}^2$

(or $\{x^2 + y^2 = 0\}$; p and first derivatives vanish, 2^{nd} derivs nondegenerate)



Calculation with resultant of polys p_0, p_∞

$\Rightarrow$ $(\leq)$ $\underline{3(n-1)^2}$ of them.

(Will give 2 proofs shortly.) First, consider easier case; $p(x)=0$ in $\mathbb{C}$.

Generically smooth (n points).

Linear 1d family $p_\lambda(x) := p_0(x) + \lambda p_\infty(x) = 0$

Singular when get double root, when <u>discriminant</u> of p_λ vanishes.

degree $2(n-1)$ in coefficients of p_λ

$\Rightarrow$ degree $2(n-1)$ in λ

$\Rightarrow$ $2(n-1)$ such double points in our 1d linear family.

Proof via Euler characteristics:

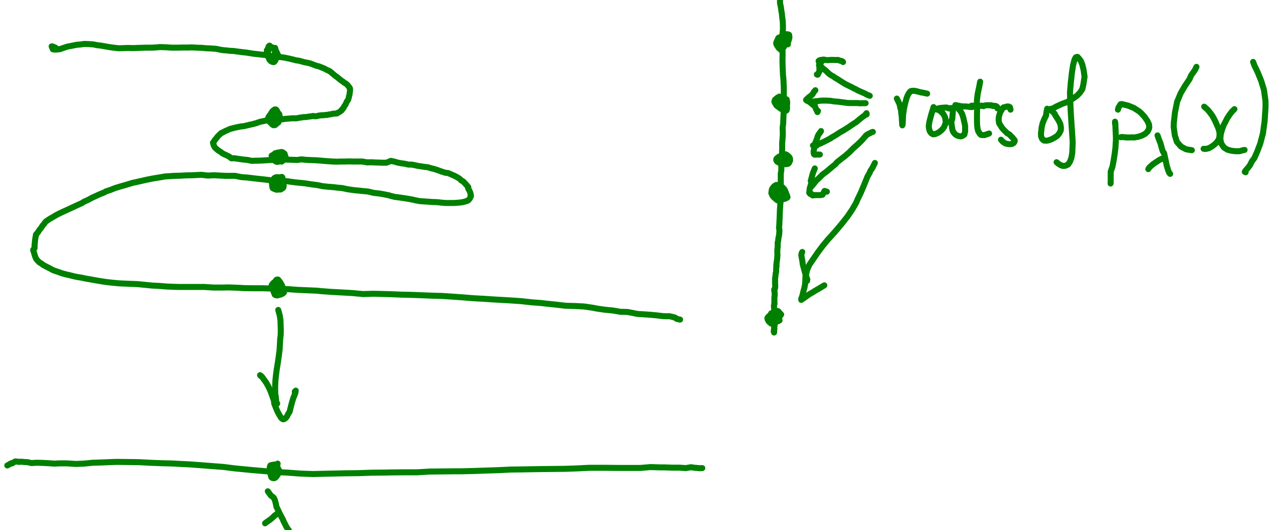
Any $x \in \mathbb{P}^1$ is root of exactly one $P_\lambda = P_0 + \lambda P_\infty$

$(\mathbb{C} \cup \{\infty\})$

(assuming P_0, P_∞ have no common roots)

Gives map $\mathbb{P}^1 \to \mathbb{P}^1$ n-fold cover.

$x \mapsto \lambda$



Inverse image of $\lambda \in \mathbb{P}^1$ is $\begin{cases} n \text{ points} & \text{generically} \\ (n-1) \text{ points} & \text{at double point of } P_\lambda \end{cases}$

Taking Euler characteristics gives

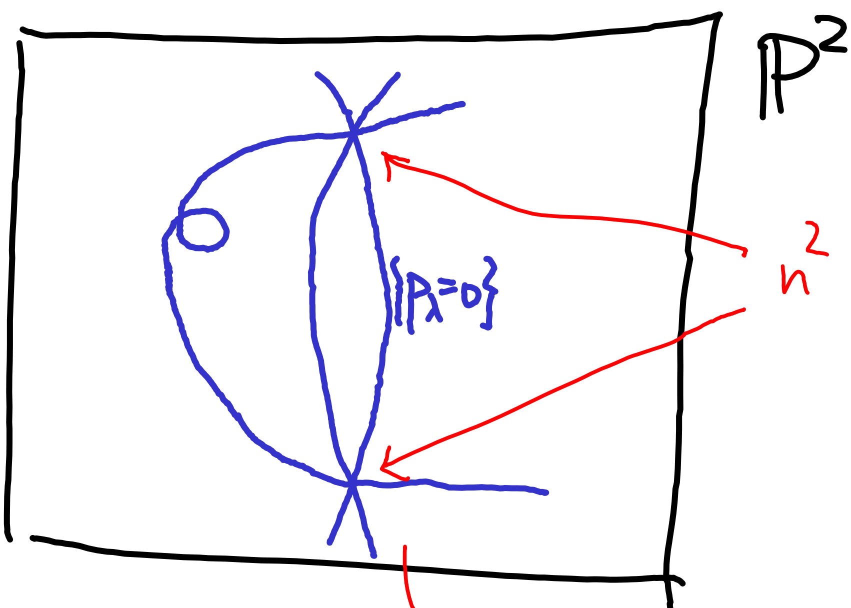
$$2 = n \cdot 2 - \#(\text{double points}) \qquad \square$$

Back to $p_\lambda(x,y)$. i.e. curves in $\mathbb{P}^2$.

$\deg p_\lambda = n \Rightarrow e(\{p_\lambda = 0\}) = 2-(n-1)(n-2)$

when $\{p_\lambda = 0\}$ smooth.

and $3-(n-1)(n-2)$ when 1-nodal



n^2 points of intersection

each punctured smooth curve has $e = 2-(n-1)(n-2) - n^2$

$$e(\mathbb{P}^2) = e(\mathbb{P}^1)\left(2-(n-1)(n-2)-n^2\right) + n^2 + \#\binom{\text{nodal}}{\text{curves}}$$

$$\Rightarrow \#\text{nodal curves} = 3n^2-6n+3 = 3(n-1)^2 \qquad \square$$

General case.

S smooth complex surface

L sufficiently ample

Göttsche: $\geq (5\delta-1)$-ample

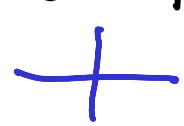
Us: can get away with $\geq \delta$-ample

$$\mathbb{P}(H^0(L)) \longleftrightarrow \text{curves } C \subset S \text{ s.t. } \mathcal{O}(C) = L$$

- Generically smooth
- Singular in codim 1

generic

- 1-d family $\mathbb{P}^1 \subset \mathbb{P}(H^0(L))$ has finite # of curves with only simplest possible singularities: nodes $\{xy=0\} \subseteq \mathbb{C}^2$



generic

• 2-d family $\mathbb{P}^2 \subset \mathbb{P}$ has finite # of curves with 2 nodes (plus finite number with cusps; 1-d family with 1 node)

generic

• δ-dim family $\mathbb{P}^\delta \subset \mathbb{P}$ has finite # of δ-nodal curves (plus much else besides; but crucially can prove all other singular curves have geometric genus $\bar{g} > g - \delta$).

How many?

Conjecturally topological, given by deg δ Universal polynomial in $c_1(L)^2$, $c_1(L).c_1(S)$, $c_1(S)^2$, $c_2(S)$.

So deg 2δ poly in $n = \deg C$ for $(\mathbb{P}^2, \mathcal{O}(n))$

Eg 1-nodal curves in pencil $\mathbb{P}^1 \subseteq \mathbb{P}(H^0(L))$.

Approach #1

Section s of $L \boxtimes \mathcal{O}(1)$ on $S \times \mathbb{P}^1$.

Nodes $\leftrightarrow$ simple zeros of (s, ds)

section of $L \boxtimes \mathcal{O}(1) \oplus T^*_{S\times\mathbb{P}^1} \otimes (L \boxtimes \mathcal{O}(1))$

(or use 1-jets if you don't like connections)

$\Rightarrow$ # nodal curves =

$$c_3\left(L \boxtimes \mathcal{O}(1) \oplus T^*_{S\times\mathbb{P}^1} \otimes (L \boxtimes \mathcal{O}(1))\right)$$

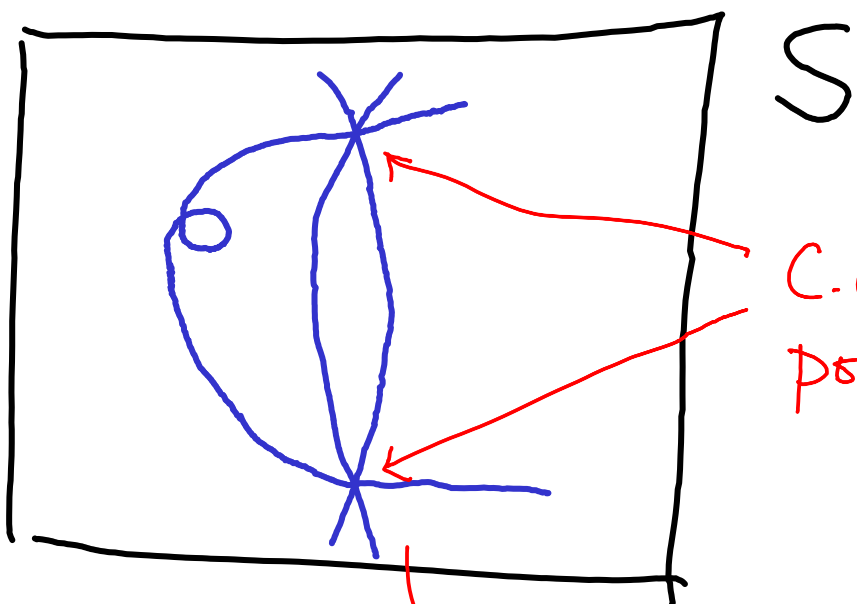
Calculate as $\boxed{c_2(S) + 3c_1(L)^2 + 2c_1(S).c_1(L)}$

Approach #2: Euler characteristics

Smooth curve C has

$$e = 2-2g = -\deg_C(K_C)$$
$$= -\deg_C(K_S \otimes L) \quad \text{by adjunction}$$
$$= (c_1(S) - c_1(L)) . c_1(L)$$

Nodal curve has $e = (2-2g)+1$.

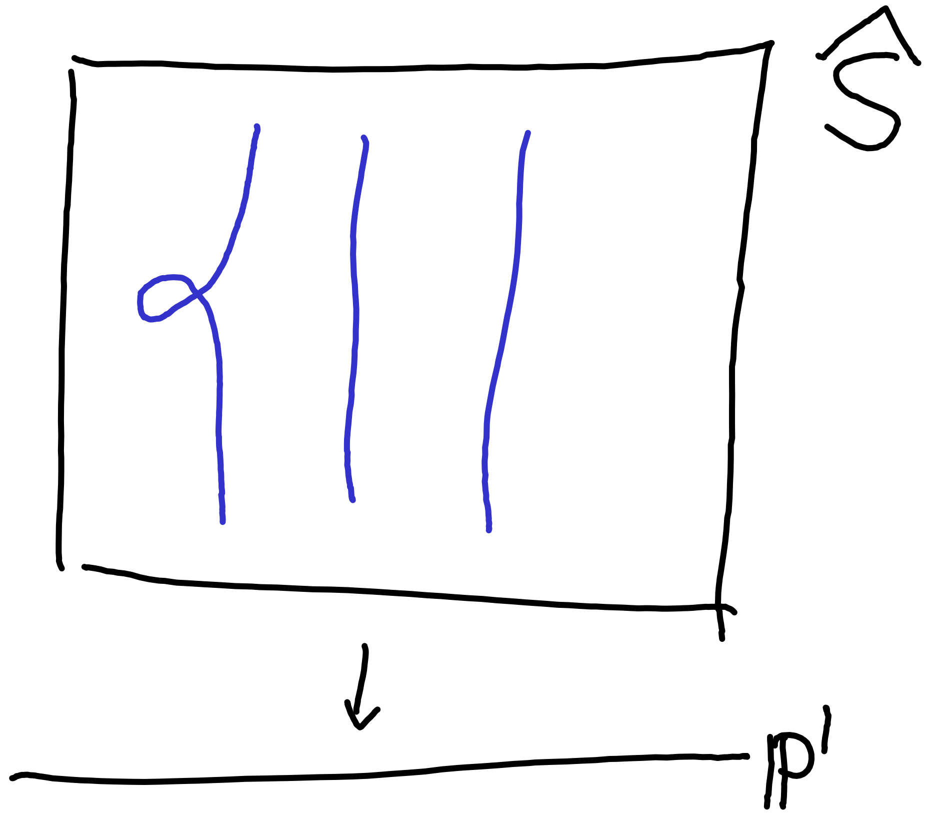


$C.C = c_1(L)^2$ points of intersection

each punctured smooth curve has $e = 2-2g-c_1(L)^2$

$$e(S) = e(\mathbb{P}^1)(2-2g-c_1(L)^2) + c_1(L)^2 + \#\begin{pmatrix}\text{nodal}\\\text{curves}\end{pmatrix}$$

Equivalently blow up S at $c_1(L)^2$ points



to get same formula

$$e(\hat{S}) = e(S) + c_1(L)^2 = e(\mathbb{P}^1)(2-2g) + \#\binom{\text{nodal}}{\text{curves}}$$

this is the <u>universal curve</u> $\mathcal{C} \downarrow \mathbb{P}^1$

$$e(\mathrm{Hilb}^1(\mathcal{C}/\mathbb{P}^1)) = (2g-2)e(\mathrm{Hilb}^0(\mathcal{C}/\mathbb{P}^1)) + \#(\text{nodal curves})$$

The key is the existence of an invariant of curves

$$e(C)-(2-2g) = \begin{cases} 0 & \text{if } C \text{ smooth} \\ 1 & \text{if } C \text{ nodal} \end{cases}$$

$= e(\mathrm{Hilb}^1 C) - (2-2g)\, e(\mathrm{Hilb}^0 C)$

"Integrating" (w.r.t. Euler characteristic) over the base gives # 1-nodal curves.

More generally, for δ-nodal curves we have the following result (joint work with R Pandharipande)

Thm [PT] $\exists$ universal linear combination

$$n_{g-\delta}(C) := \sum_{i=0}^{\delta} \alpha_i \, e(\mathrm{Hilb}^i C)$$

such that, for all reduced curves of arithmetic genus g,

$$n_{g-\delta}(C) = \begin{cases} 0 & \text{if geometric genus of } C \text{ is } > g-\delta \\ 1 & \text{if } C \text{ is } \delta\text{-nodal} \end{cases}$$

(GV BPS invt of C)

Inductively, $n_g(C) := 1$,

$$n_{g-1}(C) := e(C) - n_g(C)(2-2g),$$

$$n_{g-i}(C) := e(\mathrm{Hilb}^i C) - \sum_{j=g-i+1}^{g} n_j(C)\, e(\mathrm{Sym}^{i-(g-j)} \Sigma_j)$$

where Σ_j = smooth curve of genus j

Intuitively, $n_i(C)$ is the number of curves of arithmetic genus i "carried by" C (eg. with a map to C which is generically an iso.)

Eg for δ-nodal C, get a genus $g-i$ curve mapping to it by resolving i nodes. $\binom{\delta}{i}$ such choices, and indeed

$$n_{g-i}(C) = \binom{\delta}{i} \text{ in this case}$$

"Integrating" (w.r.t. Euler characteristic) over the base gives # δ-nodal curves as the same linear combination

$$\sum_{i=0}^{\delta} \alpha_i \, e(\mathrm{Hilb}^i \mathcal{C}/\mathbb{P}^\delta).$$

So need to compute e of

$$\mathrm{Hilb}^i \mathcal{C}/\mathbb{P}^\delta.$$

$$\mathrm{Hilb}^i \mathcal{C}/_{\mathbb{P}^\delta} \subseteq \mathrm{Hilb}^i S \times \mathbb{P}^\delta$$

Cut out by tautological section of a tautological bdl E over $\mathrm{Hilb}^i S \times \mathbb{P}^\delta$.

$$\begin{cases} \text{Fibre } E\big|_{(Z,[s])} = H^0(L|_Z) \otimes \mathcal{O}(1) \\ \qquad\qquad\qquad\qquad\qquad\quad \Psi \\ \text{Section has value } s|_Z \end{cases}$$

curve $s^{-1}(0)$

Vanishes $\Leftrightarrow s|_Z = 0 \Leftrightarrow Z \subseteq C$

Transverse! Zeros $\mathrm{Hilb}^i \mathcal{C}/_{\mathbb{P}^\delta}$ smooth of codimension $= \operatorname{rk} E$.

So we can write $e(\mathrm{Hilb}^i \mathcal{C}/\mathbb{P}^\delta)$ in terms of an integral of Chern classes of E and $T_{\mathrm{Hilb}^i S \times \mathbb{P}^\delta}$ over $\mathrm{Hilb}^i S \times \mathbb{P}^\delta$; and thus over $\mathrm{Hilb}^i S$.

Ellingsrud–Göttsche–Lehn have an algorithm to turn such integrals into integrals over S^i.

$\Rightarrow$ deg i poly in $c_1(L)^2$, $c_1(L).c_1(S)$, $c_1(S)^2$ and $c_2(S)$.

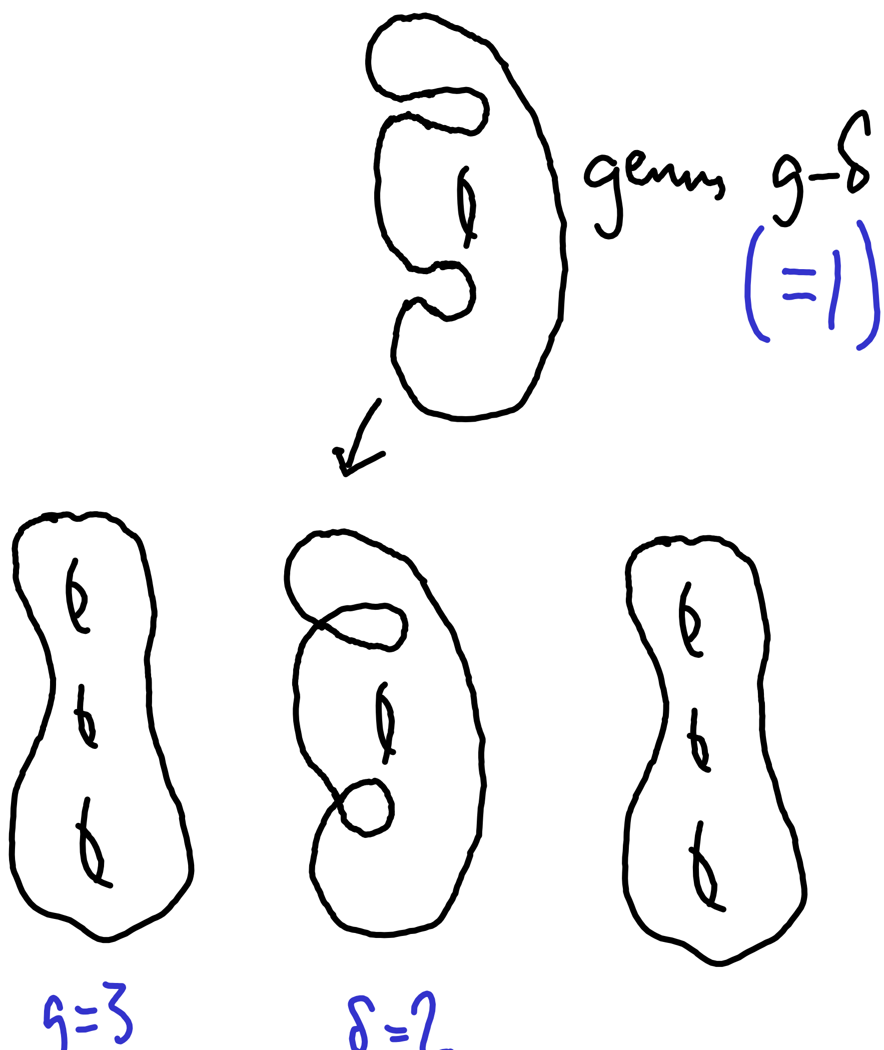
$\Rightarrow$ $n_{g-\delta}$ is deg δ poly in the same.

$\frac{c_2(S)^\delta}{\delta!}$ leading term

Computer code calculates only up to 4 nodes so far....
(correct answers though)

Motivation.

Obvious way to single out δ-nodal curves is via maps from curves of genus $g-\delta$.



Count these via ("reduced") Gromov-Witten theory and GV BPS states. Hard!

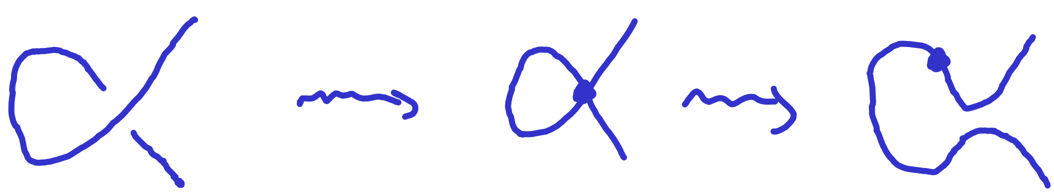
MNOP conjecture

(really for 3-folds)

GW
GV $\longleftrightarrow$ "stable pairs"

Count <u>embedded</u> curves with free points on them ("D-branes")

Change in genus $g \mapsto g-\delta$ $\longleftrightarrow$ up to δ free points



$g \mapsto g+1$

free pts $\uparrow$ $+1$

To understand $e(\mathrm{Hilb}^i)$s of curves of arithmetic genus g, geometric genus $> g-\delta$, may as well remove smooth genus $g-\delta+1$ bit and assume it's a rational curve of arithmetic genus $g=\delta-1$.

Then $AJ: \mathrm{Hilb}^i C \longrightarrow \overline{\mathrm{Pic}_i} C$

has fibre $\underbrace{\mathbb{P}(H^0(L))}_{e=h^0(L)}$ over $L \in \overline{\mathrm{Pic}_i} C$.

$(=1-g+i$ once $i>2g-2)$

$$i = \underbrace{0,1,2,\dots,g-1}, \underbrace{g,g+1,\dots,2g-2}, \underbrace{2g-1,\dots}$$

Serre dual

$L \mapsto L^* \otimes K_C$

e predetermined ($h^0(L)=1-g+i$ once $i>2g-2$)

All $e(\mathrm{Hilb}^i C)$s determined by those for $i \leq g$.

($i \leq g-1$ for AJ fibres; $i=g$ for $\overline{\mathrm{Pic}}$)

$\Rightarrow$ Can write $e(\mathrm{Hilb}^{g+1} C)$ as linear comb. of lower $e(\mathrm{Hilb}^i)$s

$e(\mathrm{Hilb}^\delta) + \alpha_{\delta-1} e(\mathrm{Hilb}^{\delta-1}) + \alpha_{\delta-2} \cdots \equiv 0 \;\; \forall C$ of geom. genus $> g-\delta$

Can check LHS applied to δ-nodal curve gives $+1$.