Multilinear Kakeya type inequalities and factorization

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Multilinear Brascamp–Lieb type inequalities and factorisation

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joint with Anthony Carbery

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Before we begin

- Let $M \geq 0$ be a function in $L^2(\mathbb{R}^2)$ (or $L^2(\mathbb{Z}^2)$ or $L^2(\mathbb{F}_q^2)$)
- Assume $\|M\|_2 = 1$

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Factorisation result

There exist $S(x,y)$, $T(x,y)$ such that

- $M(x,y) = \sqrt{S(x,y)T(x,y)}$ *and*
- $\sup_x \int_y S(x,y)dy \leq 1$
- $\sup_y \int_x T(x,y)dx \leq 1$

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- Follows from

$$\int_x \int_y f(x)g(y)dydx \leq \int_x f(x)dx \int_y g(y)dy$$

Outline

- Background
- A piece of argument from the literature in a model case
- Reversing the implication
- What is the general case
- Some consequences
- The argument for the general case

Where are we coming from

- Kakeya conjecture / inequality
- Multilinear Kakeya inequality, Bennett–Carbery–Tao
- Dvir introduces polynomial method on finite fields
- Guth adapts it to $\mathbb{R}^d$ with algebraic topology

Where are we coming from, part II

- The Brascamp–Lieb inequality

$$\int_{\mathbb{R}^n} \prod_{j=1}^{d} f_j^{r_j}(B_j x) dx \leq C \prod_{j=1}^{d} \left(\int_{\mathbb{R}^{n_j}} f_j(x_j) dx_j \right)^{r_j}$$

  - B_j are linear transformations $\mathbb{R}^n \to \mathbb{R}^{n_j}$
  - r_j are non-negative real numbers

Where are we coming from, part II

- The Brascamp–Lieb inequality
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  - B_j are linear transformations $\mathbb{R}^n \to \mathbb{R}^{n_j}$
  - r_j are non-negative real numbers
- Multilinear Hölder's inequality, if $r_1 + \cdots + r_d = 1$,
$$\int_{\mathbb{R}^n} \prod_{j=1}^{d} f_j^{r_j}(x)dx \leq \prod_{j=1}^{d} \left(\int_{\mathbb{R}^n} f_j(x_j)dx \right)^{r_j}$$
- Loomis-Whitney inequality
$$\int f(y,z)^{\frac{1}{2}} g(x,z)^{\frac{1}{2}} h(x,y)^{\frac{1}{2}} dxdydz \leq \|f\|_{L^1}^{\frac{1}{2}} \|g\|_{L^1}^{\frac{1}{2}} \|h\|_{L^1}^{\frac{1}{2}}$$
- Young's convolution inequality
$$\int f(x)^{\frac{2}{3}} g(x-y)^{\frac{2}{3}} h(y)^{\frac{2}{3}} dxdy \leq C\|f\|_{L^1}^{\frac{2}{3}} \|g\|_{L^1}^{\frac{2}{3}} \|h\|_{L^1}^{\frac{2}{3}}$$

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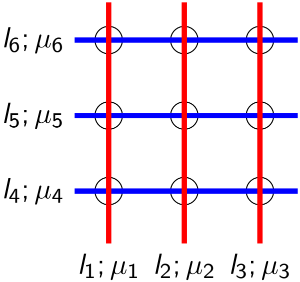
- Loomis-Whitney inequality
$$\int_x \int_y f(x)g(y)dydx \leq \int_x f(x)dx \int_y g(y)dy$$
- Young's convolution inequality
$$\int f(x)^{\frac{2}{3}} g(x-y)^{\frac{2}{3}} h(y)^{\frac{2}{3}} dxdy \leq C\|f\|_{L^1}^{\frac{2}{3}} \|g\|_{L^1}^{\frac{2}{3}} \|h\|_{L^1}^{\frac{2}{3}}$$

Model case: Loomis–Whitney on a finite field

$$\sum_{x \in \mathbb{F}_q^d} \prod_{j=1}^{d} \Bigg(\sum_{\substack{l_j \\ x \in l_j \in L_j}} \mu_{l_j} \Bigg)^{\frac{1}{d-1}} \leq K_d \prod_{j=1}^{d} \Bigg(\sum_{l_j \in L_j} \mu_{l_j} \Bigg)^{\frac{1}{d-1}}$$

- $L_1 = \{l_1, l_2, l_3\}, \ L_2 = \{l_4, l_5, l_6\}$



Model case: Loomis–Whitney on a finite field

$$\sum_{x\in\mathbb{F}_q^d}\prod_{j=1}^{d}\left(\sum_{\substack{l_j\\ x\in l_j\in L_j}}\mu_{l_j}\right)^{\frac{1}{d-1}}\leq K_d\prod_{j=1}^{d}\left(\sum_{l_j\in L_j}\mu_{l_j}\right)^{\frac{1}{d-1}}$$

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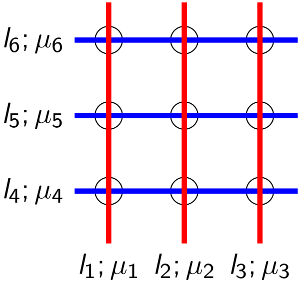

Guth's proof strategy

$$\sum_{x\in\mathbb{F}_q^d}\prod_{j=1}^{d}\left(\sum_{\substack{l_j\\ x\in l_j\in L_j}}\mu_{l_j}\right)^{\frac{1}{d-1}}\leq K_d\prod_{j=1}^{d}\left(\sum_{l_j\in L_j}\mu_{l_j}\right)^{\frac{1}{d-1}}$$

Guth's proof strategy

$$\sum_{x\in\mathbb{F}_q^d}\prod_{j=1}^{d}\Bigg(\sum_{\substack{l_j\\ x\in l_j\in L_j}}\mu_{l_j}\Bigg)^{\frac{1}{d-1}} \leq K_d^{\frac{d}{d-1}}\prod_{j=1}^{d}\Bigg(\sum_{l_j\in L_j}\mu_{l_j}\Bigg)^{\frac{1}{d-1}}$$

- Need this:

$$\begin{aligned} \text{Given } M &: \mathbb{F}_q^d \to \mathbb{R}_+, \\ \text{find } S_j &: \mathbb{F}_q^d \to \mathbb{R}_+, \quad j = 1,\dots,d \\ \text{such that } M(x) &\leq \prod_{j=1}^{d} S_j(x)^{1/d} \quad \forall x \in \mathbb{F}_q^d \\ \sum_{x\in l_j} S_j(x) &\leq K_d \|M\|_d \quad \forall j \quad \forall l_j \in L_j \end{aligned}$$

Guth's argument

$$\sum_{x\in\mathbb{F}_q^d}\prod_{j=1}^{d}\left(\sum_{x\in l_j\in L_j}\mu_{l_j}\right)^{\frac{1}{d-1}} = \sum_x M(x)^d = \sum_x M(x)M(x)^{d-1}$$

$$\leq \sum_x\left(\prod_{j=1}^{d}S_j(x)^{\frac{1}{d}}\right)\left(\prod_{j=1}^{d}\sum_{x\in l_j\in L_j}\mu_{l_j}\right)^{\frac{1}{d}}$$

$$\leq \prod_{j=1}^{d}\left(\sum_x S_j(x)\sum_{x\in l_j\in L_j}\mu_{l_j}\right)^{\frac{1}{d}}$$

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$$\leq K_d\|M\|_d\prod_{j=1}^{d}\left(\sum_{l_j\in L_j}\mu_{l_j}\right)^{\frac{1}{d}}$$

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Guth's argument

$$
\begin{aligned}
\sum_{x\in\mathbb{F}_q^d}\prod_{j=1}^{d}\Big(\sum_{x\in l_j\in L_j}\mu_{l_j}\Big)^{\frac{1}{d-1}} &= \sum_{x} M(x)^d = \sum_{x} M(x)M(x)^{d-1} \\
&\leq \sum_{x}\Big(\prod_{j=1}^{d} S_j(x)^{\frac{1}{d}}\Big)\Big(\prod_{j=1}^{d}\sum_{x\in l_j\in L_j}\mu_{l_j}\Big)^{\frac{1}{d}} \\
&\leq \prod_{j=1}^{d}\Big(\sum_{x} S_j(x)\sum_{x\in l_j\in L_j}\mu_{l_j}\Big)^{\frac{1}{d}} \\
&\leq \prod_{j=1}^{d}\Big(\sum_{l_j\in L_j}\mu_{l_j}\sum_{x\in l_j} S_j(x)\Big)^{\frac{1}{d}} \\
&\leq K_d\|M\|_d\prod_{j=1}^{d}\Big(\sum_{l_j\in L_j}\mu_{l_j}\Big)^{\frac{1}{d}}
\end{aligned}
$$

Guth's argument

$$\sum_{x\in\mathbb{F}_q^d}\prod_{j=1}^{d}\left(\sum_{x\in l_j\in L_j}\mu_{l_j}\right)^{\frac{1}{d-1}} = \sum_x M(x)^d = \sum_x M(x)M(x)^{d-1}$$

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$$\le \prod_{j=1}^{d}\left(\sum_x S_j(x)\sum_{x\in l_j\in L_j}\mu_{l_j}\right)^{\frac{1}{d}}$$

$$\le \prod_{j=1}^{d}\left(\sum_{l_j\in L_j}\mu_{l_j}\sum_{x\in l_j}S_j(x)\right)^{\frac{1}{d}}$$

$$\le K_d\|M\|_d\prod_{j=1}^{d}\left(\sum_{l_j\in L_j}\mu_{l_j}\right)^{\frac{1}{d}}$$

$$M(x)\le \prod_{j=1}^{d} S_j(x)^{1/d}$$

Guth's argument

$$\begin{aligned}
\sum_{x\in\mathbb{F}_q^d}\prod_{j=1}^{d}\left(\sum_{x\in l_j\in L_j}\mu_{l_j}\right)^{\frac{1}{d-1}} &= \sum_x M(x)^d = \sum_x M(x)M(x)^{d-1}\\
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&\le \prod_{j=1}^{d}\left(\sum_x S_j(x)\sum_{x\in l_j\in L_j}\mu_{l_j}\right)^{\frac{1}{d}}\\
&\le \prod_{j=1}^{d}\left(\sum_{l_j\in L_j}\mu_{l_j}\sum_{x\in l_j}S_j(x)\right)^{\frac{1}{d}}\\
&\le K_d\|M\|_d\prod_{j=1}^{d}\left(\sum_{l_j\in L_j}\mu_{l_j}\right)^{\frac{1}{d}}
\end{aligned}$$

Guth's argument

$$\sum_{x\in\mathbb{F}_q^d}\prod_{j=1}^{d}\left(\sum_{x\in l_j\in L_j}\mu_{l_j}\right)^{\frac{1}{d-1}} = \sum_x M(x)^d = \sum_x M(x)M(x)^{d-1}$$

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$$\sum_{x\in l_j} S_j(x) \leq K_d\|M\|_d$$

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Convex optimisation

- Given M

$$\begin{aligned} &\text{minimise } K \\ &\text{such that } M(x) \leq \prod_{j=1}^{d} S_j(x)^{1/d} \quad \forall x \in \mathbb{F}_q^d \\ &\text{and } \sum_{x \in l_j} S_j(x) \leq K \|M\|_d \quad \forall j \quad \forall l_j \in L_j \end{aligned}$$

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- $d\, q^d + 1$ variables

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- $d\, q^d + 1$ variables
- Try duality theory

The Lagrangian

$$\begin{aligned}L(K, S_j; g, \lambda_{l_j}) =& K + \sum_{x \in \mathbb{F}_q^d} g(x) \left(M(x) - \prod_{j=1}^{d} S_j(x)^{1/d} \right) \\ &+ \sum_{j=1}^{d} \sum_{l_j \in L_j} \lambda_{l_j} \left(\sum_{x \in l_j} S_j(x) - K \|M\|_d \right)\end{aligned}$$

The Lagrangian

$$L(K, S_j; g, \lambda_{l_j}) = K + \sum_{x \in \mathbb{F}_q^d} g(x) \left(M(x) - \prod_{j=1}^{d} S_j(x)^{1/d} \right) + \sum_{j=1}^{d} \sum_{l_j \in L_j} \lambda_{l_j} \left(\sum_{x \in l_j} S_j(x) - K \|M\|_d \right)$$

- Dual function $h(g, \lambda_{l_j}) = \inf_{K, S_j} L$

The Lagrangian

$$L = \sum_{x \in \mathbb{F}_q^d} g(x) M(x) + \sum_{x \in \mathbb{F}_q^d} \left(\sum_{j=1}^{d} \left(\sum_{x \in l_j \in L_j} \lambda_{l_j} \right) S_j(x) - g(x) \prod_{j=1}^{d} S_j(x)^{1/d} \right)$$

$$+ \left(1 - \|M\|_d \sum_{j=1}^{d} \sum_{l_j \in L_j} \lambda_{l_j} \right) K$$

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- Dual function $h(g, \lambda_{l_j}) = \inf_{K, S_j} L$
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- $\sum_{j=1}^{d} \left(\sum_{x \in l_j \in L_j} \lambda_{l_j} \right) S_j(x) \geq d \prod_{j=1}^{d} \left(\sum_{x \in l_j \in L_j} \lambda_{l_j} \right)^{1/d} \prod_{j=1}^{d} S_j(x)^{1/d}$

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The Lagrangian

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- $g(x) \leq d \prod_{j=1}^{d} \left(\sum_{x \in l_j \in L_j} \lambda_{l_j} \right)^{1/d}$

The Lagrangian

$$L = \sum_{x \in \mathbb{F}_q^d} g(x) M(x) + \sum_{x \in \mathbb{F}_q^d} \left(\sum_{j=1}^{d} \left(\sum_{x \in l_j \in L_j} \lambda_{l_j} \right) S_j(x) - g(x) \prod_{j=1}^{d} S_j(x)^{1/d} \right)$$

- Dual function $h(g, \lambda_{l_j}) = \inf_{K, S_j} L$
- $\|M\|_d \sum_{j=1}^{d} \sum_{l_j \in L_j} \lambda_{l_j} = 1$
- $\sum_{j=1}^{d} \left(\sum_{x \in l_j \in L_j} \lambda_{l_j} \right) S_j(x) \geq d \prod_{j=1}^{d} \left(\sum_{x \in l_j \in L_j} \lambda_{l_j} \right)^{1/d} \prod_{j=1}^{d} S_j(x)^{1/d}$
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The Lagrangian

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The Lagrangian

$$L = \sum_{x \in \mathbb{F}_q^d} d \prod_{j=1}^{d} \left(\sum_{x \in l_j \in L_j} \lambda_{l_j} \right)^{1/d} M(x)$$

- Dual function $h(g, \lambda_{l_j}) = \inf_{K, S_j} L$
- $\|M\|_d \sum_{j=1}^{d} \sum_{l_j \in L_j} \lambda_{l_j} = 1$
- $\sum_{j=1}^{d} \left(\sum_{x \in l_j \in L_j} \lambda_{l_j} \right) S_j(x) \geq d \prod_{j=1}^{d} \left(\sum_{x \in l_j \in L_j} \lambda_{l_j} \right)^{1/d} \prod_{j=1}^{d} S_j(x)^{1/d}$
- $g(x) \leq d \prod_{j=1}^{d} \left(\sum_{x \in l_j \in L_j} \lambda_{l_j} \right)^{1/d}$

The dual problem

$$\text{maximise } d \sum_{x \in \mathbb{F}_q^d} \prod_{j=1}^{d} \left(\sum_{x \in l_j \in L_j} \lambda_{l_j} \right)^{1/d} M(x)$$

$$\text{such that } \|M\|_d \sum_{j=1}^{d} \sum_{l_j \in L_j} \lambda_{l_j} = 1$$

Estimating the dual problem

$$\sum_{x\in\mathbb{F}_q^d}\prod_{j=1}^{d}\left(\sum_{x\in l_j\in L_j}\lambda_{l_j}\right)^{\frac{1}{d}} M(x) \leq \left(\sum_{x\in\mathbb{F}_q^d}\prod_{j=1}^{d}\left(\sum_{x\in l_j\in L_j}\lambda_{l_j}\right)^{\frac{1}{d-1}}\right)^{\frac{d-1}{d}} \|M\|_d$$

$$\leq \left(K_d^{\frac{d}{d-1}}\prod_{j=1}^{d}\left(\sum_{l_j\in L_j}\lambda_{l_j}\right)^{\frac{1}{d-1}}\right)^{\frac{d-1}{d}} \|M\|_d$$

$$= K_d\prod_{j=1}^{d}\left(\sum_{l_j\in L_j}\lambda_{l_j}\right)^{\frac{1}{d}} \|M\|_d$$

$$\leq K_d\frac{1}{d}\sum_{j=1}^{d}\sum_{l_j\in L_j}\lambda_{l_j}\|M\|_d$$

Estimating the dual problem

$$\begin{aligned}
\sum_{x\in\mathbb{F}_q^d}\prod_{j=1}^{d}\left(\sum_{x\in l_j\in L_j}\lambda_{l_j}\right)^{\frac{1}{d}} M(x) &\leq \left(\sum_{x\in\mathbb{F}_q^d}\prod_{j=1}^{d}\left(\sum_{x\in l_j\in L_j}\lambda_{l_j}\right)^{\frac{1}{d-1}}\right)^{\frac{d-1}{d}} \|M\|_d \\
&\leq \left(K_d^{\frac{d}{d-1}}\prod_{j=1}^{d}\left(\sum_{l_j\in L_j}\lambda_{l_j}\right)^{\frac{1}{d-1}}\right)^{\frac{d-1}{d}} \|M\|_d \\
&= K_d\prod_{j=1}^{d}\left(\sum_{l_j\in L_j}\lambda_{l_j}\right)^{\frac{1}{d}} \|M\|_d \\
&\leq K_d\frac{1}{d}\sum_{j=1}^{d}\sum_{l_j\in L_j}\lambda_{l_j}\|M\|_d
\end{aligned}$$

Estimating the dual problem

$$\sum_{x\in\mathbb{F}_q^d}\prod_{j=1}^{d}\left(\sum_{x\in l_j\in L_j}\lambda_{l_j}\right)^{\frac{1}{d}} M(x) \leq \left(\sum_{x\in\mathbb{F}_q^d}\prod_{j=1}^{d}\left(\sum_{x\in l_j\in L_j}\lambda_{l_j}\right)^{\frac{1}{d-1}}\right)^{\frac{d-1}{d}} \|M\|_d$$

$$\leq \left(K_d^{\frac{d}{d-1}}\prod_{j=1}^{d}\left(\sum_{l_j\in L_j}\lambda_{l_j}\right)^{\frac{1}{d-1}}\right)^{\frac{d-1}{d}} \|M\|_d$$

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Estimating the dual problem

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The dual problem

$$\text{maximise } d \sum_{x \in \mathbb{F}_q^d} \prod_{j=1}^{d} \left(\sum_{x \in l_j \in L_j} \lambda_{l_j} \right)^{1/d} M(x)$$

$$\text{such that } \|M\|_d \sum_{j=1}^{d} \sum_{l_j \in L_j} \lambda_{l_j} = 1$$

$$\sum_{x \in \mathbb{F}_q^d} \prod_{j=1}^{d} \left(\sum_{x \in l_j \in L_j} \lambda_{l_j} \right)^{\frac{1}{d}} M(x) \leq K_d \frac{1}{d} \sum_{j=1}^{d} \sum_{l_j \in L_j} \lambda_{l_j} \|M\|_d$$

The dual problem

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- The maximum is less that K_d

The dual problem

$$\text{maximise } d \sum_{x \in \mathbb{F}_q^d} \prod_{j=1}^{d} \left(\sum_{x \in l_j \in L_j} \lambda_{l_j} \right)^{1/d} M(x)$$

$$\text{such that } \|M\|_d \sum_{j=1}^{d} \sum_{l_j \in L_j} \lambda_{l_j} = 1$$

- The maximum is less that K_d
- Strong duality gives a solution to

$$M(x) \leq \prod_{j=1}^{d} S_j(x)^{1/d} \quad x \in \mathbb{F}_q^d$$

$$\sum_{x \in l_j} S_j(x) \leq K_d \|M\|_d \quad \forall j \quad \forall l_j \in L_j$$

Equivalence

$$\left.\begin{array}{ll} M(x) \leq \prod_j S_j(x)^{1/d} & x \in \mathbb{F}_q^d \\ \sup_{l_j \in L_j} \sum_{x \in l_j} S_j(x) \leq K \|M\|_{L^d(\mathbb{F}_q^d)} & \forall j \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} \sum_{x \in \mathbb{F}_q^d} \prod_j \sum_{x \in l_j \in L_j} \mu_{l_j}^{1/(d-1)} \\ \leq K^{\frac{d}{d-1}} \prod_j \left(\sum_{l_j \in L_j} \mu_{l_j} \right)^{\frac{1}{d-1}} \end{array}\right.$$

General version

$$\left.\begin{array}{ll} M(x) \le \prod_j S_j(x)^{1/d} & x \in \mathbb{F}_q^d \\ \sup_{l_j \in L_j} \sum_{x \in l_j} S_j(x) \le K \|M\|_{L^d(\mathbb{F}_q^d)} & \forall j \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} \sum_{x \in \mathbb{F}_q^d} \prod_j \sum_{x \in l_j \in L_j} \mu_{l_j}^{1/(d-1)} \\ \le K^{\frac{d}{d-1}} \prod_j \left(\sum_{l_j \in L_j} \mu_{l_j} \right)^{\frac{1}{d-1}} \end{array}\right.$$

$$\left.\begin{array}{ll} M(x) \le \prod_j S_j(x)^{r_j/r} & \text{a.e. } x \in X \\ \|T_j^* S_j\|_{L^{p_j'}(Y_j)} \le K \|M\|_{L^{r'}(X)} & \forall j \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} \int_X \prod_j T_j f_j(x)^{r_j} \, d\mu(x) \\ \le K^r \prod_j \|f_j\|_{L^{p_j}(Y_j)}^{r_j} \end{array}\right.$$

- $(X, d\mu)$ and $(Y_j, d\nu_j)$ for $j = 1, \ldots, d$ measure spaces
- $T_1, \ldots, T_d$ positive linear operators from functions on Y_j to functions on X
- $1 \le p_j \le \infty$
- $r_j > 0$ for $j = 1, \ldots, d$, $r = \sum_j r_j$ with $1 \le r < \infty$
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The Brascamp–Lieb inequality

$$\left.\begin{array}{l} M(x) \le \prod_j S_j(x)^{r_j/r} \quad \text{a.e. } x \in \mathbb{R}^n \\ \left\| \int_{Y_j^\perp} S_j(y,\tilde{y})d\tilde{y} \right\|_{L^\infty(\mathbb{R}^{n_j})} \le K \|M\|_{L^{r'}(\mathbb{R}^n)} \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} \int_{\mathbb{R}^n} \prod_j f_j(B_j x)^{r_j} dx \\ \le K^r \prod_j \|f_j\|_{L^1(\mathbb{R}^{n_j})}^{r_j} \end{array}\right.$$

- $X \simeq \mathbb{R}^n$, $Y_j \simeq \mathbb{R}^{n_j}$, $Y_j \subset X$
- $B_j : X \to Y_j$ projection
- $0 < r_j \le 1$, $((B_j), (r_j))$ Brascamp–Lieb datum
- $T_j f_j(x) = f_j(B_j x)$ so $T_j^* S_j(y) = \int_{Y_j^\perp} S_j(y, \tilde{y}) d\tilde{y}$
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The Brascamp–Lieb factorisation

$$\left.\begin{array}{l} M(x) \leq \prod_j S_j(x)^{r_j/r} \quad \text{a.e. } x \in \mathbb{R}^n \\ \left\| \int_{Y_j^\perp} S_j(y, \tilde{y}) d\tilde{y} \right\|_{L^\infty(\mathbb{R}^{n_j})} \leq K \|M\|_{L^{r'}(\mathbb{R}^n)} \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} \int_{\mathbb{R}^n} \prod_j f_j(B_j x)^{r_j} dx \\ \leq K^r \prod_j \|f_j\|_{L^1(\mathbb{R}^{n_j})}^{r_j} \end{array}\right.$$

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Loomis–Whitney

- Two dimensions (Fubini)
  - $M \in L^2(\mathbb{R}^2)$, $\|M\|_2 = 1$
  - $M(x,y) = \sqrt{S(x,y)T(x,y)}$.
  - $\sup_x \int_y S(x,y)dy \le 1$
  - $\sup_y \int_x T(x,y)dx \le 1$

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  - $\sup_y \int_x T(x, y)dx \leq 1$
- Three dimensions
  - $M \in L^3(\mathbb{R}^3)$, $\|M\|_3 = 1$
  - $M(x, y, z) = \sqrt[3]{S(x, y, z)T(x, y, z)U(x, y, z)}$.
  - $\sup_{x,y} \int_z S(x, y, z)dz \leq 1$
  - $\sup_{x,z} \int_y T(x, y, z)dy \leq 1$
  - $\sup_{y,z} \int_x U(x, y, z)dx \leq 1$

Geometric Brascamp–Lieb

- $Y_j \simeq \mathbb{R}^{n_j}$ subspaces of $X \simeq \mathbb{R}^n$, B_j projection onto Y_j
- $0 < r_j \leq 1$, $r = \sum r_j$
- $\displaystyle\sum_{j=1}^{d} r_j B_j^* B_j = \mathrm{Id}_{\mathbb{R}^n}$
- $\displaystyle\int \prod_{j=1}^{d} f_j^{r_j}(B_j x) dx \leq \prod_{j=1}^{d} \left(\int_{Y_j} f_j \right)^{r_j}$

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Thank you